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Divergence of Fourier series: Corrigenda. II

Masako Izumi and Shin-ichi Izumi

The proof of divergence of Fourier series which was constructed in the authors' earlier paper [*Bull. Austral. Math. Soc.* 8 (1973), 289-304] is not complete. The object of this paper is to show that the unestimated terms do not disturb the divergence.

1. Introduction

Dr Y.M. Chen has kindly pointed out to us that the formula of $s_{m_j+N}(t;\ f_n)$ in [1], p. 291, is not complete. Besides P_1, P_2, and P_3, $s_{m_j+N}(t;\ f_n)$ contains the terms P_4 and P_5 which will be written up as follows: P_4 and P_5 are the sums of terms of order $\le m_j+N$ in

$$P_4' = \frac{1}{n}\sum_{i=j+1}^{n}\ \sum_{k=m_j+1}^{m_j+N}\left(1-\frac{k}{m_i+1}\right)\cos k\left(t-c_i\right)\cdot\left(s_N(t;\ h)-s_{N+m_j-k}(t;\ h)\right)$$

and

$$P_5' = \frac{1}{n}\sum_{i=j+1}^{n}\ \sum_{k=m_j+N+1}^{m_j+2N}\left(1-\frac{k}{m_i+1}\right)\cos k\left(t-c_i\right)s_N(t;\ h)\ ,$$

respectively.

We shall estimate P_4 and P_5 and prove that the proposition stated in Section 2 of [1] is still true.

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2. Estimation of P_4

Writing

$$s_N(t;\ h) = \sum_{l=0}^{N} \left(\alpha_l \cos lt + \beta_l \sin lt\right) ,$$

the sum of terms of order $\le m_j+N$ in the term

$$\cos k\left(t-c_i\right)\left(s_N(t;\ h)-s_{N+m_j-k}(t;\ h)\right)$$

of P_4', is

$$\frac{1}{2} \sum_{l=N+m_j-k+1}^{N} \left\{\alpha_l \cos\left((k-l)t-kc_i\right)-\beta_l \sin\left((k-l)t-kc_i\right)\right\}$$

$$= \frac{1}{2\pi} \sum_{l=N+m_j-k+1}^{N} \int_0^{2\pi} h(u)\left\{\cos lu \cos\left((k-l)t-kc_i\right)-\sin lu \sin\left((k-l)t-kc_i\right)\right\} du$$

$$= \frac{1}{2\pi} \sum_{l=N+m_j-k+1}^{N} \int_0^{2\pi} h(u)\cos\left(l(u-t)+k\left(t-c_i\right)\right) du .$$

Therefore

$$P_4 = \frac{1}{2\pi n} \sum_{i=j+1}^{n} \sum_{k=m_j+1}^{m_j+N} \left(1 - \frac{k}{m_i+1}\right) \sum_{l=N+m_j-k+1}^{N} \int_0^{2\pi} h(u)\cos\left(l(u-t)+k\left(t-c_i\right)\right) du$$

$$= \frac{1}{2\pi n} \sum_{i=j+1}^{n} \sum_{k=m_j+1}^{m_j+N} \left(1 - \frac{k}{m_i+1}\right) \int_0^{2\pi} \frac{h(u)}{2\sin(u-t)/2}$$

$$\cdot \left\{\sin\left((N+1/2)(u-t)+k\left(t-c_i\right)\right)-\sin\left(\left(N+m_j-k+1/2\right)(u-t)+k\left(t-c_i\right)\right)\right\} du$$

$$= P_{41} - P_{42} ,$$

where

$$P_{41} = \frac{1}{2\pi n} \sum_{i=j+1}^{n} \sum_{k=m_j+1}^{m_j+N} \int_0^{2\pi} \frac{h(u)}{2\sin(u-t)/2}$$

$$\cdot \{\sin((N+1/2)(u-t)+k(t-c_i)) - \sin((N+m_j-k+1/2)(u-t)+k(t-c_i))\} du$$

$$= \frac{1}{2\pi n} \sum_{i=j+1}^{n} \int_0^{2\pi} \frac{h(u)}{2\sin(u-t)/2} \cdot \Big\{ \Big[\cos((N+1/2)(u-t)+(m_j+1/2)(t-c_i))$$

$$-\cos((N+1/2)(u-t)+(m_j+N+1/2)(t-c_i))\Big] \frac{1}{2\sin(t-c_i)/2}$$

$$-\Big[\cos(N(u-t)+(m_j+1/2)(t-c_i))-\cos(m_j+N+1/2)(t-c_i)\Big] \frac{1}{2\sin(2t-u-c_i)/2}\Big\} du .$$

Let n_0 be the minimum of n and $[(2n+1)t/2]$ and I_i be the interval (c_k, c_{k+1}) containing the point $2t - c_i$, then

$$P_{41} = \frac{1}{2\pi n} \sum_{i=j+1}^{n_0} \int_{I_i} \frac{h(u)}{2\sin(u-t)/2} \cdot \Big\{ \Big[\cos((N+1/2)(u-t)+(m_j+1/2)(t-c_i))$$

$$-\cos((N+1/2)(u-t)+(m_j+N+1/2)(t-c_i))\Big] \frac{1}{2\sin(t-c_i)/2}$$

$$-\Big[\cos(N(u-t)+(m_j+1/2)(t-c_i)-\cos(m_j+N+1/2)(t-c_i))\Big] \frac{1}{2\sin(2t-u-c_i)/2}\Big\} du$$

$$+ \frac{1}{2\pi n} \sum_{i=j+1}^{n} \int_{c_j}^{c_{j+1}} \frac{h(u)}{2\sin(u-t)/2}$$

$$\cdot \{ [\cos((N+1/2)(u-t)+(m_j+1/2)(t-c_i))\sin(2t-u-c_i)/2$$

$$-\cos(N(u-t)+(m_j+1/2)(t-c_i))\sin(t-c_i)/2]$$

$$-[\cos((N+1/2)(u-t)+(m_j+N+1/2)(t-c_i))\sin(2t-u-c_i)/2$$

$$-\cos(m_j+N+1/2)(t-c_i)\sin(t-c_i)/2] \}$$

$$\cdot \frac{1}{2\sin(t-c_i)/2\sin(2t-u-c_i)/2} du + O(1)$$

$$= Q_{41} + Q_{42} + O(1) .$$

Since

$$\cos((N+1/2)(u-t)+(m_j+1/2)(t-c_i)) - \cos((N+1/2)(u-t)+(m_j+N+1/2)(t-c_i))$$
$$= 2\sin N(t-c_i)/2 \cdot \sin((N+1/2)(u-t)+(m_j+(N+1)/2)(t-c_i))$$

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and

$$\cos\left(N(u-t)+(m_j+1/2)(t-c_i)\right) - \cos(m_j+N+1/2)(t-c_i)$$
$$= \cos\left(N(u-(2t-c_i))+(m_j+N+1/2)(t-c_i)\right) - \cos(m_j+N+1/2)(t-c_i)$$
$$= \left(\cos N(u-(2t-c_i))-1\right)\cos(m_j+N+1/2)(t-c_i)$$
$$- \sin(m_j+N+1/2)(t-c_i)\sin N(u-(2t-c_i)) ,$$

we get

$$Q_{41} = \frac{1}{2\pi n}\sum_{i=j+1}^{n_0}\int_{I_i}\frac{h(u)}{2\sin(u-t)/2}$$
$$\cdot\left\{\frac{\cos N(u-(2t-c_i))-1}{2\sin(2t-u-c_i)/2}\cos(m_j+N+1/2)(t-c_i)\right.$$
$$\left.-\frac{\sin N(u-(2t-c_i))}{2\sin(2t-u-c_i)/2}\sin(m_j+N+1/2)(t-c_i)\right\}du + O(1)$$

$$= \frac{1}{2n}\sum_{i=j+1}^{n_0}\frac{1}{2\sin(t-c_i)/2}\cdot\{\cos(m_j+N+1/2)(t-c_i)\bar{s}_N(2t-c_i;\ h)$$
$$+\sin(m_j+N+1/2)(t-c_i)s_N(2t-c_i;\ h)\}du + O(1)$$

$$= \frac{1}{2n}\{\cos(m_j+N+1/2)t\cdot\bar{s}_N(2t-c_i;\ h)+\sin(m_j+N+1/2)t\cdot s_N(2t-c_i;\ h)\}$$
$$\cdot\sum_{i=j+1}^{n_0}\frac{1}{2\sin(t-c_i)/2} + O(1)$$

$$= \frac{1}{2n}\{\cos(m_j+N+1/2)t\cdot\bar{h}(2t-c_j)+\sin(m_j+N+1/2)t\}$$
$$\cdot\sum_{i=j+1}^{n_0}\frac{1}{2\sin(t-c_i)/2} + O(1) ,$$

since $h(2t-c_j) = 1$.

We shall now estimate Q_{42} . Since

$$\cos\left((N+1/2)(u-t)+(m_j+1/2)(t-c_i)\right)\sin(2t-u-c_i)/2$$
$$- \cos\left(N(u-t)+(m_j+1/2)(t-c_i)\right)\sin(t-c_i)/2$$
$$= - \cos\left((N+1/2)(u-t)+(m_j+1)(t-c_i)\right)\sin(u-t)/2$$

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and

$$\cos\big((N+1/2)(u-t)+(m_j+N+1/2)(t-c_i)\big)\sin(2t-u-c_i)/2$$
$$-\cos(m_j+N+1/2)(t-c_i)\sin(t-c_i)/2$$
$$=(\cos N(u-t)-1)\cos(m_j+N+1/2)(t-c_i)\sin(t-c_i)/2$$
$$-\sin(m_j+N+1/2)(t-c_i)\sin(N+1/2)(u-t)\cos(t-u)/2\sin(t-c_i)/2$$
$$-\cos(m_j+N+1/2)(t-c_i)\cos(N+1/2)(u-t)\sin(t-u)/2\cos(t-c_i)/2$$
$$+\sin(N+1/2)(u-t)\sin(u-t)/2\sin(m_j+N)(t-c_i)\ ,$$

we have

$$Q_{42}=\frac{-1}{2\pi n}\sum_{i=j+1}^{n}\int_{c_j}^{c_{j+1}}\frac{h(u)}{2\sin(2t-u-c_i)/2}\cdot\Big\{\cos(m_j+N+1/2)(t-c_i)\frac{\cos N(u-t)-1}{2\sin(u-t)/2}$$
$$-\sin(m_j+N+1/2)(t-c_i)\frac{\sin(N+1/2)(u-t)}{2\sin(u-t)/2}\Big\}du+O(1)$$
$$=\frac{1}{2n}\sum_{i=j+1}^{n}\frac{1}{2\sin(t-c_i)/2}$$
$$\cdot\{\cos(m_j+N+1/2)(t-c_i)\bar{s}_N(t;\ h)+\sin(m_j+N+1/2)(t-c_i)s_N(t;\ h)\}+O(1)$$
$$=\frac{1}{2n}\{\cos(m_j+N+1/2)t\cdot\bar{h}(t)+\sin(m_j+N+1/2)t\}\cdot\sum_{i=j+1}^{n}\frac{1}{2\sin(t-c_i)/2}+O(1)\ .$$

Therefore

$$P_{41}=\frac{1}{2n}\{\cos(m_j+N+1/2)t\cdot\bar{h}(2t-c_i)+\sin(m_j+N+1/2)t\}\cdot\sum_{i=j+1}^{n_0}\frac{1}{2\sin(t-c_i)/2}$$
$$+\frac{1}{2n}\{\cos(m_j+N+1/2)t\cdot\bar{h}(t)+\sin(m_j+N+1/2)t\}\cdot\sum_{i=j+1}^{n}\frac{1}{2\sin(t-c_i)/2}+O(1)\ .$$

By similar extimation, we can see that

$$P_{42}=O\big((\log\log n)^2\big)\ .$$

3. Estimation of P_5

The sum of terms of order $\le m_j+N$ in

$$\cos k(t-c_i)s_N(t;\ h)$$

of P_5' , is

$$\frac{1}{2}\sum_{l=k-m_j-N}^{N}\left\{\alpha_l\cos\left((k-l)t-kc_i\right)-\beta_l\sin\left((k-l)t-kc_i\right)\right\}$$

$$= \frac{1}{2\pi}\sum_{l=k-m_j-N}^{N}\int_0^{2\pi} h(u)\left\{\cos\left((k-l)t-kc_i\right)\cos lu-\sin\left((k-l)t-kc_i\right)\sin lu\right\}du$$

$$= \frac{1}{2\pi}\sum_{l=k-m_j-N}^{N}\int_0^{2\pi} h(u)\cos\left(l(u-t)+k\left(t-c_i\right)\right)du \ ;$$

and then

$$P_5 = \frac{1}{2\pi n}\sum_{i=j+1}^{n}\sum_{k=m_j+N+1}^{m_j+2N}\left(1-\frac{k}{m_i+1}\right)\cdot\sum_{l=k-m_j-N}^{N}\int_0^{2\pi} h(u)\cos\left(l(u-t)+k\left(t-c_i\right)\right)du$$

$$= \frac{1}{2\pi n}\sum_{i=j+1}^{n}\sum_{k=m_j+N+1}^{m_j+2N}\left(1-\frac{k}{m_i+1}\right)\int_0^{2\pi}\frac{h(u)}{2\sin(u-t)/2}$$

$$\cdot\left\{\sin\left((N+1/2)(u-t)+k\left(t-c_i\right)\right)-\sin\left(\left(k-m_j-N-1/2\right)(u-t)+k\left(t-c_i\right)\right)\right\}du$$

$$= P_{51} - P_{52} \ ,$$

where

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$$P_{51} = \frac{1}{2\pi n} \sum_{i=j+1}^{n} \sum_{k=m_j+N+1}^{m_j+2N} \int_0^{2\pi} \frac{h(u)}{2\sin(u-t)/2}$$

$$\cdot \{\sin((N+1/2)(u-t)+k(t-c_i)) - \sin((k-m_j-N-1/2)(u-t)+k(t-c_i))\}du$$

$$= \frac{1}{2\pi n} \sum_{i=j+1}^{n} \int_0^{2\pi} \frac{h(u)}{2\sin(u-t)/2}$$

$$\cdot \Big\{\Big[\cos((N+1/2)(u-t)+(m_j+N+1/2)(t-c_i))$$

$$-\cos((N+1/2)(u-t)+(m_j+2N+1/2)(t-c_i))\Big] \frac{1}{2\sin(t-c_i)/2}$$

$$-\Big[\cos(m_j+N+1/2)(t-c_i) - \cos(N(u-t)+(m_j+2N+1/2)(t-c_i))\Big] \frac{1}{2\sin(u-c_i)/2}\Big\}du$$

$$= \frac{1}{2\pi n} \sum_{i=j+1}^{n} \int_0^{2\pi} \frac{h(u)}{2\sin(u-t)/2}$$

$$\cdot \{[\cos(N+1/2)(u-t) - \cos(u-t)/2]\cos(m_j+N+1/2)(t-c_i)\sin(u-c_i)/2$$

$$-\sin(N+1/2)(u-t)\sin(m_j+N+1/2)(t-c_i)\sin(t-c_i)/2$$

$$-\cos(m_j+N+1/2)(t-c_i)\cos(u-c_i)/2\sin(t-u)/2$$

$$-\cos((N+1/2)(u-t)+(m_j+2N+1)(t-c_i))\sin(u-t)/2\}$$

$$\cdot \frac{du}{2\sin(t-c_i)/2 \cdot \sin(u-c_i)/2}$$

$$= \frac{1}{2n} \sum_{i=j+1}^{n} \{\cos(m_j+N+1/2)(t-c_i)\bar{s}_N(t;\, h)$$

$$-\sin(m_j+N+1/2)(t-c_i)s_N(t;\, h)\} \frac{1}{2\sin(t-c_i)/2} + O(1)$$

$$= \frac{1}{2n} \{\cos(m_j+N+1/2)t \cdot \bar{h}(t) - \sin(m_j+N+1/2)t\} \sum_{i=j+1}^{n} \frac{1}{2\sin(t-c_i)/2} + O(1) .$$

Similarly we can prove that

$$P_{52} = O((\log \log n)^2) .$$

4.

In [1], p. 294, the l-th Fourier coefficient of $J(v)$ must be $1/\pi l$ instead of $1/2\pi l$ (see [2]). Therefore the formula in [1], p. 300, must be changed as follows:

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$$P_1 + P_2 + P_3 = \frac{1}{2n}\left\{3\sin\left(m_j+N+1/2\right)t-\sin\left(m_j+1/2\right)t\right.$$

$$\left.-18\sin\left(m_j+N+1/2\right)t/\log\log n\right\}\sum_{i=j+1}^{n}\frac{1}{2\sin\left(t-c_i\right)/2}+O\left((\log\log n)^2\right) .$$

By the estimations in Sections 2 and 3,

$$P_4 + P_5 = \frac{1}{2n}\left\{\sin\left(m_j+N+1/2\right)t+\cos\left(m_j+N+1/2\right)t\cdot\bar{h}\left(2t-c_j\right)\right\}\cdot\sum_{i=j+1}^{n_0}\frac{1}{2\sin\left(t-c_i\right)/2}$$

$$+\frac{1}{n}\cos\left(m_j+N+1/2\right)t\cdot\bar{h}(t)\sum_{i=j+1}^{n}\frac{1}{2\sin\left(t-c_i\right)/2}+O\left((\log\log n)^2\right) .$$

Therefore, if we suppose $j > n/\log n$, then

$$s_{m_j+N}\left(t;\ f_n\right) = P_1+P_2+P_3+P_4+P_5$$

$$= \frac{1}{2n}\left[\left\{3\sin\left(m_j+N+1/2\right)t-\sin\left(m_j+1/2\right)t+2\cos\left(m_j+N+1/2\right)t\cdot\bar{h}(t)\right.\right.$$

$$\left.-18\sin\left(m_j+N+1/2\right)t/\log\log n\right\}\sum_{i=j+1}^{n}\frac{1}{2\sin\left(t-c_i\right)/2}$$

$$+\left\{\sin\left(m_j+N+1/2\right)t+\cos\left(m_j+N+1/2\right)t\cdot\bar{h}\left(2t-c_i\right)\right\}$$

$$\left.\cdot\sum_{i=j+1}^{n_0}\frac{1}{2\sin\left(t-c_i\right)/2}\right]+O\left((\log\log n)^2\right)$$

$$= \frac{1}{2n}\left\{4\sin\left(m_j+N+1/2\right)t-\sin\left(m_j+1/2\right)t\right.$$

$$+2\cos\left(m_j+N+1/2\right)t\cdot\bar{h}(t)+\cos\left(m_j+N+1/2\right)t\cdot\bar{h}\left(2t-c_j\right)$$

$$\left.-18\sin\left(m_j+N+1/2\right)t/\log\log n\right\}\cdot\sum_{i=j+1}^{n_0}\frac{1}{2\sin\left(t-c_i\right)/2}$$

$$+O\left((\log\log n)^2\right)$$

$$= \frac{1}{2n}K(t)\sum_{i=j+1}^{n_0}\frac{1}{2\sin\left(t-c_i\right)/2}+O\left((\log\log n)^2\right) .$$

Let E_{jn} be the set of t satisfying the conditions (1) and (2) in [1] and such that

$$|K(t)| > 1/\log\log n .$$

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If we put

$$E_n = \bigvee \left(E_{jn};\ n/\log n < j < n-\sqrt{n}\right) ,$$

then E_n and f_n satisfy the conditions of the proposition stated at the beginning of Section 2 of [1].

Section 3 of [1] remains unchanged.

References

[1] Masako Izumi and Shin-ichi Izumi, "Divergence of Fourier series", *Bull. Austral. Math. Soc.* 8 (1973), 289-304.

[2] Masako Izumi and Shin-ichi Izumi, "Divergence of Fourier series: Corrigenda", *Bull. Austral. Math. Soc.* 9 (1973), 319-320.

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.