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INTEGRALLY CLOSED TORSIONLESS RINGS

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ABSTRACT. A characterization of torsionless rings is given which shows that the integral closure of a torsionless ring need not be Prüfer.

In [3, p. 57] it was conjectured that the integral closure of a torsionless ring is Prüfer. The purpose of this note is to give a (negative) solution to this conjecture. This will be accomplished by a new characterization of torsionless rings which follows easily from results in [3].

All rings in this note will be integral domains. We begin by recalling the terminology. An R-module M is said to be *torsionless* if the canonical map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is injective. The module M is said to be *reduced* if it has no divisible submodules. Following Matlis we say that a domain R is *torsionless* if each reduced torsion-free R-module of finite rank is torsionless. The *rank* of a torsion-free R-module M is defined as the dimension of the Q-vector space $Q \otimes_R M$ where Q denotes the quotient field of R. If J is an R submodule of Q we will denote $(R:_Q J) = \{x \in Q | xJ \subseteq R\}$ by J^{-1}.

By [3, Theorem 42] an integral domain R is torsionless if and only if R is complete in the R-topology and each reduced torsion-free R module of rank one is torsionless. The following characterization of torsionless rings parallels the characterization [3, Theorem 79] of rings which do not have remote quotient field.

THEOREM 1. *The following properties of an integral domain R with quotient field Q are equivalent.*

(i) *R is torsionless.*

(ii) *Each valuation overring $V \neq Q$ of R satisfies $V^{-1} \neq \{0\}$ and is complete in the V-topology.*

(iii) *There exists a valuation overring V of R which satisfies $V^{-1} \neq \{0\}$ and is complete in the V-topology.*

PROOF. It follows from [3, Theorems 14, 15] that if V is an overring with

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$V^{-1} \neq \{0\}$, then R is complete in the R-topology if and only if V is complete in the V-topology. The result follows from this and [3, Theorem 79].

It follows that torsionless domains are characterized as pullbacks

$$\begin{array}{ccc} R & \longrightarrow & V \\ \downarrow & & \downarrow \\ A & \longrightarrow & V/I \end{array} \tag{1}$$

where V is a valuation overring of R which is complete in the V-topology, $I \neq 0$, and $A \to V/I$ is injective. Since every overring of a torsionless ring is torsionless [3, Theorem 43], the conjecture of Matlis is equivalent to the assertion that each integrally closed torsionless ring is Prüfer. The conjecture is now settled in the negative by the following result on the pullback diagram (1). For our purposes it suffices to consider the case that $I = m =$ the maximal ideal of V.

THEOREM 2. *If R is constructed as in* (1) *above with* (V, m) *a valuation ring and* $I = m$, *then*

(a) *R is integrally closed if and only if A is integrally closed in V/m, and*

(b) *R is Prüfer if and only if A is Prüfer with quotient field V/m.*

PROOF. The proofs are similar to the corresponding results about the $D + M$ construction [2, Part II, p. 560], which is a special case of the pullback construction. Statement (a) follows from the observation that an element of V is integral over R if and only if its image in V/m is integral over A. The details of part (b) are given in [1, Theorem 2.4].

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