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HIGHLY SYMMETRIC HOMOGENEOUS SPACES

L. N. MANN

We consider effective homogeneous spaces $M = G/H$ where G is a compact connected Lie group, H is a closed subgroup and G acts effectively on M (i.e., H contains no non-trivial subgroup normal in G). It is known that $\dim G \leqq m^2/2 + m/2$ where $m = \dim M$ and that if $\dim G = m^2/2 + m/2$, then M is diffeomorphic to the standard sphere S^m or the standard real projective space RP^m [**1**]. In addition it has been shown that for fixed m there are gaps in the possible dimensions for G below the maximal bound [**4**; **5**]. Using the Main Lemma of [**3**], we give a classification of all effective homogeneous spaces $M = G/H$ with $\dim G \geqq m^2/4 + m/2$, $m \geqq 19$. In particular, if M is simply-connected we conclude that $M = CP^n$, $m = 2n$, or $M = S^{m-k} \times V^k$, $0 \leqq k \leqq m/2$, where V^k is a k-dimensional simply-connected homogeneous space.

THEOREM. *Let* $M^m = G/H$ *be an effective homogeneous space with* $\dim G \geqq m^2/4 + m/2$, $m \geqq 19$. *Then exactly one of the following holds:*

(1) $M = CP^n$ $(m = 2n)$ *and* G *is locally isomorphic to* $SU(n + 1)$.

(2) $M = CP^n \times S^1$ $(m = 2n + 1)$ *and* G *is locally isomorphic to* $U(n + 1)$.

(3) M *is a simple lens space finitely covered by* S^{2n+1} $(m = 2n + 1)$ *and* G *is locally isomorphic to* $U(n + 1)$.

(*In possibilities* (4) *through* (6), $V^k = G_2/H_2$ *denotes a* k*-dimensional homogeneous space,* $0 \leqq k \leqq m/2$, *and* G *is locally isomorphic to*

$$\mathrm{Spin}(m - k + 1) \times G_2$$

where G_2 *is a compact connected Lie group with* $\dim G_2 \leqq k(k + 1)/2$.)

(4) $M = RP^{m-k} \times V^k$.

(5) $M = S^{m-k} \times V^k$.

(6) $M = S^{m-k} \times_K V^k$ *where* K *is a group of order two acting freely on* $S^{m-k} \times V^k$ *and generated by* (A_1, A_2) *where* A_1 *is the antipodal map on* S^{m-k} *and* A_2 *is an element of order two of* $N(H_2, G_2)/H_2$.

COROLLARY. *If* M *is simply-connected, then* $M = CP^n$ *or* $M = S^{m-k} \times V^k$ *where* V^k *is simply-connected.*

Proof of the Theorem. Since G acts effectively on M, we may apply the Main Lemma of [**3**]. Possibilities (1), (2) and (3) then correspond to cases (α), (β) and (γ) of the Main Lemma. We are left with case (δ) of the Main Lemma. Therefore G is locally isomorphic to $\mathrm{Spin}(m - k + 1) \times G_2$, $0 \leqq k \leqq m/2$,

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where $G_1 = \text{Spin}(m - k + 1)$ acts *almost effectively*, i.e. with finite kernel, on M with orbits which are some combination of fixed points, standard $(m - k)$-spheres and standard real projective $(m - k)$-spaces. It follows moreover from [**4**, Theorem 1] that $\dim G_2 \leqq k(k + 1)/2$. Since the almost effective actions of G_1 and G_2 on M commute it is easily verified that if $y = g_2 g_1 x$ for $x, y \in M$ and $g_1 \in G_1$, $g_2 \in G_2$ then

$$(G_1)_y = g_1 (G_1)_x g_1^{-1}$$

where $(G_1)_y$ denotes the isotropy or stability subgroup of G_1 at y. But $G_1 \times G_2$ is transitive on M and it follows from the observation above that all the orbits of the action of G_1 on M are of the same type. Let $H_1 = (G_1)_x$ for some $x \in M$. By Borel [**2**], M is a fibre bundle over M/G_1 with fibre G_1/H_1 (either S^{m-k} or RP^{m-k}) and structural group $N(H_1, G_1)/H_1$. If $G_1/H_1 = RP^{m-k}$, then the structural group $N(H_1, G_1)/H_1$ is trivial and we have possibility (4). So we assume $G_1/H_1 = S^{m-k}$. If the bundle is still trivial then of course we have possibility (5). So we are left with the case where M is the total space of a non-trivial S^{m-k} bundle over M/G_1. We can describe the bundle as follows [**2**]. Let F be the fixed point set of H_1 on M and let $K = N(H_1, G_1)/H_1 \cong Z_2$. Now K acts freely on F and we have the principal K-bundle : $F \rightarrow M/G_1$. The associated $G_1/H_1 = S^{m-k}$ bundle is

$$M = S^{m-k} \times_K F \rightarrow M/G_1.$$

We show G_2 acts transitively on F. The orbits of G_2 on M are all of the same type and G_2 leaves the subset F invariant. Since the actions of G_2 and K commute on F and G_2 acts transitively on $M/G_1 = F/K$, K acts transitively on F/G_2. Therefore F/G_2 consists of either one or two points and, hence, there are either one or two orbits for the action of G_2 on F. But if there are two orbits, K permutes these two orbits and

$$M = S^{m-k} \times_K F = RP^{m-k} \times F_0$$

where F_0 is one of these two orbits. However, this would place us back in possibility (4). Therefore $F = G_2/H_2$.

Since K acts freely on F, it is easily verified that K is isomorphic to a subgroup S of $N(H_2, G_2)/H_2$ and, in fact, the action of K on F is equivalent to the action of S on G_2/H_2 induced from the standard action of $N(H_2, G_2)/H_2$ on G_2/H_2. This completes the proof of the theorem.

For $k \leqq k_0 = [\frac{1}{2}((1 + 8m)^{1/2} - 3)]$,

$$\tfrac{1}{2}(m - k)(m - k + 1) + \tfrac{1}{2}k(k + 1) < \tfrac{1}{2}(m - k + 1)(m - k + 2).$$

Therefore it follows from the theorem that to list the effective homogeneous spaces G/H with $\dim G \geqq (m - k)(m - k + 1)/2$, $k \leqq k_0$, it is sufficient to list all homogeneous spaces of dimension less than or equal to k. For small values of k this program is not difficult. As an example we list below all homogeneous spaces of dimensions one, two and three:

$k = 1$: S^1

$k = 2$: S^2, RP^2, T^2

$k = 3$: S^3, RP^3, T^3, $RP^2 \times S^1$, $S^2 \times S^1$, $S^2 \times_{Z_2} S^1$,

S^3/F (where F is any finite subgroup of S^3).

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University of Massachusetts,
Amherst Massachusetts