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# EXISTENCE RESULT FOR NONUNIFORMLY DEGENERATE SEMILINEAR ELLIPTIC SYSTEMS IN $\mathbb{R}^N$

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**Abstract.** We study the existence of solutions for a class of nonuniformly degenerate elliptic systems in $\mathbb{R}^N$, $N \geq 3$, of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) + a(x)u = f(x, u, v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(h_2(x)\nabla v) + b(x)v = g(x, u, v) & \text{in } \mathbb{R}^N, \end{cases}$$

where $h_i \in L^1_{loc}(\mathbb{R}^N)$, $h_i(x) \geqq \gamma_0 |x|^\alpha$ with $\alpha \in (0, 2)$ and $\gamma_0 > 0$, $i = 1, 2$. The proofs rely essentially on a variant of the Mountain pass theorem (D. M. Duc, Nonlinear singular elliptic equations, *J. Lond. Math. Soc.* **40**(2) (1989), 420–440) combined with the Caffarelli–Kohn–Nirenberg inequality (First order interpolation inequalities with weights, *Composito Math.* **53** (1984), 259–275).

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**1. Introduction.** This paper deals with the existence of solutions to the nonuniformly degenerate elliptic systems in $\mathbb{R}^N$, $N \geqq 3$, of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) + a(x)u = f(x, u, v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(h_2(x)\nabla v) + b(x)v = g(x, u, v) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.1}$$

Note that in the case when $h_1(x) \equiv h_2(x) \equiv 1$ in $\mathbb{R}^N$, system (1.1) was studied by D. G. Costa [**7**]. In that paper, using variational methods the author proved the existence of a weak solution in a subspace of the Sobolev space $H^1(\mathbb{R}^N, \mathbb{R}^2)$. This was extended by N. T. Chung [**6**], in which the author considered the situation that $h_i \in L^1_{loc}(\mathbb{R}^N)$, $h_i(x) \geqq 1$ for a.e. $x \in \mathbb{R}^N$ with $i = 1, 2$. Then, system (1.1) now was nonuniformly elliptic and an existence result was obtained by using a variant of the Mountain pass theorem in [**8**]. We also find that in the scalar case, the degenerate elliptic problem of the form

$$-\operatorname{div}(|x|^\alpha \nabla u) = f(x, u) \text{ in } \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in (0, 2)$ and the nonlinearity term $f$ has special structures, was studied in many works (see [**4**, **9**, **10**, **12**–**14**]). Such problems in anisotropic media can be regarded as equilibrium solutions of the evolution equations. For instance, in describing the behaviour of a bacteria culture, the state variable $u$ represents the number of mass of the bacteria.

In the present paper, we extend the results in [**6**, **7**, **10**, **12**, **13**] to a class of nonuniformly degenerate semilinear elliptic systems in $\mathbb{R}^N$. In order to state our main theorem, we first introduce some hypotheses.

Assume that the functions $a, b : \mathbb{R}^N \to \mathbb{R}$ and $h_i : \mathbb{R}^N \to [0, \infty)$, $i = 1, 2$, satisfy the following hypotheses:

**(A − B)** $a(x), b(x) \in L^\infty_{loc}(\mathbb{R}^N)$, there exist $a_0, b_0 > 0$ such that $a(x) \geqq a_0$, $b(x) \geqq b_0$ for all $x \in \mathbb{R}^N$.

**(H)** $h_i \in L^1_{loc}(\mathbb{R}^N)$, $i = 1, 2$, and there exist constants $\alpha \in (0, 2)$, $\gamma_0 > 0$ such that $h_i(x) \geqq \gamma_0 |x|^\alpha$ for all $x \in \mathbb{R}^N$.

Next, we assume that the functions $F, f, g : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are of $C^1$ class, $\frac{\partial F}{\partial u} = f(x, w)$, $\frac{\partial F}{\partial v} = g(x, w)$, $\nabla F(x, w) = \left(\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right)$ for all $x \in \mathbb{R}^N$ and all $w = (u, v) \in \mathbb{R}^2$. In addition, the following hypotheses are satisfied:

**($F_1$)** $f(x, 0, 0) = g(x, 0, 0) = 0$ for all $x \in \mathbb{R}^N$.

**($F_2$)** There exist nonnegative functions $\tau_1, \tau_2$ with $\tau_1 \in L^{r_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\tau_2 \in L^{s_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where $r, s \in \left(1, \frac{N+2-\alpha}{N-2+\alpha}\right)$, $r_0 = \frac{2N}{2N-(r+1)(N-2+\alpha)}$, $s_0 = \frac{2N}{2N-(s+1)(N-2+\alpha)}$, $\alpha \in (0, 2)$ such that

$$|\nabla f(x, w)| + |\nabla g(x, w)| \leqq \tau_1(x)|w|^{r-1} + \tau_2(x)|w|^{s-1}$$

for all $x \in \mathbb{R}^N$, $w = (u, v) \in \mathbb{R}^2$.

**($F_3$)** There exists a constant $\mu > 2$ such that

$$0 < \mu F(x, w) \leqq w \cdot \nabla F(x, w)$$

for all $x \in \mathbb{R}^N$ and $w \in \mathbb{R}^2 \backslash \{(0, 0)\}$.

Let $E$ and $H$ be the spaces defined as the completion of $C_0^\infty(\mathbb{R}^N, \mathbb{R}^2)$ with respect to the norms

$$\|w\|_\alpha^2 = \int_{\mathbb{R}^N} [|x|^\alpha |\nabla u|^2 + |x|^\alpha |\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2]\, dx$$

and

$$\|w\|_H^2 = \int_{\mathbb{R}^N} [h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2]\, dx$$

for $w = (u, v)$. Then, it is clear that $E$ and $H$ are Hilbert spaces with respect to the inner products

$$\langle w_1, w_2 \rangle_\alpha = \int_{\mathbb{R}^N} [|x|^\alpha \nabla u_1 \nabla u_2 + |x|^\alpha \nabla v_1 \nabla v_2 + a(x) u_1 u_2 + b(x) v_1 v_2]\, dx$$

for $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2) \in E$ and

$$\langle w_1, w_2 \rangle_H = \int_{\mathbb{R}^N} [h_1(x) \nabla u_1 \nabla u_2 + h_2(x) \nabla v_1 \nabla v_2 + a(x) u_1 u_2 + b(x) v_1 v_2]\, dx$$

for $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2) \in H$. Moreover, by the condition **(H)**, the embedding $H \hookrightarrow E$ is continuous.

DEFINITION 1.1. We say that $w = (u, v) \in H$ is a weak solution of system (1.1) if

$$\int_{\mathbb{R}^N} [h_1(x)\nabla u \nabla \varphi_1 + h_2(x)\nabla v \nabla \varphi_2 + a(x)u\varphi_1 + b(x)v\varphi_2]\, dx -$$

$$- \int_{\mathbb{R}^N} [f(x, u, v)\varphi_1 + g(x, u, v)\varphi_2]\, dx = 0$$

for all $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2)$.

Our main result is given by the following theorem.

THEOREM 1.2. *Assume that the hypotheses* **(A − B)**, **(H)** *and* **(F$_1$) − (F$_3$)** *are satisfied. Then system (1.1) has at least one non-trivial weak solution.*

Note that by hypothesis **(H)**, the problem which was considered here contains the situations in [**6**] and [**7**]. We also do not require the coercivity for the functions $a(x)$ and $b(x)$ as in [**12**]. Theorem 1.2 will be proved by using variational techniques based on a variant of the Mountain pass theorem [**8**]. But the key in our arguments is the following lemma which can be obtained essentially by interpolating between Sobolev's and Hardy's inequalities (see [**3**, **5**]).

LEMMA 1.3 (Caffarelli–Kohn–Nirenberg). *Let $N \geq 2$, $\alpha \in (0, 2)$. Then there exists a constant $C_\alpha > 0$ such that*

$$\left( \int_{\mathbb{R}^N} |\varphi|^{2_\alpha^*} dx \right)^{\frac{2}{2_\alpha^*}} \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |\nabla \varphi|^2 dx$$

*for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $2^\star = \frac{2N}{N-2+\alpha}$.*

**2. Proof of the main result.** Let us define the functional $\mathcal{I} : H \to \mathbb{R}$ given by

$$\mathcal{I}(w) = \frac{1}{2} \int_{\mathbb{R}^N} [h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2]\, dx - \int_{\mathbb{R}^N} F(x, u, v)\, dx$$
$$= \mathcal{H}(w) - \mathcal{F}(w), \tag{2.1}$$

where

$$\mathcal{H}(w) = \frac{1}{2} \int_{\mathbb{R}^N} [h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2]\, dx, \tag{2.2}$$

$$\mathcal{F}(x) = \int_{\mathbb{R}^N} F(x, u, v)\, dx \text{ for all } w = (u, v) \in H. \tag{2.3}$$

In general, as $h_i \in L^1_{loc}(\mathbb{R}^N)$, $i = 1, 2$, the functional $\mathcal{H}$ (and thus $\mathcal{I}$) may not belong to $C^1(H)$ as usual (in this work, we are not completely interested in the case whether the functional $\mathcal{I}$ belongs to $C^1(H)$ or not). This means that we cannot apply directly the Mountain pass theorem by Ambrosetti and Rabinowitz [**1**]. To overcome this difficulty, we need to recall the following useful concept of weakly continuous differentiablity.

DEFINITION 2.1. Let $J$ be a functional from a Banach space $Y$ into $\mathbb{R}$. We say that $J$ is weakly continuously differentiable on $Y$ if and only if the following conditions are satisfied:

(i) For any $u \in Y$ there exists a linear map $DJ(u)$ from $Y$ into $\mathbb{R}$ such that

$$\lim_{t\to 0} \frac{J(u+tv)-J(u)}{t} = \langle DJ(u), v\rangle\,, \forall v \in Y.$$

(ii) For any $v \in Y$, the map $u \mapsto \langle DJ(u), v\rangle$ is continuous on $Y$.

We denote by $C^1_w(Y)$ the set of weakly continuously differentiable functionals on $Y$. It is clear that $C^1(Y) \subset C^1_w(Y)$, where $C^1(Y)$ is the set of all continuously Fréchet differentiable functionals on $Y$. With similar arguments as those used in the proof of Proposition 2.2 in [**6**], we conclude the following lemma which concerns the smoothness of the functional $\mathcal{I}$.

LEMMA 2.2. *The functional $\mathcal{I}$ given by (2.1) is weakly continuously differentiable on $H$ and we have*

$$\langle D\mathcal{I}(w), \varphi\rangle = \int_{\mathbb{R}^N} [h_1(x)\nabla u \nabla \varphi_1 + h_2(x)\nabla v \nabla \varphi_2 + a(x)u\varphi_1 + b(x)v\varphi_2]\,dx - \int_{\mathbb{R}^N} [f(x,u,v)\varphi_1 + g(x,u,v)\varphi_2]\,dx \quad (2.4)$$

*for all $w = (u, v)$, $\varphi = (\varphi_1, \varphi_2) \in H$.*

By Lemma 2.2, weak solutions of system (1.1) correspond to the critical points of the functional $\mathcal{I}$. Our approach is based on a weak version of the Mountain pass theorem by D. M. Duc [**8**].

LEMMA 2.3. *The functional $\mathcal{H}$ given by (2.2) is weakly lower semicontinuous on the space $H$.*

*Proof.* By the convexity of the functional $\mathcal{H}$, in order to prove the weak lower semicontinuity of $\mathcal{H}$ on $H$ we shall prove that for any $w_0 \in H$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{H}(w) \geq \mathcal{H}(w_0) - \epsilon \quad \forall w \in H : \|w - w_0\|_H < \delta.$$

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Since $\mathcal{H}$ is convex, for all $w \in H$ we have

$$\begin{aligned}
\mathcal{H}(w) &\geq \mathcal{H}(w_0) + \langle D\mathcal{H}(w_0), w - w_0\rangle \\
&\geq \mathcal{H}(w_0) - \int_{\mathbb{R}^N} [h_1(x)|\nabla u_0||\nabla u - \nabla u_0| + h_2(x)|\nabla v_0||\nabla v - \nabla v_0|]\,dx \\
&\quad - \int_{\mathbb{R}^N} [a(x)|u_0\|u - u_0| + b(x)|v_0\|v - v_0|]\,dx \\
&\geq \mathcal{H}(w_0) - \left(\int_{\mathbb{R}^N} h_1(x)|\nabla u_0|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^N} h_1(x)|\nabla u - \nabla u_0|^2 dx\right)^{\frac{1}{2}} \\
&\quad - \left(\int_{\mathbb{R}^N} h_2(x)|\nabla v_0|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^N} h_2(x)|\nabla v - \nabla v_0|^2 dx\right)^{\frac{1}{2}} \\
&\quad - \left(\int_{\mathbb{R}^N} a(x)|u_0|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^N} a(x)|u - u_0|^2 dx\right)^{\frac{1}{2}} \\
&\quad - \left(\int_{\mathbb{R}^N} b(x)|v_0|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^N} b(x)|v - v_0|^2 dx\right)^{\frac{1}{2}} \\
&\geq \mathcal{H}(w_0) - c\,\|w - w_0\|_H, \qquad \text{where } c = 4\,\|w_0\|_H.
\end{aligned}$$

Taking $\delta = \frac{\epsilon}{c}$ we obtain that

$$\mathcal{H}(w) \geq \mathcal{H}(w_0) - \epsilon, \quad \forall w \in H : \|w - w_0\|_H < \delta.$$

Thus, we have proved that $\mathcal{H}$ is strongly lower semicontinuous on $H$. Since $\mathcal{H}$ is convex, by Corollary III.8 in [**2**] we conclude that $\mathcal{H}$ is weakly lower semicontinuous on $H$. □

LEMMA 2.4. *The functional $\mathcal{I}$ given by (2.1) satisfied the Palais-Smale condition in $H$.*

*Proof.* Let $\{w_m\} = \{(u_m, v_m)\}$ be a sequence in $H$ such that

$$\lim_{m\to\infty} \mathcal{I}(w_m) = \overline{c}, \quad \lim_{m\to\infty} \|D\mathcal{I}(w_m)\|_{H^\star} = 0.$$

We first prove that $\{w_m\}$ is bounded in $H$. By $(\mathbf{F_3})$ we have

$$\begin{aligned}
\mathcal{I}(w_m) - \frac{1}{\mu}\langle D\mathcal{I}(w_m), w_m\rangle &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|w_m\|_H^2 + \left(\frac{1}{\mu}\langle D\mathcal{F}(w_m), w_m\rangle - \mathcal{F}(w_m)\right) \\
&\geq \gamma\,\|w_m\|_H^2,
\end{aligned}$$

where $\gamma = \frac{1}{2} - \frac{1}{\mu}$. It yields that

$$\begin{aligned}\mathcal{I}(w_m) &\geq \gamma \|w_m\|_H^2 + \frac{1}{\mu}\langle D\mathcal{I}(w_m), w_m\rangle \\ &\geq \gamma \|w_m\|_H^2 - \frac{1}{\mu}\|D\mathcal{I}(w_m)\|_{H^\star}.\|w_m\|_H \\ &= \|w_m\|_H\left(\gamma \|w_m\|_H - \frac{1}{\mu}\|D\mathcal{I}(w_m)\|_{H^\star}\right). \end{aligned} \tag{2.5}$$

Letting $m \to \infty$, since $\|D\mathcal{I}(w_{m_j})\|_{H^\star} \to 0$ and $\mathcal{I}(u_m) \to \overline{c}$, we deduce that $\{w_m\}$ is bounded in $H$. Since $H$ is a Hilbert space and $\{w_m\}$ is bounded, there exists a subsequence of $\{w_m\}$, denoted by $\{w_m\}$, such that $\{w_m\}$ converges weakly to some $w = (u, v)$ in $H$. Then, by Lemma 2.3 we find that

$$\mathcal{H}(w) \leq \liminf_{m\to\infty} \mathcal{H}(w_m). \tag{2.6}$$

Furthermore, by Lemma 1.3 and the condition **(H)** we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |\varphi_i|^{2_\alpha^\star} dx\right)^{\frac{2}{2_\alpha^\star}} &\leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |\nabla\varphi_i|^2 dx \\ &\leq \frac{C_\alpha}{\gamma_0}\int_{\mathbb{R}^N} h_i(x)|\nabla\varphi_i|^2 dx, \text{ for any } \varphi_i \in C_0^\infty(\mathbb{R}^N), i = 1, 2.\end{aligned}$$

It follows that the embeddings $H \hookrightarrow E \hookrightarrow L^{2_\alpha^\star}(\mathbb{R}^N, \mathbb{R}^2)$ are continuous. Therefore, $\{w_m\}$ converges weakly to $w$ in $L^{2_\alpha^\star}(\mathbb{R}^N, \mathbb{R}^2)$ and $w_m(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$. Then, it is clear that the sequence $\{|w_{m_k}|^{r-1}w_{m_k}\}$ converges weakly to $|w|^{r-1}w$ in $L^{\frac{2_\alpha^\star}{r}}(\mathbb{R}^N, \mathbb{R}^2)$. Using the method as in **[11]** we define the map $K(w) : L^{\frac{2_\alpha^\star}{r}}(\mathbb{R}^N, \mathbb{R}^2) \to \mathbb{R}$ by

$$\langle K(w), \Phi\rangle = \int_{\mathbb{R}^N} \tau_1(x) w\varphi dx, \quad \varphi = (\varphi_1, \varphi_2) \in L^{\frac{2_\alpha^\star}{r}}(\mathbb{R}^N, \mathbb{R}^2).$$

Since $\tau_1 \in L^{p_0}(\mathbb{R}^N)$, $w \in L^{2_\alpha^\star}(\mathbb{R}^N, \mathbb{R}^2)$, $\varphi \in L^{\frac{2_\alpha^\star}{r}}(\mathbb{R}^N, \mathbb{R}^2)$ and $\frac{1}{r_0} + \frac{1}{2_\alpha^\star} + \frac{r}{2_\alpha^\star} = 1$, the map $K(w)$ is linear and continuous. Hence,

$$\left\langle K(w), |w_m|^{r-1}w_m\right\rangle \to \left\langle K(w), |w|^{r-1}w\right\rangle \text{ as } m \to \infty$$

i.e.

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} \tau(x)|w_m|^{r-1}w_m w dx = \int_{\mathbb{R}^N} \tau(x)|w|^{r+1} dx. \tag{2.7}$$

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With the same arguments we can show that

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}\tau_2(x)|w_m|^{s-1}w\,dx=\int_{\mathbb{R}^N}\tau_2(x)|w|^{s+1}dx, \tag{2.8}$$

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}\tau_1(x)|w_m|^{r+1}dx=\int_{\mathbb{R}^N}\tau_1(x)|w|^{r+1}dx, \tag{2.9}$$

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}\tau_2(x)|w_m|^{s+1}dx=\int_{\mathbb{R}^N}\tau_2(x)|w|^{s+1}dx. \tag{2.10}$$

Relations (2.7) and (2.9) imply that

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}\tau_1(x)|w_m|^{r-1}w_m(w_m-w)\,dx=0. \tag{2.11}$$

Similarly we obtain

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}\tau_2(x)|w_m|^{s-1}w_m(w_m-w)\,dx=0. \tag{2.12}$$

By (2.11), (2.12) and the condition $(\mathbf{F_2})$ we get

$$\lim_{m\to\infty}\langle D\mathcal{F}(w_m),w_m-w\rangle=\lim_{m\to\infty}\int_{\mathbb{R}^N}\nabla F(x,w_m)(w_m-w)=0, \tag{2.13}$$

which implies that

$$\lim_{m\to\infty}\langle D\mathcal{H}(w_m),w_m-w\rangle=0. \tag{2.14}$$

Using (2.14) and the convexity of $\mathcal{H}$ we infer that

$$\begin{aligned}\mathcal{H}(w)-\lim_{m\to\infty}\sup\mathcal{H}(w_m)&=\lim_{m\to\infty}\inf\,[\mathcal{H}(w)-\mathcal{H}(w_m)]\\&\geq\lim_{m\to\infty}\langle D\mathcal{H}(w_m),w-w_m\rangle=0.\end{aligned} \tag{2.15}$$

Relations (2.6) and (2.15) imply that

$$\mathcal{H}(w)=\lim_{m\to\infty}\mathcal{H}(w_m). \tag{2.16}$$

We now prove that $\{w_m\}$ converges strongly to $w$ in $H$. Indeed, we assume by contradiction that $\{w_m\}$ is not strongly convergent to $w$ in $H$. Then there exist a constant $\epsilon_0>0$ and a subsequence of $\{w_m\}$, denoted by $\{w_m\}$, such that $\|w_m-w\|_H\geq\epsilon_0>0$ for all $m=1,2,\ldots$ Hence,

$$\frac{1}{2}\mathcal{H}(w_m)+\frac{1}{2}\mathcal{H}(w)-\mathcal{H}\left(\frac{w_m+w}{2}\right)=\frac{1}{4}\|w_m-w\|_H^2\geq\frac{1}{4}\epsilon_0^2. \tag{2.17}$$

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Remark that the sequence $\{\frac{w_m+w}{2}\}$ also converges weakly to $w$ in $H$, applying Lemma 2.3 again we get

$$\mathcal{H}(w) \leq \lim_{j\to\infty}\inf \mathcal{H}\left(\frac{w_m+w}{2}\right). \tag{2.18}$$

Hence, letting $m \to \infty$ from (2.17) we infer

$$\mathcal{H}(w) - \lim_{j\to\infty}\inf \mathcal{H}\left(\frac{w_m+w}{2}\right) \geq \frac{1}{4}\epsilon_0^2, \tag{2.19}$$

which contradicts (2.18). Therefore, we conclude that $\{w_m\}$ converges strongly to $w$ in $H$. Thus, $\mathcal{I}$ satisfies the Palais-Smale condition in $H$. □

In order to apply the Mountain pass theorem we shall prove the following lemma which shows that the functional $\mathcal{I}$ has the geometry of the Mountain pass theorem.

LEMMA 2.5.

(i) *There exist two positive constants $\beta$ and $\rho$ such that $\mathcal{I}(w) \geq \beta$ $\forall w \in H$ with $\|w\|_H = \rho$.*
(ii) *There exists $w_0 \in H$ such that $\|w_0\|_H > \rho$ and $\mathcal{I}(w_0) < 0$.*

*Proof.* (i) We follow the method used in the proof of Theorem 1.2 in [**7**]. From condition (**F**$_3$) it is easy to see that

$$F(x, z) \geq \min_{|s|=1} F(x, s)|z|^{\mu} \quad \forall x \in \mathbb{R}^N \text{ and } z = (z_1, z_2) \in \mathbb{R}^2, |z| \geq 1, \tag{2.20}$$

$$0 < F(x, z) \leq \max_{|s|=1} F(x, s)|z|^{\mu} \quad \forall x \in \mathbb{R}^N \text{ and } z = (z_1, z_2) \in \mathbb{R}^2, |z| \leq 1, \tag{2.21}$$

where $\max_{|s|=1} F(x, s) \leq c$ in view of $(H_2)$.

Since $\mu > 2$, it follows from (2.21) that

$$\lim_{|z|\to 0} \frac{F(x, z)}{|z|^2} = 0 \text{ uniformly for } x \in \mathbb{R}^N. \tag{2.22}$$

From (2.22) we deduce that for every $\epsilon > 0$ there exists $\delta \in (0, 1)$ such that

$$0 < F(x, z) < \epsilon|z|^2 \tag{2.23}$$

for all $z$ with $|z| < \delta$. Therefore, by using the continuous embeddings $H \hookrightarrow E \hookrightarrow L^2(\mathbb{R}^N, \mathbb{R}^2)$, a simple calculation implies from (2.23) that $\inf_{\|w\|_H=\rho} \mathcal{I}(w) = \alpha > 0$ for all $\rho > 0$ small enough.

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(ii) Besides, by (2.14), for any given compact set $\Omega \subset \mathbb{R}^N$ there exists $\overline{c} = \overline{c}(\Omega)$ such that

$$F(x, z) \geq \overline{c}|z|^{\mu} \text{ for all } x \in \Omega, |z| \geq 1. \tag{2.24}$$

Let $\varphi \in C_0^{\infty}(\mathbb{R}^N, \mathbb{R}^2)$, $\varphi \not\equiv 0$, for $t > 0$ large enough, from (2.24) we have

$$\begin{aligned} \mathcal{I}(t\varphi) &= \frac{1}{2}t^2\|\varphi\|_H^2 - \int_{\mathbb{R}^N} F(x, t\varphi)\, dx \\ &\leqq \frac{1}{2}t^2\|\varphi\|_H^2 - t^{\mu}\overline{c}\int_{\mathbb{R}^N} |\varphi|^{\mu}\, dx. \end{aligned} \tag{2.25}$$

This and the condition $\mu > 2$ help us to conclude (ii). □

*Proof of Theorem 1.2.* It is clear that $\mathcal{I}(0) = 0$. Furthermore, the acceptable set

$$\mathbf{G} = \{\gamma \in C([\mathbf{0}, \mathbf{1}], H) : \gamma(0) = 0, \gamma(1) = \omega_0\},$$

where $w_0$ is given in Lemma 2.5, is not empty since clearly the function $\gamma(t) = t\omega_0 \in \mathbf{G}$. Besides, by Lemmas 2.2, 2.4 and 2.5, all assumptions of the Mountain pass theorem in [**8**] are satisfied. Therefore, there exists $\hat{w} \in H$ such that

$$0 < \alpha < \mathcal{I}(\hat{w}) = \inf\{\max \mathcal{I}(\gamma([0, 1])) : \gamma \in \mathbf{G}\}$$

and $\langle D\mathcal{I}(\hat{w}), \varphi\rangle = 0$ for all $\varphi \in C_0^{\infty}(\mathbb{R}^N, \mathbb{R}^2)$. Thus $\hat{w}$ is a weak solution of system (1.1). The solution $\hat{w}$ is not trivial since $\mathcal{I}(\hat{w}) \geq \alpha > 0$. Theorem 1.2 is completely proved. □

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