1 Vector Calculus-I

Marine applications encompass a wide spectrum of engineering problems involving solid rigid bodies, solid flexible bodies, and fluids; sometimes solids within fluids and sometimes fluids within solids. Most situations require analysis of the underlying statics and dynamics that reveal important design drivers and methods that could be used in evaluating our designs for successful operation. Insightful and efficient analysis depends on physically and mathematically consistent representations of variables and phenomena. Vectors and matrices are the fundamental building blocks of such representations. We begin with a brief treatment of vectors in two and three dimensions. Vectors are intuitive and quantitative. Vectors have magnitudes, represented by a number (having the same units as the quantity represented, e.g., Newton or N for force), and a direction [represented by another number between 0 and 2π rad (360°)]. Thus, a force pulling an object in a certain direction may be specified using a magnitude F and a direction θ. An opposite force pushing (rather than pulling) the object has an opposite direction $\theta + \pi$. Vectors with a magnitude 1 are unit vectors and are convenient for defining directions. Vectors can be interrelated to each other and to scalars within the consistent frameworks of algebra and calculus, ultimately enabling design-critical decisions involving forces, masses, stiffnesses, velocities, accelerations, volumes, areas, and time. Our goal in this chapter is to review some of the algebra and differential calculus of vectors. We attempt informally to understand more than we formally prove. More detailed treatments of topics covered in this chapter can be found in Kreyszig, Kreyszig, and Norminton (2011) and Greenberg (1978, 1998), and so forth.

1.1 Basic Properties of Vectors

Consider a ship in calm sea conditions that is tied to bollards at 2 or 3 points around the vessel (see, for instance, Figure 1.1). Clearly, the net force on the vessel will depend on the locations of these points relative to the vessel as well as the magnitudes of the forces applied by the cables. In other words, the force directions as determined by the locations of the bollards relative to the vessel are important. For N cables pulling at the vessel from N points, the net force on the vessel is given by

$$\mathbf{F} = \sum_{n=1}^{N} \mathbf{F}_n. \tag{1.1}$$

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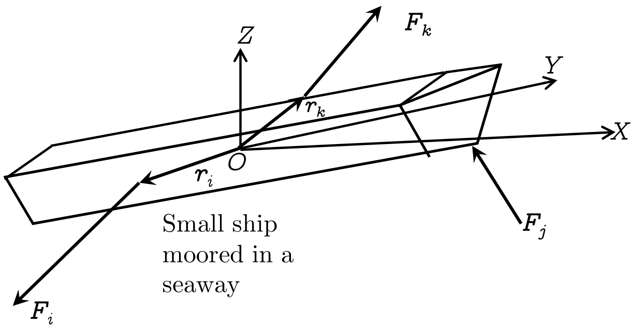

Figure 1.1 A small ship moored at sea, experiencing forces from different directions. Forces may include tension in the mooring lines between the buoys or platforms to which the ship may be moored, as well as any forces from the current flow past the ship at that site. Also included, though not shown here, among the forces and the effective moments would be the wave forces and moments due to wave effects.

Equation (1.1) represents vector summation in three-dimensional Cartesian space. Denoting the directions of the three axes (x, y, z) as chosen in our particular example using the unit vectors **i**, **j**, and **k**, we have

$$F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} = \left(\sum_{n=1}^{N} F_{xi}\right)\mathbf{i} + \left(\sum_{n=1}^{N} F_{yi}\right)\mathbf{j} + \left(\sum_{n=1}^{N} F_{zi}\right)\mathbf{k}, \tag{1.2}$$

where F_x, F_y, and F_z represent the components of the resultant vector along the three coordinate directions. The resultant force vector can thus be obtained simply by adding the respective components of the N force vectors, and

$$\begin{aligned} F_x &= \sum_{n=1}^{N} F_{xi}, \\ F_y &= \sum_{n=1}^{N} F_{xi}, \\ F_z &= \sum_{n=1}^{N} F_{zi}. \end{aligned} \tag{1.3}$$

The components of each vector are related to its direction cosines. Thus, the magnitude of a vector **F** in terms of its components F_x, F_y, and F_z is given by

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \left[F_x^2 + F_y^2 + F_z^2\right]^{1/2}. \tag{1.4}$$

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The direction cosines $[\alpha\ \beta\ \gamma]$ can then be expressed as

$$\cos\alpha = \frac{F_x}{F},$$
$$\cos\beta = \frac{F_y}{F},$$
$$\cos\gamma = \frac{F_z}{F}. \tag{1.5}$$

It is not difficult to see that an equal and opposite vector will have the same magnitude F but opposite direction, as given by direction cosines $\pi + \alpha$, $\pi + \beta$, and $\pi + \gamma$.

One property of vector addition that is important to recognize (but easy to pass over lightly) is that it is commutative. Indeed, sometimes it is this property that allows us to decide whether a given quantity is a vector or not. Angular displacement through finite angles (i.e., $\gg 0$) is not a vector since the sequence of rotations matters. This can be illustrated using a book (or another object shaped like a parallelepiped). A rotation through 90° about the long axis followed by a rotation through 90° about the short axis does not place the book in the same orientation as when an opposite sequence is employed. On the other hand, infinitesimal rotations do commute, for which reason, angular velocities also commute and therefore can be considered vectors.

The property of commutativity also plays an important role in vector multiplication. Returning to the aforementioned example of the ship pulled from different directions, so far we have tacitly assumed that the lines of action of all forces pass through the ship's center of mass. In such a situation, there is no net moment generated by the forces that would cause the ship to rotate. However, if the line of action of a force does not pass through the center of mass, then the force will also produce a moment that acts to rotate the body in the direction of the moment. Thus, if different forces $\mathbf{F}_n$ pass through points P_n on the body where each P_n is at a position vector $\mathbf{r}_n$ from the center of mass, then the moment produced by $\mathbf{F}_i$ about the mass center is

$$\mathbf{M}_n = \mathbf{r}_n \times \mathbf{F}_n. \tag{1.6}$$

Note that, as expected, moment is a vector. If both force and position vector lie in a plane, positive moment is counterclockwise and is thought to be out of the plane ("right-hand screw rule"), while negative moment is clockwise and into the plane. The net moment on the ship above is simply the vector sum of the individual moments $\mathbf{M}_i$, such that

$$\mathbf{M} = \sum_{n=1}^{N} \mathbf{r}_n \times \mathbf{F}_n. \tag{1.7}$$

The product above is referred to as "cross product." It is also known as "vector product," since the result of this multiplication of one vector with another is also a vector (see, for instance, Figure 1.2). The magnitude of this product can be evaluated as, continuing the example in equation (1.6),

$$M_n = r_n F_n \sin\theta_n, \tag{1.8}$$

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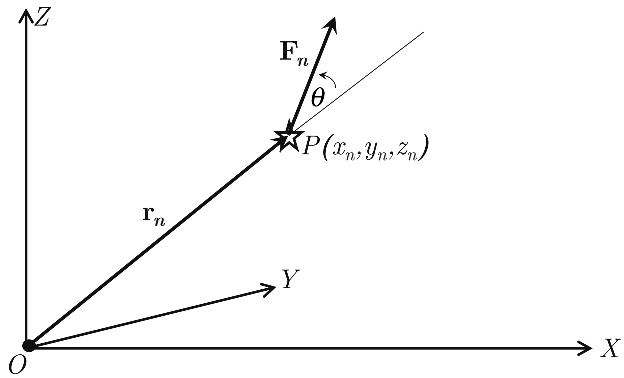

Figure 1.2 A schematic showing the relative disposition of the two vectors $\mathbf{r}_n$ and $\mathbf{F}_n$ involved in the cross product. The result will be a vector, perpendicular to the plane containing the two vectors being multiplied together.

where θ_n is the angle between $\mathbf{r}_n$ and $\mathbf{F}_n$. The vector $\mathbf{M}_n$ resulting from this multiplication is perpendicular to the plane that contains both $\mathbf{r}_n$ and $\mathbf{F}_n$. Further, if one could take the first vector in the product (e.g., here $\mathbf{r}_n$) and literally turn it toward the second vector as if turning a screw, a right-hand screw would advance out of the plane containing $\mathbf{r}_n$ and $\mathbf{F}_n$. This is intuitively consistent with what one expects of moments. It is also easy to see from equation (1.8) that when the two vectors involved in a cross product are collinear (i.e., the angle between them is zero or π radians), their cross product is zero. On the other hand, when the two vectors are perpendicular (i.e., the angle between them is $\pi/2$ or $3\pi/2$ radians), their cross product is the greatest it can be for the given vector magnitudes. Note that the product $\mathbf{F}_n \times \mathbf{r}_n$ also has the same magnitude as shown in equation (1.8). However, if one could now turn $\mathbf{F}_n$ toward $\mathbf{r}_n$ as if turning a screw, a right-hand screw recedes into the plane containing $\mathbf{r}_n$ and $\mathbf{F}_n$. Thus,

$$\mathbf{r}_n \times \mathbf{F}_n = -\mathbf{F}_n \times \mathbf{r}_n. \tag{1.9}$$

We see thus that order is important in a cross product, and that a cross product is not commutative. A more formal way to evaluate a cross product is to evaluate the following determinant, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors along the three Cartesian coordinate axes.

$$\mathbf{r}_n \times \mathbf{F}_n = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_n & y_n & z_n \\ F_{nx} & F_{ny} & F_{nz} \end{vmatrix}, \tag{1.10}$$

where $\mathbf{r}_n = x_n\mathbf{i} + y_n\mathbf{j} + z_n\mathbf{k}$ and $\mathbf{F}_n = F_{nx}\mathbf{i} + F_{ny}\mathbf{j} + F_{nz}\mathbf{k}$. The relation in equation (1.10) is more convenient to use in three-dimensional static and dynamic analyses, and also makes it clear why order is important in a cross product.

In the following section, we examine the inner product or dot product, which leads to a scalar result and is therefore commutative.

1.2 Inner Product

When trying to evaluate the work done by a force in causing a displacement of a body, such as perhaps, the work done by the thrust to be applied on a ship in order to move through a certain trajectory, we may want to know how much energy is spent in performing that operation. Alternatively, when one wants to understand the contribution of a force in a particular direction, or the amount of displacement in a particular direction given a displacement vector, one resorts to the so-called inner product or dot product of vectors (see, for instance, Figure 1.3). Thus, the component of a force **F** in a direction **e** could be represented as

$$F_e = \mathbf{F} \cdot \mathbf{e}, \tag{1.11}$$

where **e** is the unit vector in the direction of interest. Since the direction is already defined, F_e only needs to be, and is, a scalar. The result of an inner product is thus a scalar. It follows then that order can be reversed in an inner product, since either way, the same scalar results. The inner product of two vectors **a** and **b** can be found using

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\theta, \tag{1.12}$$

where a and b denote the magnitudes of **a** and **b**, and θ is the angle between them. It is easy to see from equation (1.12) that when two vectors are orthogonal, i.e., when $\theta = \pi/2$, the inner product between them is zero. Intuitively, it is not surprising that the component of a vector along a direction normal to itself is zero. While the expression in equation (1.12) is convenient to use when the two vectors are coplanar, a more general expression for evaluating them is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3, \tag{1.13}$$

where the subscripts 1, 2, and 3 represent the three coordinate directions x, y, and z, respectively.

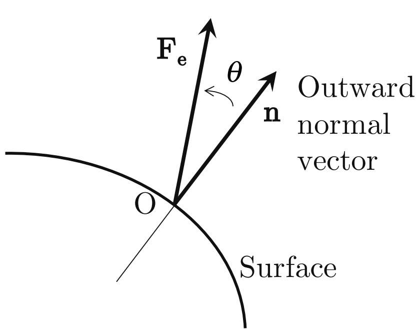

Figure 1.3 A schematic showing the relative disposition of the two vectors involved in the inner product. Here we are interested in knowing the component of the force in a direction normal to the surface.

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1.3 General Observations on Cross, Inner Products, and Their Combinations

It is interesting to consider, based on equations such as (1.6) and (1.11), that the cross product of a vector with itself is zero, while the inner product of a vector with itself equals the square of its magnitude. Further, it is also interesting to consider how differently the two products operate on the unit vectors defining the three Cartesian coordinate directions. Thus,

$$\begin{aligned} \mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0, \\ \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1. \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} \mathbf{i}\times\mathbf{j} = \mathbf{k},\ \ \mathbf{j}\times\mathbf{k} = \mathbf{i},\ \ \mathbf{k}\times\mathbf{i} = \mathbf{j}, \\ \mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0. \end{aligned} \tag{1.15}$$

The vector product relations among the unit vectors can be visualized as a cyclic permutation. Combinations of cross and inner products also lead to interesting results. Thus, an inner product between a vector **A** and another vector representing the cross product of two other vectors **B** and **C** can be represented as $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$. It can be seen that this combination results in a scalar. Further, any cyclic permutation of the three vectors leads to the same scalar result. Thus,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A}. \tag{1.16}$$

The scalar triple product could arise, for instance, when we need to compute the instantaneous power provided by a force on a body that is constrained to rotate about a fixed point in space. In this case, the cross product of the angular velocity of the body and the radius vector to its center of mass gives the linear velocity vector for the center of mass. The inner product of the force vector and the velocity vector determines the power transmitted by the force. The scalar triple product can be written in determinant form as

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & y_n & z_n \\ C_x & C_y & C_z \end{vmatrix}. \tag{1.17}$$

The vector product of a vector **A** with a vector product of two other vectors **B** and **C** is a vector quantity, and therefore sensitive to the order in which the vectors appear. Thus,

$$\mathbf{A}\times\mathbf{B}\times\mathbf{C} = -\mathbf{C}\times\mathbf{A}\times\mathbf{B} = \mathbf{C}\times\mathbf{B}\times\mathbf{A}. \tag{1.18}$$

A useful identity involving the vector triple product is

$$\mathbf{A}\times\mathbf{B}\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \tag{1.19}$$

Vector triple products often arise in the study of dynamics of bodies in three dimensions, when transforming quantities expressed in a coordinate system at the centroid of a rotating body to quantities written with respect to an inertial reference frame. Recall that an inertial reference frame is either fixed in space or moving at constant velocity (hence by extension, nonrotating).

1.4 Differential Calculus of Vectors

In rigid-body dynamics, one deals with time derivatives of vectors as a routine matter. In flexible-body dynamics, as well as in fluid dynamics, however, quantities can vary in time as well as space. Often the quantities to be observed are scalars. Density, for instance, is a scalar quantity that in an inhomogeneous fluid may be a function of position of a point within a fluid. Similarly, velocity of fluid particles is a vector that may be different at different points. When we need to keep track of the spatial and temporal dependence of quantities such as density, velocity, and pressure, the term "field" is commonly employed. Thus, we have a velocity field described as $\mathbf{v}(x, y, z; t)$ or a pressure field represented as $p(x, y, z; t)$. Note that since pressure at a point within a fluid is independent of direction, it is a scalar. Derivatives of field variables such as pressure and velocity take on a new significance when their rates of change relative to spatial coordinates need to be analyzed. Here, we get our early introduction to the concepts and notations that are frequently used in marine applications, as we will see in Chapters 10 and 11, later in this text. We begin by reviewing an important symbol, ∇, the so-called "del" operator, which is defined as

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}. \tag{1.20}$$

We have here assumed the Cartesian rectangular coordinate system. Note that ∇ is a vector and combines partial derivatives along the three coordinate axes in a consistent manner, using the unit vectors **i**, **j**, and **k** to represent the three coordinate directions with which the individual partial derivatives are associated. Note further that ∇ represents an operation, inasmuch as the individual partial derivatives are operations. Since pressure is a scalar, the ∇ operation on pressure p can be represented as

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}. \tag{1.21}$$

Here the three partial derivatives represent the rates of change of pressure in the three coordinate directions. When ∇ operates on a scalar, the result is a vector and is referred to as the "gradient" of the scalar pressure field at each point. The gradient is an important operation, because we can use it to find the rates of change of a field variable in particular directions. An arbitrary direction in space can be described using a unit vector $\mathbf{n}$, where $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$. The rate of change of pressure above, for instance, in the $\mathbf{n}$ direction is simply the component of its gradient vector along the vector $\mathbf{n}$, as given by the inner product

$$\nabla p \cdot \mathbf{n} = \frac{\partial p}{\partial x}n_x + \frac{\partial p}{\partial y}n_y + \frac{\partial p}{\partial z}n_z. \tag{1.22}$$

What if the field variable that ∇ is to operate on is also a vector, such as velocity $\mathbf{v}(x, y, z; t)$? We can perform this operation in two ways: (i) define it as an inner product between ∇ and $\mathbf{v}$, or (ii) define it as a cross product between ∇ and $\mathbf{v}$. In the former case, the result is a scalar, while in the latter case, it is a vector. Both are meaningful physically. The inner product is referred to as the "divergence" of the vector field $\mathbf{v}$, while the cross product is referred to as the "curl" of the vector field $\mathbf{v}$. We begin with a discussion of the inner product:

$$\mathrm{div}\mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}. \tag{1.23}$$

We note again that divergence is a scalar quantity, and since it also is a function of the spatial coordinate, it represents a scalar field in its own right. In physical terms, divergence of velocity represents the net outward flow per unit volume at a point represented by the coordinates (x, y, z) (see, for instance, (Greenberg 1978) for a further discussion of divergence). This interpretation of divergence leads naturally into the following discussion.

Another scalar field was mentioned above, namely, the fluid density $\rho(x, y, z; t)$. By combining density ρ with velocity $\mathbf{v}$, one can discuss mass flow rates and examine what conservation of mass means in fluids. If we consider a small hypothetical elemental volume within the fluid, we can argue that the rate of change of mass (increase or decrease) within the small element must equal the net inflow or outflow of mass into or from the element. This can be expressed in the form

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0. \tag{1.24}$$

With the help of equation (1.23), we find that equation (1.24) implies that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0. \tag{1.25}$$

Expanding the second term,

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v} + \rho \nabla \cdot \mathbf{v} = 0. \tag{1.26}$$

The ∇ in the second term represents the gradient of the density field, given by

$$\nabla \rho = \frac{\partial \rho}{\partial x}\mathbf{i} + \frac{\partial \rho}{\partial y}\mathbf{j} + \frac{\partial \rho}{\partial z}\mathbf{k}. \tag{1.27}$$

Thus,

$$\nabla \rho \cdot \mathbf{v} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}. \tag{1.28}$$

Here, we have used $\mathbf{v} = (u, v, w)$, or in other words, utilized the three Cartesian components of velocity $\mathbf{v}$. Equation (1.26) can thus be rewritten as

$$\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right] + \rho \nabla \cdot \mathbf{v} = 0. \tag{1.29}$$

The terms enclosed by the square bracket represent the so-called "material derivative" or "Lagrangian derivative" in a fluid flow, represented using the notation D/Dt. This

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is the rate of change of a quantity (here density) that we would see if we followed the fluid, flowing with it at velocity $\mathbf{v}$. Thus, here

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0. \tag{1.30}$$

If the fluid is incompressible, then

$$\frac{D\rho}{Dt} = 0, \tag{1.31}$$

which implies (since ρ is not in general, zero)

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.32}$$

Equation (1.30) represents the continuity equation (or the conservation of mass principle) in compressible fluids, while when the fluid is incompressible, equation (1.31) represents conservation of mass principle. In general, when flow velocities are much smaller than the speed of sound in the fluid (i.e., two or three orders of magnitude smaller), the effect of compressibility becomes relatively negligible, and the fluid can be treated as nearly incompressible.

We mention here that an important application of the material derivative arises when one wants to study the dynamics of flowing fluids, where the first step is the application of Newton's second law on an arbitrary small fluid element of volume dV. With the condition that the mass of the element itself remains constant, we can express its momentum as $\rho\mathbf{v}dV$. The rate of change of momentum, for incompressible fluids, as we track the element through its motion is understood to be

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right]. \tag{1.33}$$

Let us next consider the cross product of the operator ∇ with another vector. Using the fluid velocity $\mathbf{v}$ again as an example, we can express the cross product $\nabla \times \mathbf{v}$ as

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}. \tag{1.34}$$

This can be rewritten as

$$\nabla \times \mathbf{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j}$$
$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}. \tag{1.35}$$

This product thus takes a vector field such as velocity and defines another, closely associated, vector field. The product in equations (1.34) and (1.35) is referred to as the curl of velocity $\mathbf{v}$. It is easier to see the physical significance of curl in a two-dimensional case, where $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$. In this case, $w = 0$, and there is no variation of

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u and v in the z direction in purely two-dimensional flow. Thus, the curl of this vector becomes

$$\nabla \times \mathbf{v} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{k}. \tag{1.36}$$

First of all, we notice that the curl in equation (1.36) is perpendicular to the plane occupied by the **v** vector. On a little more specific level, we observe that the first term represents the rate of change of the y-component of velocity with respect to x, while the second term denotes the rate of change of the x-component of velocity relative to y. Figure 1.4 explains what this means for a small fluid element of dimensions dx and dy. Counterclockwise rotation is taken to be positive. If the element were to rotate about the fixed point O as a rigid body, the horizontal side would rotate by the same amount as the vertical side, in the same sense/direction. Then, the horizontal side OA would rotate counterclockwise at an angular velocity $\partial v/\partial x = \omega_H$, and the vertical side OC would rotate counterclockwise at an angular velocity $\partial u/\partial y = \omega_V$, where the two are numerically equal, or $\omega_H = \omega_V = \omega_R$, out of the plane of the page. Since u is positive to the left, the angular velocity of the element could then be expressed as

$$\omega = \frac{1}{2}(\omega_H + \omega_V) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \mathbf{k} = \omega_R \mathbf{k}. \tag{1.37}$$

The first term after the equality denotes an average of the two angular velocities, which in this case happens to be the actual angular velocity of the element. The above, of course, assumes that the element is rotating like a rigid body. At small scales, fluid elements tend to be deformable, however, and the sides OA and OC may not rotate at the same angular velocity, or in the same sense, for that matter. Thus, the side OA could rotate counterclockwise, while the side OC rotates clockwise, and a square-shaped element may not look like a square upon deformation. This is the case of shear

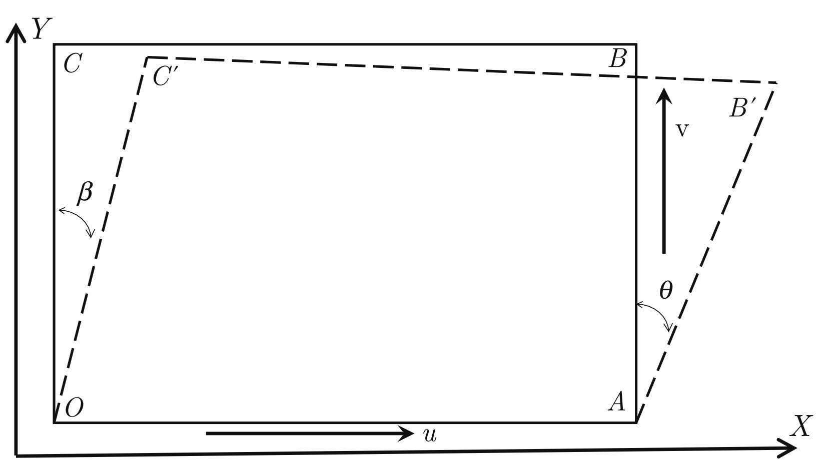

Figure 1.4 A schematic showing the deformation of a small fluid element through non-rigid-body rotation of the element, where its vertical and adjacent sides undergo different rotational deformations.

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deformation, as would be caused by viscous effects in a fluid, which we will study in detail in Chapter 10. In that case, we would need to use the original expression in equation (1.36) to define the average angular velocity of the element, such that

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k} = \frac{1}{2}\left(\nabla \times \mathbf{v}\right). \tag{1.38}$$

In either case, we see that

$$\nabla \times \mathbf{v} = \frac{1}{2}\omega. \tag{1.39}$$

In terms of its physical significance, curl of the velocity is thus related to the angular velocity in a fluid and is referred to as "vorticity." An example of flow such as in equation (1.37) can occur in a fluid that is stirred into rotation about a certain axis at a uniform angular velocity. We make a slight digression here to point out that fluid particle motion under a small-amplitude propagating surface gravity wave in deep water is circular. Yet such waves and the particle motion under them can be described using velocity potentials, which presumes irrotational motion. Particle motion here is therefore such that the individual particle itself does not rotate about its own axis as it travels over its circular trajectory. In other words, a hypothetical rectangular particle with a horizontal long dimension travels such that the long horizontal dimension remains horizontal throughout the trajectory. If the particle were to undergo rotational motion at angular ω, all sides of the particle would rotate at ω. As pointed out by Greenberg (1978), particle would undergo irrotational motion if it traveled as an individual seat in a Ferris wheel would. Some interesting observations follow:

$$\nabla \times \nabla f = 0, \tag{1.40}$$

where f denotes a scalar field. In words, this is to say that the curl of a gradient of a scalar field is zero. In addition,

$$\nabla \cdot \nabla \times \mathbf{v} = 0. \tag{1.41}$$

Equation (1.41) states that the divergence of a curl of a vector is zero. Both statements can be verified using the determinant expansion form for cross product and working through the entire operations, realizing that

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y}, \quad \text{etc.} \tag{1.42}$$

Some other statements and interesting interrelationships involving the del operator are included below:

$$\begin{aligned}
\nabla^2 &= \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},\\
\nabla \cdot (f\mathbf{v}) &= \nabla f \cdot \mathbf{v} + f \nabla \cdot \mathbf{v},\\
\nabla \times (f\mathbf{v}) &= \nabla f \times \mathbf{v} + f \nabla \times \mathbf{v},\\
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}).
\end{aligned} \tag{1.43}$$

The ∇^2 operator in the first of equations (1.43) is generally referred to as the "Laplacian" and will be used frequently in the hydrodynamics discussions of the subsequent chapters.

1.5 Indicial Notation

Here we introduce a system for representing three-dimensional (and higher) quantities. The indicial notation is widely used in Physics, Hydrodynamics, and Elasticity and represents a very compact way to represent, manipulate, and understand higher-dimensional quantities (i.e., vectors, tensors, etc.). Indicial notation will be used frequently in this text, for instance, in Chapters 9, 10, 11, and so forth. The representations for inner products (dot products), derivatives, and so forth, require some attention and are also discussed here.

Under the rules of indicial notation, when an index occurs unrepeated in a term, that index is understood to take on the values 1, 2, . . ., K, where K is a specified integer. When an index appears twice in a term, that index is understood to take on all the values of its range, and the resulting terms summed. This often is called "Einstein's Summation Convention." In this notation, the repeated indices are called "dummy indices" and the unrepeated ones are called "free indices." For example, the material derivative can be written, in this notation, as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}, \qquad j = 1,2,3,$$

and material derivative of a vector $\mathbf{u} = [u_1, u_2, u_3]$ can be written as

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}, \quad i = 1,2,3, \quad j = 1,2,3 \text{ (sum over } j). \tag{1.44}$$

Note also that $\partial u_i / \partial x_j$ can be written as $u_{i,j}$ in indicial notation, in other words, comma denotes differentiation.

The most important rules of the indicial notation are:

1. A letter index may occur either once or twice in a given term. If it occurs once, it is a free index. If it occurs twice, it is a dummy index. For example, the term a_{ijj} has one free index, i, and one dummy index, j.
2. If an index is a free index, it is understood to take on the values $1, 2, \ldots, K$, where K is an integer that defines the range of the index. For example, a_{ijj}, $i, j = 1,2,3$, indicates that i and j run from 1 through 3. The tensorial rank of a term is determined by the number of free indices in that term. For example, a_{ijj} indicates that the term is a tensor of rank 1, since there is only one free index, i. This will be discussed in Section 2.6.
3. When an index is a dummy index, i.e., it occurs twice in a given term, it is understood that the term is summed over the dummy index from 1 through K, the entire range of the index. For example, $a_{ijj} = 0$, $i, j = 1,2,3$, represents $K = 3$

equations, one for each free index i, and each equation has 3 terms on the left-hand side, e.g., $a_{1jj} = a_{111} + a_{122} + a_{133} = 0$, for $i = 1$, $j = 1, 2, 3$, and so forth.

4. A dummy (or repeated) index can be replaced by another dummy index in any one or more terms in an equation since the meaning of the dummy index does not change, e.g., $a_{ijj} = a_{ikk}$ is acceptable or is equivalent. But a free index can only be replaced by another letter index if the same free index in all other terms in the equation is replaced with the same letter index. For example, $a_{ijj} + b_{ill} = 0$ can also be written as $a_{kjj} + b_{kll} = 0$.

1.5.1 Examples

(**a**) Velocity vector:

$$\boldsymbol{V} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3} \quad \text{or} \quad \boldsymbol{V} = u_j\boldsymbol{e_j}.$$

(**b**) Equation of a plane in Ox_1, x_2, x_3:

$$ax_1 + bx_2 + cx_3 = d \quad \text{or} \quad a_1x_1 + a_2x_2 + a_3x_3 = d \quad \text{or} \quad \sum_{j=1}^{3} a_jx_j = d \quad \text{or} \quad a_jx_j = d.$$

(**c**) Dot product:

$$\begin{aligned}\boldsymbol{q} \cdot \boldsymbol{q} &= (q_1\boldsymbol{e_1} + q_2\boldsymbol{e_2} + q_3\boldsymbol{e_3}) \cdot (q_1\boldsymbol{e_1} + q_2\boldsymbol{e_2} + q_3\boldsymbol{e_3}),\\ &= q_1^2 + q_2^2 + q_3^2 = \sum_{j=1}^{3} q_jq_j\\ &= q_jq_j, \qquad j = 1, 2, 3.\end{aligned}$$

(**d**) Derivative of a scalar:

$$\frac{\partial^2\phi}{\partial {x_1}^2} + \frac{\partial^2\phi}{\partial {x_2}^2} + \frac{\partial^2\phi}{\partial {x_3}^2} = \nabla^2\phi = \Delta\phi = \phi_{,jj} = 0,$$

which is the Laplace equation. Note that $\nabla^2 \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \equiv \Delta$ (Delta).

We now introduce the "Kronecker delta" and the "permutation symbol." Kronecker delta is defined by:

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{1.45}$$

We also introduce the "permutation symbol," defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ are an even permutation of } 1, 2, 3, \\ -1 & \text{if } i, j, k \text{ are an odd permutation of } 1, 2, 3, \\ 0 & \text{if } i, j, k \text{ are not a permutation of } 1, 2, 3. \end{cases} \tag{1.46}$$

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1.5.2 Examples

a) $\delta_{11} = 1,\ \delta_{12} = 0,\ \delta_{31} = 0,\ \delta_{22} = 1,$
b) $\epsilon_{123} = 1, \epsilon_{321} = -1, \epsilon_{112} = 0, \epsilon_{213} = -1,$
c) $\delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3,$
d) Verify that $\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}$.

Note first that the number of free indices on the left and on the right are the same, as they should be. Let us now expand $\epsilon_{ijk}\epsilon_{ijm}$ for a range of $i, j, k, m = 1, 2, 3$:

$$\epsilon_{ijk}\epsilon_{ijm} = \epsilon_{12k}\epsilon_{12m} + \epsilon_{13k}\epsilon_{13m} + \epsilon_{21k}\epsilon_{21m} + \epsilon_{23k}\epsilon_{23m} + \epsilon_{31k}\epsilon_{31m} + \epsilon_{32k}\epsilon_{32m},$$

since if any two indices are the same, then $\epsilon_{ijk} = 0$, e.g., $\epsilon_{11k} = 0$, and therefore,

$$\text{for} \quad k = 1, \quad \epsilon_{ijk}\epsilon_{ijm} = \epsilon_{231}\epsilon_{23m} + \epsilon_{321}\epsilon_{32m} = \begin{cases} 2 & \text{if } m = 1 \\ 0 & \text{if } m \neq 1 \end{cases}$$

$$\text{for} \quad k = 2, \quad \epsilon_{ijk}\epsilon_{ijm} = \epsilon_{132}\epsilon_{13m} + \epsilon_{312}\epsilon_{31m} = \begin{cases} 2 & \text{if } m = 2 \\ 0 & \text{if } m \neq 2 \end{cases}$$

$$\text{for} \quad k = 3, \quad \epsilon_{ijk}\epsilon_{ijm} = \epsilon_{123}\epsilon_{12m} + \epsilon_{213}\epsilon_{21m} = \begin{cases} 2 & \text{if } m = 3 \\ 0 & \text{if } m \neq 3 \end{cases}$$

As a result,

$$\epsilon_{ijk}\epsilon_{ijm} = \begin{cases} 2 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$$

and thus,

$$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}, \tag{1.47}$$

as we were asked to verify.

The Kronecker delta is generally used in cases such as an orthogonal transformation of a coordinate system to another, or as a substitution operator to obtain compact forms of some equations as we shall see later on. δ_{ij} is a second-order tensor (see Section 2.6).

The permutation symbol (or alternator) is generally used for the cross product of vectors. For instance, if $\boldsymbol{a} = a_1\boldsymbol{e_1} + a_2\boldsymbol{e_2} + a_3\boldsymbol{e_3}$ and $\boldsymbol{b} = b_1\boldsymbol{e_1} + b_2\boldsymbol{e_2} + b_3\boldsymbol{e_3}$ then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$ can be written as

$$\epsilon_{ijk}a_j b_k = c_i. \tag{1.48}$$

Equation (1.48) can be expanded to see that this is indeed the case. ϵ_{ijk} is a third-order tensor (see Section 2.6).

Note also that if two indices of the permutation symbol are exchanged, then the value of the permutation symbol will change the sign. For example,

$$\epsilon_{ijk} = -\epsilon_{jik}, \tag{1.49}$$

and this can be shown by simply expanding the indices for their range. Therefore, ϵ_{ijk} is a skew-symmetric tensor. Equation (1.49) is frequently used in expanding equations written in indicial notation or to prove identities. Also note that the product of a skew symmetric tensor and a symmetric tensor is zero.

Another identity commonly used is the $\epsilon - \delta$ identity given by

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \tag{1.50}$$

Note also that

$$\delta_{ij}x_j = x_i, \qquad \delta_{kl}v_l = v_k, \tag{1.51}$$

1.6 Concluding Remarks

In this chapter, we reviewed some of the basic descriptions and relationships important to the use of vectors. Part of our focus here was on vector algebra, specifically, the different ways in which vectors can be multiplied with each other, to give either scalar or vector results, namely, the inner product, and the cross product, respectively. We considered some examples representative of situations often encountered in marine applications, and the fluid mechanics basics that one needs to be familiar with when working with bodies in water and fluid flows. Part of our attention here was on the differential calculus relations that are important to solving the problems of interest to marine situations. Such problems may require analysis of the combined effects of multiple vector quantities, particularly, how their interaction determines behavior, of a body in water or the water itself. In Chapter 2, we will review the integral calculus of vectors, with a special emphasis on some of the classical relationships and theorems of vector integral calculus that are frequently used in problem-solving.

1.7 Self-Assessment

1.7.1

Consider the vectors

$$\mathbf{c} = [3, 2, -1] \text{ and } \mathbf{d} = [-2, 3, 1]. \tag{1.52}$$

Evaluate: $\mathbf{c} \times \mathbf{d}$ and $\mathbf{d} \times \mathbf{c}$.

1.7.2

Given the two vectors $\mathbf{c}$ and $\mathbf{d}$ above, and a third vector $\mathbf{e} = [1, 0, 10]$, consider the three-dimensional volume defined by the three vectors.

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(a) What is the area of the parallelogram formed by $\mathbf{c}$ and $\mathbf{d}$ (given by $\|\mathbf{c} \times \mathbf{d}\|$)?
(b) What is the volume defined by the three vectors, given by $|\mathbf{c} \cdot \mathbf{d} \times \mathbf{e}|$? What would the volume be if the three vectors are coplanar? Thus, suggest a test for coplanarity of three vectors.

1.7.3

For $\phi = x^2 y^3 z^3$, obtain

(a) $\nabla\phi$,
(b) $\nabla\phi$ at $(2,1,0)$ along the y axis (i.e., in the $\mathbf{j}$ direction).

1.7.4

Verify the following relationships listed in Section 1.4:

$$\begin{aligned}
&\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\
&\nabla \cdot (f\mathbf{v}) = \nabla f \cdot \mathbf{v} + f\nabla \cdot \mathbf{v} \\
&\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v} \\
&\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})
\end{aligned} \tag{1.53}$$

1.7.5

Show that

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}, \quad \boldsymbol{\nabla^2} = (\nabla \cdot \nabla)\mathbf{a}. \tag{1.54}$$

1.7.6

If $\mathbf{a} = (2,1,3)$, what is $\nabla \times (\nabla \times \mathbf{a})$?

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