NEAR-RINGS WITH IDENTITIES ON DIHEDRAL GROUPS

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(Received 27th March 1972)

1. Introduction

Normal right submodules and right ideals need not coincide in an arbitrary near-ring. Berman and Silverman (**1**) have shown that in a near-ring $(N, +, \cdot)$ with a two-sided zero (i.e. $x \cdot 0 = 0 \cdot x = 0$, for all $x \in N$) a right ideal is also a right submodule. If $(N, +, \cdot)$ is in fact a distributively generated near-ring, then all normal right submodules are also right ideals. (See (**5**).)

$T_0(G)$, the near-ring of all transformations from a group $(G, +)$ into itself which map 0 into 0, is a near-ring with a two-sided zero. In (**6**) it is shown that normal right submodules and right ideals coincide if and only if all normal right submodules are sums of annihilator right ideals.

In this paper we examine the following more general question. Under what conditions do normal right submodules and right ideals coincide in an arbitrary near-ring $(G, +, \cdot)$ with a two-sided zero? In Section 3 we reach the conclusion that the answer to this question depends more on the particular multiplication $\cdot$ than on the structure of the group $(G, +)$.

In Section 4 we consider near-rings defined on $(D_n, +)$, the dihedral group of order $2n$. We show that if $(D_n, +, \cdot)$ is a near-ring with multiplicative identity, then normal right submodules and right ideals of $(D_n, +, \cdot)$ coincide if and only if all normal right submodules are annihilator right ideals.

In Section 5 we prove the existence of a near-ring with multiplicative identity on $(D_n, +)$ where $n = 2p$, p a prime. In (**2**) Clay determined all near-rings with identity on $(D_4, +)$. We show here that if p is an odd prime, then there is, up to isomorphism, a unique near-ring with multiplicative identity on $(D_n, +)$.

2. Definitions

A *(left) near-ring* is a triple $(N, +, \cdot)$ such that $(N, +)$ is a group, $(N, \cdot)$ is a semigroup, and $\cdot$ is left distributive over $+$.

An *ideal* H of N is a normal subgroup of $(N, +)$ such that:

(1) $(n+h)m - nm \in H$ for all $h \in H$; $n, m \in N$

(2) $nh \in H$ for all $h \in H$; $n \in N$.

A *right ideal* of N is a normal subgroup of $(N, +)$ which satisfies condition (1). A *left ideal* of N is a normal subgroup of $(N, +)$ which satisfies condition (2).

A *right submodule* K of N is a subgroup of $(N, +)$ such that $k \cdot n \in K$, for all $k \in K$, $n \in N$. K is a *normal right submodule* if, in addition, $(K, +)$ is a normal subgroup of $(N, +)$.

Let C be a subset of N. It is easy to show that

$$A(C) = \{x \in N \mid cx = 0, \text{ for all } c \in C\}$$

is a right ideal of N. Right ideals of this type are called *annihilator right ideals* (6).

3. General results

Let $(G, +)$ be a group. If $(G, +)$ contains no proper normal subgroups, then the only normal right submodules or right ideals of a near-ring with two-sided zero whose additive group is $(G, +)$ are $\{0\}$ and G itself. Hence, normal right submodules and right ideals must coincide in this case.

Now suppose $(G, +)$ is not a simple group. From (6) we note that the structure of the group $(G, +)$ determines whether or not normal right submodules and right ideals of $T_0(G)$ coincide. This is not the case if we consider an arbitrary near-ring with a two-sided zero defined on $(G, +)$. Instead, as the following theorem illustrates, it is the structure of the particular multiplication that determines whether or not normal right submodules are right ideals.

We use the following notation. Let G be a group and let C be a subset of G. We denote the complement of C in G by $G-C$, i.e. $G-C = \{x \in G \mid x \notin C\}$. If $g \in G$, then $|g|$ denotes the order of g.

Theorem 3.1. *Let $(G, +)$ be a group which contains a proper normal subgroup H. Then two near-rings, each containing a two-sided zero, can be defined on $(G, +)$ such that normal right submodules and right ideals coincide in one near-ring, but not in the other.*

Proof. Clearly zero multiplication on $(G, +)$ defines a near-ring $(G, +, *)$ in which normal right submodules and right ideals coincide.

Now define a binary operation $\cdot$ on G as follows. Choose an element $n \in G-H$. For all $y \in G$, define $n \cdot y = y$ and $x \cdot y = 0$ if $x \neq n$. Then by Theorem 1.8 of (3) $(G, +, \cdot)$ is a near-ring. Clearly H is a normal right submodule of $(G, +, \cdot)$. However, H is not a right ideal, since if h is a non-zero element of H, then

$$(n+h)n - n \cdot n = -n \notin H.$$

Annihilator right ideals play a major role in the study of the structure of right ideals of near-rings. From (6) every right ideal of $T_0(G)$ is the sum of annihilator right ideals. Also, the significance of annihilator right ideals in near-rings defined on the dihedral group $(D_n, +)$ of order $2n$ was stated in the introduction. Theorem 3.3 indicates the frequent occurrence of annihilator right ideals in near-rings with multiplicative identity.

For easy reference we state the following result from (**4**):

Lemma 3.2. *Let* $(G, +)$ *be a finite group. If* $(G, +, *)$ *is a near-ring and* $e \in G$ *is an identity with respect to* $*$, *then* $|x|$ *divides* $|e|$, *for all* $x \in G$.

Theorem 3.3. *Let* $(G, +)$ *be a finite group of composite order. Suppose* G *contains two non-zero elements* x *and* y *such that* $|x| \neq |y|$. *Then any near-ring* $(G, +, \cdot)$ *with multiplicative identity must contain a proper annihilator right ideal.*

Proof. Let e be the multiplicative identity of $(G, +, \cdot)$. Let q be a prime divisor of the order of G and let $g \in G$ such that $|g| = q$. Note that q is a proper divisor of $|e|$, since G contains two non-zero elements of different order. Let $A = \{z \in G \mid gz = 0\}$. Then A is an annihilator right ideal of $(G, +, \cdot)$. The element $qe \in A$, so $A \neq \{0\}$, and since e is a multiplicative identity, $A \neq G$.

Clearly, if G is a p-group of composite order, for some prime p, then G is not simple. Hence as an immediate corollary of Theorem 3.3 we obtain the following result of (**4**).

Corollary 3.4. *A simple group of composite order cannot be the additive group of a near-ring with identity.*

4. Ideal and submodule structure of near-rings with identities on dihedral groups

The dihedral group $(D_n, +)$, $n > 2$, is a group of order $2n$ generated by two elements e and b which satisfy the relations $ne = 0$, $2b = 0$, $b+e = (n-1)e+b$. In this section we examine normal right submodules and right ideals of near-rings defined on $(D_n, +)$. We consider only near-rings with multiplicative identity.

Using Lemma 3.2, the following result is immediate.

Proposition 4.1. *If* n *is an odd integer, then* $(D_n, +)$ *cannot be the additive group of a near-ring with multiplicative identity.*

Hence, we restrict our attention to dihedral groups $(D_n, +)$, where n is an even integer. Since the order of every element of D_n must divide the order of a multiplicative identity, we can assume without loss of generality that the multiplicative identity of a near-ring $(D_n, +, \cdot)$ is e.

The next result follows immediately from Theorem 1.1 of (**3**).

Lemma 4.2. *Let* $(D_n, +, \cdot)$ *be a near-ring and let* $x, y \in D_n$. *Then* $|x \cdot y|$ *divides* $|y|$.

Note that every element of $(D_n, +)$ of the form $me+b$, $1 \leqq m \leqq n$, has order 2.

Proposition 4.3. *If* $(D_n, +, \cdot)$ *is a near-ring with multiplicative identity* e, *then* $0 \cdot x = x \cdot 0 = 0$, *for all* $x \in D_n$.

Proof. That $x \cdot 0 = 0$, for all $x \in D_n$, follows from the definition of (left) near-ring. We show $0 \cdot x = 0$.

Clearly, $0(me) = 0$, for all m, $1 \leqq m \leqq n$. Then $0 \cdot b = 0(me+b)$, $1 \leqq m \leqq n$. Hence it suffices to show that $0 \cdot b = 0$.

By Lemma 4.2 the order of $0 \cdot b$ must divide 2. Since $(e+0b)(e+0b) = e$, then by Lemma 4.2, n must divide the order of $e+0b$. Hence either $0b = 0$ or $0b = (n/2)e$. Since $0(0b) = 0b$, then $0b = 0$.

Proposition 4.4. *If $(D_n, +, \cdot)$ is a near-ring with multiplicative identity, then $n = 2p$, where p is a prime number.*

Proof. Let $r = 2cd$, where $c > 1$, $d > 1$. Suppose that $(D_r, +, \cdot)$ is a near-ring with multiplicative identity e. Let $X = \{x \in D_r \mid (ce)x = 0\}$. Then X is a right ideal of $(D_r, +, \cdot)$ and $X \neq \{0\}$, since $2de \in X$.

By Lemma 4.2, $|2de \cdot b|$ divides 2. Hence $2de \cdot b$ is either 0, cde, or $me+b$, for some integer m, $0 < m \leqq r$. We examine these three possibilities.

If $2de \cdot b = 0$, then $2de(e+b) = 2de$. Then by Lemma 4.2, $|2de| = 2$ and hence $c = 2$. Thus $(ce)(de \cdot b) = 0$, so $de \cdot b \in X$. Consider the element $de \cdot b$. By Lemma 4.2, $|de \cdot b|$ divides 2. If $de \cdot b = 0$, then $de(e+b) = de$, which is a contradiction to Lemma 4.2. Similarly $de \cdot b \neq cde$. If $de \cdot b = me+b$ for some m, $0 < m \leqq r$, then $me+b \in X$ and so $ce[(m+1)e+b] = ce$, which is a contradiction to Lemma 4.2. We have reached a contradiction for every possible value for $de \cdot b$. Hence, $2de \cdot b \neq 0$. Also $2de \cdot b \neq me+b$, for any m, since from above, $me+b \notin X$.

If $2de \cdot b = cde$, then $2de(e+b) = 2de+cde$. Then by Lemma 4.2 $|2de+cde|$ divides 2 and hence it follows that $c = 2$. Thus $ce[de(e+b)] = 0$, so $de(e+b) \in X$. But then, using arguments similar to the above, we reach a contradiction for every possible value for $de(e+b)$. Hence, $2de \cdot b \neq cde$.

So we have eliminated all possible values of $2de \cdot b$. This contradiction tells us that $(D_r, +)$ cannot be the additive group of a near-ring with multiplicative identity.

Since we are interested in near-rings with multiplicative identity, from now on $(D_n, +)$ will denote a dihedral group where $n = 2p$, p a prime, and $(D_n, +, \cdot)$ will denote a near-ring with multiplicative identity e on the above group.

The following lemma is routine.

Lemma 4.5. *The only proper normal subgroups of $(D_n, +)$ are*

$$A = \{0, 2e, 4e, \ldots, (n-2)e, b, 2e+b, \ldots, (n-2)e+b\},$$
$$B = \{0, 2e, 4e, \ldots, (n-2)e, e+b, 3e+b, \ldots, (n-1)e+b\},$$

and subgroups of the group generated by e.

Hence, the only candidates for proper normal right submodules of a near-ring $(D_n, +, \cdot)$ with multiplicative identity are A, B, and proper subgroups of the group generated by e.

Let $C = \{0, 2e, 4e, \ldots, (n-2)e\}$.

Proposition 4.6. *C is a right submodule of $(D_n, +, \cdot)$ if and only if $n = 4$.*

Proof. By Lemma 4.2 $|2e\cdot b|$ divides 2. Hence $2e\cdot b$ is either 0, $(n/2)e$, or $me+b$, for some integer m, $0<m\leqq n$.

Suppose $n=4$. If $2e\cdot b=me+b$, then

$$0=[(2e)(2e)]\cdot b=2e(2e\cdot b)=2e(me+b)\neq 0.$$

This contradiction shows that $2e\cdot b\in C$. It follows easily that C is a right submodule.

Now suppose $n\neq 4$. If $2e\cdot b=0$, then $2e(e+b)=2e$. But this is a contradiction, since $|2e|$ does not divide 2. Hence, $2e\cdot b\notin C$, so C is not a right submodule.

We consider the two cases, $n=4$ and $n=2q$, q an odd prime, separately. We show if $n=2q$, q an odd prime, then normal right submodules and right ideals of a near-ring with multiplicative identity on $(D_n,+)$ will always coincide. However, if $(D_4,+,\cdot)$ is a near-ring with multiplicative identity, normal right submodules and right ideals coincide if and only if all normal right submodules are annihilator right ideals.

First we consider the case $n=2q$, q an odd prime. Throughout we assume $(D_n,+,\cdot)$ is a near-ring with multiplicative identity e. Let $F=\{0, qe\}$.

Proposition 4.7. *F is a right ideal of $(D_n,+,\cdot)$, $n=2q$.*

Proof. Clearly F is a normal subgroup of $(D_n,+)$. Let

$$X=\{x\in D_n \mid 2e\cdot x=0\}.$$

Then X is a right ideal and $F\subset X$. Since $2e$ is an element of all non-zero normal subgroups of $(D_n,+)$ which contain F as a proper subgroup, but $2e\notin X$, then $F=X$.

Proposition 4.8. *Either A or B, but not both, is a right submodule, and in particular, an annihilator right ideal of $(D_n,+,\cdot)$, $n=2q$.*

Proof. A and B cannot both be right submodules, since by Proposition 4.6, $C=A\cap B$ is not a right submodule.

Let $X=\{x\in D_n \mid yx=0$, for all $y\in D_n$ such that $|y|=2\}$. Then X is a right ideal of $(D_n,+,\cdot)$ and $C\subset X$. Since C is not a right submodule, $C\neq X$. Hence, either $X=A$ or $X=B$.

Combining Lemma 4.5 and Propositions 4.6, 4.7, 4.8, we see that the only normal right submodules of $(D_n,+,\cdot)$ are either A and F or B and F. Hence we obtain the following theorem.

Theorem 4.9. *Normal right submodules and right ideals of $(D_{2q},+,\cdot)$ coincide. In addition all right ideals are annihilator right ideals.*

Now consider near-rings on $(D_4,+)$. Again we will assume that $(D_4,+,\cdot)$ denotes a near-ring with multiplicative identity e. In Proposition 4.6 we showed that C is a right submodule of $(D_4,+,\cdot)$. Now we show that both A and B are also right submodules.

Proposition 4.10. *A is a normal right submodule of* $(D_4, +, \cdot)$.

Proof. From Lemma 4.5 we already know that A is a normal subgroup of $(D_4, +)$. We show that A is a right submodule.

Let $m \in \{0, 2\}$. Since C is a right submodule, $me \cdot x \in C \subset A$, for all $x \in D_4$. By Lemma 4.2, $|(me+b)\cdot b|$ divides 2. If $(me+b)\cdot b = te+b$, where t is an odd integer, then $(me+b)(e+b)$ is an odd multiple of e, contradicting Lemma 4.2. Hence $(me+b)\cdot b \in A$. It follows easily that $(me+b)\cdot x \in A$, for all $x \in D_4$.

The proof of the next result is analogous.

Proposition 4.11. *B is a normal right submodule of* $(D_4, +, \cdot)$.

Inspection of the normal subgroups A and B yields the following.

Lemma 4.12. *If $x \in A-B$, then $e+x \in B-A$. If $y \in B-A$, then $e+y \in A-B$.*

Lemma 4.13. *Suppose both A and B are right ideals of $(D_4, +, \cdot)$. Let $x \in A-B$, $y \in B-A$. Then*

(1) $(e+x)x \in A \cap B$
(2) $(e+x)y \in B-A$
(3) $(e+y)y \in A \cap B$
(4) $(e+y)x \in A-B$.

Proof. Since A is a right ideal, then $(e+x)x-x = (e+x)x-ex \in A$. But $x \in A$, so $(e+x)x \in A$. From Lemma 4.12 $(e+x) \in B$ and hence $(e+x)x \in B$. Therefore, $(e+x)x \in A \cap B$. Since $y \in B-A$ and $(e+x)y-y \in A$, then

$$(e+x)y \in B-A.$$

The proofs of (3) and (4) are analogous.

Proposition 4.14. *If both A and B are right ideals of $(D_4, +, \cdot)$, then both A and B are annihilator right ideals.*

Proof. Let $x \in A-B$. Then by Lemma 4.13, $(2e+x)(e+x) \in A \cap B = \{0, 2e\}$, so $x[(2e+x)(e+x)] = 0$. Let $Y = \{y \in D_4 \mid [x(2e+x)]y = 0\}$. Using Lemma 4.12, we can show that $x \cdot x \neq 0$, and hence $x(2e+x) \neq 0$. Then Y is a proper right ideal and $e+x \in Y$. Hence $Y = B$, so B is an annihilator right ideal.

Similarly, A is an annihilator right ideal.

If both A and B are right ideals of $(D_4, +, \cdot)$, then $A \cap B = C$ is also a right ideal. In addition we have the following result.

Proposition 4.15. *If both A and B are right ideals of $(D_4, +, \cdot)$, then C is an annihilator right ideal.*

Proof. $C = \{x \in D_4 \mid yx = 0$, for all $y \in D_4$ such that $|y| = 2\}$.

Combining Propositions 4.14 and 4.15, we obtain:

Theorem 4.16. *Normal right submodules and right ideals of $(D_4, +, \cdot)$ coincide if and only if all normal right submodules are annihilator right ideals.*

We will need the following result in Section 5.

Proposition 4.17. *Let $n = 2p$, p a prime. If A, B, or C is a right ideal of a near-ring $(D_n, +, \cdot)$ with multiplicative identity, then it is also a left ideal.*

Proof. It is easy to show that if C is a right ideal, then it is also a left ideal.

Assume A is a right ideal. Let $y \in A$ and let $x \in D_n - A$. Then $x = e+z$, where $z \in A$. Since A is a right ideal, $(e+z)y - ey \in A$. But $-ey \in A$, so $xy = (e+z)y \in A$. Hence A is also a left ideal.

Similarly, if B is a right ideal, then B is also a left ideal.

5. Existence of near-rings on dihedral groups

In (2) Clay showed that there are seven non-isomorphic classes of near-rings with multiplicative identity on the dihedral group $(D_4, +)$. These seven classes of near-rings are listed in Table 3 of (2). Normal right submodules and right ideals coincide in the near-rings in Classes 2, 4, and 7. However, there exists a normal right submodule which is not a right ideal in each near-ring in Classes 1, 3, 5 and 6.

In this section we show the existence of a near-ring with multiplicative identity on $(D_n, +)$, *where $n = 2q$, q an odd prime*. Thus, we need to define a binary operation $*$ on D_n such that $*$ is left distributive over $+$. Using Clay's terminology, we call such an operation $*$ a multiplication. From (3), if $(G, +)$ is an arbitrary group, then $*$ is a multiplication on G if and only if there exists a function f from G into the endomorphisms of G such that $x*y = f(x)(y)$ for all $x, y \in G$.

By Proposition 4.8 either A or B must be a right submodule of a near-ring on $(D_n, +)$. First consider the case when A is a right submodule. We wish to define an associative multiplication $\cdot_f$ on $(D_n, +)$ such that e is an identity for $\cdot_f$ and A is a right submodule of $(D_n, +, \cdot_f)$. Denote the endomorphisms which define $\cdot_f$ by $f(x)$, for all $x \in D_n$. These endomorphisms must satisfy the following criteria.

Since e is to be an identity for $\cdot_f$, then $f(x)(e) = x$, for all $x \in D_n$, $f(e)(b) = b$, and $f(0)$ is the zero endomorphism. Since A is to be a right submodule, then from the proof of Proposition 4.8 we see that every element of A annihilates every element in D_n of order 2. Hence $f(me+b)(b) = 0$, for all m, $1 \leqq m \leqq n$, and $f(qe)(b) = 0$. Clearly, $f(e)$, $f(me+b)$, and $f(qe)$ extended to all of D_n are endomorphisms. In order to have a multiplication on $(D_n, +)$, we need only define $f(me)(b)$, for all m, $1 < m < n$, $m \neq q$, in such a way that $f(me)$ extended to all of D_n is also an endomorphism.

Lemma 5.1. *Let r be an even integer. If A is a right submodule of a near-ring $(D_n, +, \cdot)$, $n = 2q$, then $me(re+b) = [(q+m)e](re+b)$, for all m, $1 \leqq m \leqq n$.*

Proof. Let $[(q+m)e](re+b) = x$. Since

$$(2e)x = (2e)[(q+m)e](re+b) = 2me(re+b),$$

then $2e[me(re+b)-x] = 0$. Hence $me(re+b)-x \in F = \{0, qe\}$. But A is also a left ideal by Proposition 4.17, so both x and $me(re+b)$ are elements of A. Therefore, $x = me(re+b)$.

From number theory (7) we know that since q is a prime, then q has a primitive root, say h. Then $0, h, h^2, \ldots, h^{q-1}$ are all distinct modulo q. Hence every integer m, $1 \leqq m \leqq q-1$, can be written as a power of h. Since $\cdot_f$ must be associative, once $f(he)(b)$ is defined, then $f(h^r e)(b)$ is uniquely determined for all r, $1 \leqq r \leqq q-1$. Using Lemma 5.1, it follows that $f(me)(b)$ is uniquely determined for all m, $1 \leqq m \leqq n$. Hence the definition of $\cdot_f$ depends only upon the value of $f(he)(b)$.

Since $b \in A$, a left ideal, then $f(he)(b) \in A$. It is routine to prove that $f(he)$ will extend to an endomorphism of D_n if and only if $f(he)(b) = xe+b$, for some integer x, $1 \leqq x \leqq n$. Hence there are at most q near-rings definable on $(D_n, +)$ with e as multiplicative identity and A as right submodule. Now we show there are exactly q such near-rings.

Theorem 5.2. *Let $x \in A-C$. Define $f(he)(b) = x$. Let $\cdot_f$ be the multiplication on D_n determined by $f(he)$ as stated above. Then $(D_n, +, \cdot_f)$ is a near-ring.*

Proof. It suffices to show that $\cdot_f$ is an associative operation. Choose integers u, v, $0 \leqq u, v<n$. Then $u \equiv 0, q, h^r$, or $h^r+q \pmod n$, for some r, and $v \equiv 0$, q, h^s, or $h^s+q \pmod n$, for some s. By considering all possible cases, it is easy to show that $(ue\cdot_f ve)\cdot_f b = ue\cdot_f(ve\cdot_f b)$. Since b annihilates all elements of order 2, if either c or d is of order 2, then $(c\cdot_f d)\cdot_f b = c\cdot_f(d\cdot_f b) = 0$. Hence $(z\cdot_f y)\cdot_f b = z\cdot_f(y\cdot_f b)$ for all $y, z \in D_n$.

Since e is an identity for $\cdot_f$, then $(z\cdot_f y)\cdot_f e = z\cdot_f(y\cdot_f e)$ for all $y, z \in D_n$. It follows that $\cdot_f$ is associative.

By Theorem 5.2 there are q distinct near-rings with multiplicative identity e and normal right submodule A. Using Theorem B of (2) and the automorphisms $\alpha_i: D_n \to D_n$ defined by $\alpha_i(e) = e$ and $\alpha_i(b) = ie+b$, for all odd integers i, $1 \leqq i \leqq n-1$, we obtain q distinct near-rings with multiplicative identity e and normal right submodule B. Hence there are a total of n distinct near-rings on $(D_n, +)$ with multiplicative identity e. Now we show that all of these n near-rings are in fact isomorphic.

Lemma 5.3. *Let $(D_n, +, \cdot)$, $n = 2q$, be a near-ring with multiplicative identity e and normal right submodule A. Then there exists a unique even integer k, $0 \leqq k \leqq n-1$, such that $he(ke+b) = ke+b$.*

Proof. Since the subgroup generated by e is not a proper right submodule, then $he\cdot b = me+b$ for some integer m, $1 \leqq m \leqq n$. Since q is prime, there exists a unique integer s, $0 \leqq s \leqq q-1$, such that $(h-1)s \equiv (n-m) \pmod q$. Since q is an odd prime, then exactly one of s or $s+q$ is even. Let k represent this even integer. Then either $he(ke+b) = ke+b$ or $he(ke+b) = (k+q)e+b$. Since $ke+b \in A$, which is a left ideal by Proposition 4.17, then $he(ke+b) = ke+b$.

Uniqueness of k follows from uniqueness of s.

Theorem 5.4. *There is, up to isomorphism, a unique near-ring with multiplicative identity e on $(D_n, +)$, $n = 2q$.*

Proof. Let $(D_n, +, \cdot_f)$ be the near-ring with multiplicative identity e and normal right submodule A, defined by $f(he)(2e+b) = 2e+b$. By the remark preceding Lemma 5.3 it suffices to show that any other near-ring $(D_n, +, \cdot_g)$ with multiplicative identity e and normal right submodule A is isomorphic to $(D_n, +, \cdot_f)$.

By Lemma 5.3 there exists a unique even integer k, $0 \leqq k < n$, such that $he \cdot_g (ke+b) = ke+b$. Define an automorphism $\beta: D_n \to D_n$ by $\beta(e) = e$ and $\beta(2e+b) = ke+b$. By Theorem B of (2) the set of endomorphisms

$$\{t(\beta(x)) = \beta \circ f(x) \circ \beta^{-1} \mid x \in D_n\}$$

defines a near-ring N isomorphic to $(D_n, +, \cdot_f)$. But $\beta \circ f(he) \circ \beta^{-1} = g(he)$, so $N = (D_n, +, \cdot_g)$.

These n near-rings all have multiplicative identity e. Using the automorphism δ defined by $\delta(e) = me$, m odd, $m \neq q$, and $\delta(b) = b$, we obtain a near-ring with me as multiplicative identity. Hence, for all odd integers m, $1 \leqq m \leqq n$, $m \neq q$, there exist n distinct near-rings with multiplicative identity me. So there are exactly $n(n/2-1)$ distinct near-rings with multiplicative identity on $(D_n, +)$, but all of them are isomorphic.

Added in Proof: John Krimmel has also (independently) observed that there is, up to isomorphism, a unique near-ring with multiplicative identity on $(D_n, +)$, $n = 2q$.

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