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A NOTE ON THE $R_{n,k}$ PROPERTY FOR $L^1(\mu, E)$

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ABSTRACT. In this paper we show that for a Banach space X having the $R_{n,k}$ property and such that X is constrained in a dual space having the Radon-Nikodým property, the space of X-valued Bochner integrable functions has the $R_{n,k}$ property.

Introduction. In [3], Å. Lima has introduced a decomposition property for Banach spaces called the $R_{n,k}$ property (for positive integers $n > k \geqq 2$). According to this definition a Banach space E has the $R_{n,k}$ property if for any n-tuple $(x_1, \ldots, x_n)$ of elements of E with $\sum_{i=1}^n x_i = 0$, there exist $m = \binom{n}{k}$-many n-tuples $(x_{i1}, \ldots, x_{in})$, $1 \leqq i \leqq m$ such that each of these n-tuples has at most k-many non-zero components, $\sum_{j=1}^n x_{ij} = 0$ for each i and $(x_1, \ldots, x_n) = \sum_{i=1}^m (x_{i1}, \ldots, x_{in})$ with

$$\|x_j\| = \sum_{i=1}^m \|x_{ij}\| \quad \text{for} \quad 1 \leqq j \leqq n.$$

The object of this note is to consider the $R_{n,k}$ property for the space of Bochner integrable functions $L^1(X, \Sigma, \mu, E)((X, \Sigma, \mu)$ is any measure space) when E has the $R_{n,k}$ property. We prove a theorem which is more general than the one in (4, Theorem 4.1) and arrived at by a completely different approach. As a by-product we also get some information about when $L^1(X, \Sigma, \mu, E)$ is constrained, that is, complemented by a contractive projection, in its bidual.

Our proof makes use of the following characterization of the $R_{n,k}$ property for a dual Banach space E^*, obtained by Lima in [3].

E^* has the $R_{n,k}$ property iff E has the almost n, k intersection property (abbreviated as a.n.k.I.P.), that is, for any family of n-closed balls $\{B(a_i, r_i)\}$ in E such that the intersection of any k-many of them is non-empty then for any $\epsilon > 0$,

$$\bigcap_{i=1}^n B(A_i, r_i + \epsilon) \neq \phi.$$

We follow the notation of [4] and our results are valid for real or complex Banach spaces.

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At the suggestion of the referee we have included a couple of paragraphs of [4] in this paper to make it self contained.

Main Result To treat the problem of $R_{n,k}$ property for $L^1(X, \Sigma, \mu, E)$, since one needs to decompose only finitely many functions at a time and since integrable functions have σ-finite support there is no loss of generality in assuming that the measure space (X, Σ, μ) is σ-finite (See [4], Section 4). It is a well-known trick in measure theory to get hold of a finite measure η on Σ such that the corresponding Bochner function spaces are isometric. Hence we may and do assume that (X, Σ, μ) is a finite measure space and will write $L^1(\mu, E)$ for $L^1(X, \Sigma, \mu, E)$.

A straightforward approach to prove the $R_{n,k}$ property for $L^1(\mu, E)$ when E has the $R_{n,k}$ property, is to go via simple functions. It is not difficult to see from the definition that if E has the $R_{n,k}$ property then the space of simple functions in $L^1(\mu, E)$ has the $R_{n,k}$ property. So the problem reduces to showing that a Banach space Z with a dense subspace Y having the $R_{n,k}$ property, will itself have the $R_{n,k}$ property. Notice that in this case since $Y^* = Z^*$ an easy application of Theorems 1.2 and 1.1 of Lima [2] shows that Y^* has the a.n.k.I.P. (Though Lima's results were stated for Banach spaces, the parts that are required for the above argument hold for normed linear spaces as well). And hence by Theorems 3.1 of [3], Z^{**} has the $R_{n,k}$ property. Unfortunately *it is still unknown whether $R_{n,k}$ property for Z^{**} implies the same for Z or not.* However notice that if Z is constrained in Z^{**} (we always consider Z as a subspace of its bidual Z^{**} under the natural embedding), then it follows from the easily verifiable fact: "$R_{n,k}$ property is hereditary for constrained subspaces", that Z has the $R_{n,k}$ property when Z^{**} does.

THEOREM . *Suppose E is a Banach space having the $R_{n,k}$ property and E is constrained in a dual space F^* and F^* has the Radon-Nikodým property w.r.t. the finite measure space (X, Σ, μ), then $L^1(\mu, E)$ has the $R_{n,k}$ property.*

PROOF. In view of the above discussion, we only need to show that $L^1(\mu, E)$ is constrained in its bidual. For this purpose we need the following:

FACT: A Banach space Z is constrained in its bidual iff it is isometric to a constrained subspace of a dual space Y^*.

This we believe is well-known and is a part of the folklore. However as we are unable to track down a reference we give below its easy proof.

PROOF OF FACT: Let Z be a constrained subspace of Y^* and let P be the corresponding projection. Let $R(\tau) = \tau/Y$ be the standard contractive projection from Y^{***} onto Y^*. Since $Z \subseteq Y^*$, using natural identifications it is easy to see that $Z \subseteq Z^{**} = Z^{\perp\perp} \subseteq Y^{***}$ and $P \circ R/Z^{\perp\perp}$ is a contractive projection from Z^{**} onto Z.

Notice that this makes the property "being constrained in the bidual" a hereditary property for constrained subspaces.

If P denotes the contractive projection from F^* onto E, then the maping $f \longrightarrow P \circ f$ is a contractive projection from $L^1(\mu, F^*)$ onto $L^1(\mu, E)$. Hence our theorem will be proved once we show that $L^1(\mu, F^*)$ is isometric to a constrained subspace of a dual space.

It follows from the proof of Lemma 1 in [1] that if $\mathbf{Q}$ denotes the maximal ideal space of $L^\infty(\mu)$ (the space of scalar valued, essentially bounded functions) and $\mathcal{B}$ the Borel σ-field on $\mathbf{Q}$ and $\hat{\mu}$ a finite regular Borel measure on $\mathcal{B}$ whose values at a clopen set $\hat{A} \subset \mathbf{Q}$ are determined by $\hat{\mu}(\hat{A}) = \mu(A)$ ($\hat{}$ denotes the usual Gelfand map) then $L^1(\mu, F^*)$ is isometric $L^1(\hat{\mu}, F^*)$. We will now proceed to show that F^* has the Radon-Nikodým property w.r.t. $(\mathbf{Q}, \mathcal{B}, \hat{\mu})$. Let G be a countably additive F^*-valued vector measure of bounded variation on $\mathcal{B}$ such that $G \ll \hat{\mu}$. Define $\bar{G}$ on Σ by $\bar{G}(A) = G(\hat{A})$. Since $\hat{\mu}$ vanishes on nowere dense Borel sets and since $G \ll \hat{\mu}$, it follows that $\bar{G}$ is a countably additive F^*-valued vector measure on Σ, of bounded variation and $\bar{G} \ll \mu$. Hence by hypothesis, the Radon-Nidodým derivative $d\bar{G}/d\mu$ exists in $L^1(\mu, F^*)$. Now the way the isometry between the L^1-spaces was defined in the proof of Lemma 1 in [1], via simple functions, it is quite clear that the element in $L^1(\hat{\mu}, F^*)$ corresponding to $d\bar{G}/d\mu$ is the required derivative. Hence F^* has the Radon-Nidokým property w.r.t. $(\mathbf{Q}, \mathcal{B}, \hat{\mu})$,

If one identifies $C(\mathbf{Q}, F)^*$ as the space of F^*-valued countably additive vector measures of bounded variation with the variation norm then the natural map which associates to each f in $L^1(\hat{\mu}, F^*)$ the measure $\{\int_E f d\hat{\mu}\}_{E \in \mathcal{B}}$ is an isometric embedding of $L^1(\hat{\mu}, F^*)$ in $C(Q, F)^*$. Now as in Step 1 of the proof of Theorem 4.1 in (4), if one considers the standard Lebesgue decomposition of a $G \in C(Q, F)^*$ as $G = G_a + G_s$ where $G_a \ll \hat{\mu}$ and G_s is 'singular' w.r.t. $\hat{\mu}$ then since F^* has the Radon-Nikodým property w.r.t. $\hat{\mu}$, we get that $G \longrightarrow \int_E (dG_a/d\hat{\mu}) d\hat{\mu}$ is a contractive projection from $C(Q, F)^*$ onto $L^1(\hat{\mu}, F^*)$. Hence $L^1(\mu, E)$ is constrained in its bidual and consequently has the $R_{n,k}$ property.

Corollary : *If a Banach space E is constrained in a dual space F^* and F^* has the Radon-Nikodým property w.r.t a σ-finite measure space (X, Σ, μ) then $L^1(\mu, E)$ is constrained in its bidual.*

Remarks: (1) For a σ-finite measure space (X, Σ, μ), since $L^1(\mu, E)$ is isometric to a $L^1(\tau, E)$ for a suitable finite measure τ on Σ, E is clearly isometric to a constrained subspace of $L^1(\mu, E)$ and hence for $L^1(\mu, E)$ to be constrained in its bidual it is necessary that E be constrained in its bidual. We do not know whether this condition is also sufficient. Notice that one only needs to consider spaces of the form $L^1(\mu, F^*)$ and decide if they are always constrained in their bidual.

(2) We also do not know if $R_{n,k}$ property for E is a sufficient condition for $L^1(\mu, E)$ to have the $R_{n,k}$ property. It is fairly easy to see that this is so, if μ is a discrete measure. Also when $n = 4$ and $k = 3$ Å. Lima has proved in [3] that $R_{4,3}$ property for E implies that E is isometric to $L^1(\tau)$ and it can be shown that $L^1(\mu, L^1(\tau))$ is isometric to an L^1 of a measure space (scalar valued). Hence by the same result of Lima, $L^1(\mu, E)$ has the $R_{4,3}$ property.

NOTE (added June 88).

We briefly indicate two significant improvements on the results presented above.

1. It is enough to assume the Radon-Nikodým property for E (rather than for F^*) in the statement of the Theorem. To see this consider $L^1(\hat{\mu}, E)$ as a subspace of $C(Q, F)^*$ and for any $G \in C(Q, F)^*$, take the Lebesgue decomposition of $P \circ G$ and then the Randon-Nikodým derivative.

2. The Corollary is true for any measure space. Let (Ω, Σ, μ) be any measure space. We claim that there are finite measure spaces $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ such that $L^1(\mu, E)$ is isometric to an l^1 direct product of $L^1(\mu_\alpha, E)$. From this it is fairly easy to see the validity of the claim for $L^1(\mu, E)$. A well known construction due to Pelczyński [A. Pelcsyński: On Banach spaces containing $L_1(\mu)$, Studia Math, (30) (1969) 231–245], shows that the Kakutani's representation of $L^1(\mu)$ as the l^1-direct product of $L^1(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ is identifiable as : $L^1(\nu)$ where ν is defined on $\{A \subset \cup\Omega_\alpha : A \cap \Omega_\alpha \in \Sigma_\alpha\}$ by $\nu(A) = \Sigma\, \mu_\alpha(A \cap \Omega_\alpha)$ and the identification is via the map $f \longrightarrow f/\Omega_\alpha$. It is not too difficult to see that this isometry extends via simple functions to an isometry between $L^1(\nu, E)$ and $\oplus_1 L^1(\mu_\alpha, E)$. Note that $L^1(\mu, E)$ is isometric to $L^1(\nu, E)$ since the corresponding scalar valued spaces $L^1(\mu)$ and $L^1(\nu)$ are isometric.

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