BIPOWER VARIATION FOR GAUSSIAN PROCESSES WITH STATIONARY INCREMENTS

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Abstract

Convergence in probability and central limit laws of bipower variation for Gaussian processes with stationary increments and for integrals with respect to such processes are derived. The main tools of the proofs are some recent powerful techniques of Wiener/Itô/Malliavin calculus for establishing limit laws, due to Nualart, Peccati, and others.

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1. Introduction

The theory of bipower, and more generally multipower, variation has been developed out of questions raised in mathematical finance; for motivation, some first results, and applications, see [2], [3], [4], [5], [6], [8], [29], [30], [31], and [32]. It is natural, therefore, that initially the focus was on Brownian semimartingales, for which a rather complete and comprehensive theory is now available; cf. [10] (see also [9] and [21]). Extensions of the theory to Lévy processes and Itô semimartingales have been obtained, particularly by Jacod [18] (cf. also [5]), and applications to finance of such extensions are discussed in [20] and [28].

A further avenue of generalisation is to stochastic integrals with respect to Gaussian processes having stationary increments. Starting points in this direction are [7] and [14], which treated the power variation case, providing, in particular, a feasible central limit theorem for inference on the integrands in question. (As discussed in [7], an important early forerunner of that paper is a paper by Guyon and Leon [16], in which quadratic variation limit results for stationary Gaussian processes were derived.) The techniques used there, as well as in the present paper which considers the bipower case, come from very powerful recent results developed in the context of Wiener/Itô/Malliavin calculus, especially by Nualart, Peccati, and coauthors; see [24], [25], and [26] (cf. also [22]). (In fact, we believe that there are no other tools available that would allow derivation of the conclusions in the present paper.)

The structure of the paper is as follows. In Section 2 we list a number of background results needed in Appendix A for the proofs of the main results, which are presented in Sections 3 and 4. In Sections 3 and 4 we discuss limit laws of bipower variation for Gaussian processes with stationary increments and for integrals with respect to such processes, respectively. In Section 5 we conclude the paper. The proofs are given in Appendix A.

2. Background

In this section we review the basic concepts of the Wiener chaos expansion. In particular, we present a multiplication formula (Proposition 1) and a multivariate central limit theorem for a sequence of random variables which admit a chaos representation (Theorem 1). The latter is based on the theory for multiple stochastic integrals developed in [17], [24], and [26].

Consider a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a Gaussian subspace $\mathcal{H}_1$ of $L^2(\Omega, \mathcal{F}, \mathrm{P})$ whose elements are zero-mean Gaussian random variables. Let $\mathbb{H}$ be a separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $\| \cdot \|_{\mathbb{H}}$. We will assume that there is an isometry,

$$W : \mathbb{H} \to \mathcal{H}_1,$$
$$h \mapsto W(h),$$

in the sense that

$$\mathrm{E}[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathbb{H}}.$$

It is easy to see that this map has to be linear.

For any $m \geq 2$, we denote by $\mathcal{H}_m$ the mth Wiener chaos, that is, the closed subspace of $L^2(\Omega, \mathcal{F}, \mathrm{P})$ generated by the random variables $H_m(X)$, where $X \in \mathcal{H}_1$, $\mathrm{E}[X^2] = 1$, and H_m is the mth Hermite polynomial, i.e. $H_0(x) = 1$ and

$$H_m(x) = (-1)^m \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\exp\left(-\frac{x^2}{2}\right)\right).$$

Suppose that $\mathbb{H}$ is infinite-dimensional, and let $\{e_i, \ i \geq 1\}$ be an orthonormal basis of $\mathbb{H}$. Denote by Λ the set of all sequences $a = (a_1, a_2, \ldots)$, $a_i \in \mathbb{N}$, such that all the terms, except a finite number of them, vanish. For $a \in \Lambda$, we set $a! = \prod_{i=1}^{\infty} a_i!$ and $|a| = \sum_{i=1}^{\infty} a_i$. For any multi-index $a \in \Lambda$, we define

$$\Phi_a = \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

The family of random variables $\{\Phi_a, \ a \in \Lambda\}$ is an orthonormal system. In fact,

$$\mathrm{E}\left[\prod_{i=1}^{\infty} H_{a_i}(W(e_i)) \prod_{i=1}^{\infty} H_{b_i}(W(e_i))\right] = \delta_{ab} a!,$$

where δ_{ab} denotes the Kronecker symbol. Moreover, $\{\Phi_a \mid a \in \Lambda, \ |a| = m\}$ is a complete orthonormal system in $\mathcal{H}_m$.

Let $a \in \Lambda$ with $|a| = m$. The mapping

$$I_m : \mathbb{H}^{\odot m} \to \mathcal{H}_m,$$
$$\overset{\infty}{\underset{i=1}{\widetilde{\bigotimes}}} e_i^{\otimes a_i} \mapsto \prod_{i=1}^{\infty} H_{a_i}(W(e_i)),$$

between the symmetric tensor product $\mathbb{H}^{\odot m}$, equipped with the norm $\sqrt{m!}\,\|\cdot\|_{\mathbb{H}^{\otimes m}}$, and the mth chaos $\mathcal{H}_m$ is a linear isometry. Here '$\tilde{\otimes}$' denotes the symmetrization of the tensor product '$\otimes$' and I_0 is the identity in $\mathbb{R}$.

For any $h = h_1 \otimes \cdots \otimes h_m$ and $g = g_1 \otimes \cdots \otimes g_m \in \mathbb{H}^{\otimes m}$, we define the pth contraction of h and g, denoted by $h \otimes_p g$, as the element of $\mathbb{H}^{\otimes 2(m-p)}$ given by

$$h \otimes_p g = \langle h_1, g_1 \rangle_{\mathbb{H}} \cdots \langle h_p, g_p \rangle_{\mathbb{H}} h_{p+1} \otimes \cdots \otimes h_m \otimes g_{p+1} \otimes \cdots \otimes g_m.$$

This definition can be extended by linearity to any element of $\mathbb{H}^{\otimes m}$. The element $h \otimes_p g$ does not necessarily belong to $\mathbb{H}^{\odot(2m-p)}$, even if h and g belong to $\mathbb{H}^{\odot m}$. We denote by $h\tilde{\otimes}_p g$ the symmetrization of $h \otimes_p g$.

Proposition 1. *For any $h \in \mathbb{H}^{\otimes p}$ and $g \in \mathbb{H}^{\otimes q}$, we have*

$$I_p(h)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(h\tilde{\otimes}_r g). \tag{1}$$

Proof. First, note that

$$I_1(e_i) = W(e_i).$$

Let $a \in \Lambda$ with $\lfloor a \rfloor = p$ and $q = 1$. Owing to the linearity of I_p, it suffices to consider the case in which $h = \tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i}$ and $g = e_j$. It holds that

$$I_p\left(\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i}\right) I_1(e_j) = \prod_{i=1}^{\infty} H_{a_i}(W(e_i)) W(e_j).$$

Assume that j is an index such that $a_j = 0$. Then

$$\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i} \tilde{\otimes}_1 e_j = 0$$

and

$$\prod_{i=1}^{\infty} H_{a_i}(W(e_i)) W(e_j) = I_{p+1}\left(\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i} \tilde{\otimes} e_j\right);$$

so we have

$$I_p\left(\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i}\right) I_1(e_j) = I_{p+1}\left(\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i} \tilde{\otimes} e_j\right) + p I_{p-1}\left(\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i} \tilde{\otimes}_1 e_j\right).$$

Now assume that $a_j \neq 0$. Then we obtain the identity

$$\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i} \tilde{\otimes}_1 e_j = \frac{a_j}{p} \tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i'}$$

with $a_i' = a_i$ if $i \neq j$ and $a_j' = a_j - 1$. Furthermore,

$$\prod_{i=1}^{\infty} H_{a_i}(W(e_i)) W(e_j) = \prod_{i=1,\, i \neq j}^{\infty} H_{a_i}(W(e_i))(H_{a_j+1}(W(e_j)) + a_j H_{a_j-1}(W(e_j)))$$

$$= I_{p+1}\left(\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i} \tilde{\otimes} e_j\right) + p I_{p-1}\left(\tilde{\bigotimes}_{i=1}^{\infty} e_i^{\otimes a_i} \tilde{\otimes}_1 e_j\right),$$

since the Hermite polynomials verify that

$$H_{n+1}(x) = x H_n(x) - n H_{n-1}(x).$$

Hence, relationship (1) is true for $q = 1$. The general formula follows by induction through the lines of the proof of Proposition 1.1.3 of [23].

Remark 1. Note that if we take $h = e_i^{\otimes p}$ and $g = e_i^{\otimes q}$ then we obtain the well-known identity

$$H_p(W(e_i)) H_q(W(e_i)) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} H_{p+q-2r}(W(e_i)).$$

Now, let $\mathcal{G}$ be the σ-field generated by the random variables $\{W(h) \mid h \in \mathbb{H}\}$. Any square-integrable random variable $F \in L^2(\Omega, \mathcal{G}, P)$ has a unique chaos decomposition

$$F = \sum_{m=0}^{\infty} I_m(h_m),$$

where $h_m \in \mathbb{H}^{\odot m}$ (see [23, p. 13] for more details).

Finally, we present a multivariate central limit theorem for sequences of functionals $F_n \in L^2(\Omega, \mathcal{G}, P)$.

Theorem 1. *Consider a sequence of d-dimensional random vectors $F_n = (F_n^1, F_n^2, \ldots, F_n^d)$, such that $F_n^k \in L^2(\Omega, \mathcal{G}, P)$ and*

$$F_n^k = \sum_{m=0}^{\infty} I_m(h_{m,n}^k),$$

where $h_{m,n}^k \in \mathbb{H}^{\odot m}$. Assume that the following conditions hold.

(i) *For any $k = 1, \ldots, d$, we have*

$$\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{m=N+1}^{\infty} m! \, \|h_{m,n}^k\|_{\mathbb{H}^{\otimes m}}^2 = 0.$$

(ii) *For $k, l = 1, \ldots, d$, we have*

$$m! \lim_{n \to \infty} \|h_{m,n}^k\|_{\mathbb{H}^{\otimes m}}^2 = \Sigma_{kk}^m,$$

$$m! \lim_{n \to \infty} \langle h_{m,n}^k, h_{m,n}^l \rangle_{\mathbb{H}^{\otimes m}} = \Sigma_{kl}^m, \qquad k \neq l,$$

and $\sum_{m=1}^{\infty} \Sigma^m = \Sigma \in \mathbb{R}^{d \times d}$.

(iii) *For any $m \geq 1$, $k = 1, \ldots, d$, and $r = 1, \ldots, m - 1$,*

$$\lim_{n \to \infty} \|h_{m,n}^k \otimes_r h_{m,n}^k\|_{\mathbb{H}^{\otimes 2(m-r)}}^2 = 0.$$

Then we have

$$F_n - h_{0,n} \xrightarrow{D} N_d(0, \Sigma) \quad \text{as } n \to \infty, \tag{2}$$

where '$\xrightarrow{D}$' denotes convergence in distribution, and, for any natural number N and $k = 1, \ldots, d$,

$$\lim_{n \to \infty} E\left[\left(\sum_{m=1}^{N} I_m(h_{m,n}^k) \right)^4 \right] = 3 \left(\sum_{m=1}^{N} \Sigma_{kk}^m \right)^2. \tag{3}$$

Proof. Under conditions (ii) and (iii), the weak convergence (2) of the vector

$$(I_{m_1}(h^1_{m_1,n}), I_{m_2}(h^2_{m_2,n}), \ldots, I_{m_d}(h^d_{m_d,n}))$$

is shown in [26] (moreover, these authors proved that (2) implies (3)). Under the additional condition (i), this result can be extended to general multivariate sequences F_n with square-integrable components (see [7]).

3. Asymptotic theory for bipower variation of Gaussian processes with stationary increments

We consider a Gaussian process $(G_t)_{t \geq 0}$, defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathrm{P})$, with centred and stationary increments. The variance function R of the increments of G is defined as

$$R(t) = \mathrm{E}[|G_{s+t} - G_s|^2], \qquad t \geq 0. \tag{4}$$

In this section we study the asymptotic behaviour of the *bipower variation processes*

$$V(G; p, q)^n_t = \frac{1}{n\tau_n^{p+q}} \sum_{i=1}^{[nt]} |\Delta^n_i G|^p |\Delta^n_{i+1} G|^q, \qquad p, q \geq 0,$$

where $\Delta^n_i G = G_{i/n} - G_{(i-1)/n}$ and $\tau^2_n = R(1/n) = \mathrm{E}[|\Delta^n_i G|^2]$, using the multiplication formula (1) and the central limit theorem discussed in the previous section. For this purpose, we introduce the representation

$$|x|^p = \sum_{m=0}^{\infty} a_{p,m} H_m(x), \tag{5}$$

where the H_m are Hermite polynomials as defined in Section 2.

In order to give a statement about the asymptotic behaviour of the bipower variation process $V(G; p, q)^n_t$, we require the following assumptions on the variance function R defined in (4), which were introduced by Guyon and Leon [16].

(A1) $R(t) = t^\beta L_0(t)$ for some $\beta \in (0, 2)$ and some positive slowly varying (at 0) function L_0, which is continuous on $(0, \infty)$.

(A2) $R''(t) = t^{\beta-2} L_2(t)$ for some slowly varying function L_2, which is continuous on $(0, \infty)$.

(A3) There exists a $b \in (0, 1)$ with

$$K = \limsup_{x \to 0} \sup_{y \in [x, x^b]} \left| \frac{L_2(y)}{L_0(x)} \right| < \infty.$$

Recall that a function $L \colon (0, \infty) \to \mathbb{R}$ is called slowly varying at 0 when the identity

$$\lim_{x \searrow 0} \frac{L(tx)}{L(x)} = 1$$

holds for any fixed $t > 0$. Provided that L is continuous on $(0, \infty)$, we have

$$|L(x)| \leq Cx^{-\alpha}, \qquad x \in (0, T],$$

for any $\alpha > 0$ and any $T > 0$ (where the constant $C > 0$ depends on α and T).

Finally, we introduce the correlation function of the increments of G, i.e.

$$r_n(j) = \text{cov}\left(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\right), \qquad j \geq 0.$$

By the polarization identity, and due to the stationarity of the increments of G, we know that $r_n(0) = 1$ and

$$r_n(j) = \frac{R((j+1)/n) + R((j-1)/n) - 2R(j/n)}{2R(1/n)}, \qquad j \geq 1. \tag{6}$$

We start with the weak law of large numbers for the sequence $V(G; p, q)_t^n$. Throughout this paper, we write $Y^n \xrightarrow{\text{ucp}} Y$ when $\sup_{t \in [0,T]} |Y_t^n - Y_t| \xrightarrow{P} 0$ for any $T > 0$, where '$\xrightarrow{P}$' denotes convergence in probability.

Theorem 2. *Assume that conditions (A1)–(A3) are satisfied. Then we have*

$$\frac{V(G; p, q)_t^n}{\rho_{p,q}^{(n)}} \xrightarrow{\text{ucp}} t,$$

where the quantity $\rho_{p,q}^{(n)}$ is given by

$$\rho_{p,q}^{(n)} = \sum_{m=0}^{\infty} a_{p,m} a_{q,m} m! \, r_n^m(1). \tag{7}$$

Proof. See Appendix A.

Remark 2. Note that, by the orthogonality of Hermite polynomials and (5), the identity

$$\rho_{p,q}^{(n)} = \text{E}\left[\left|\frac{\Delta_i^n G}{\tau_n}\right|^p \left|\frac{\Delta_{i+1}^n G}{\tau_n}\right|^q\right]$$

holds. Moreover, since the function L_0 is slowly varying at 0, assumption (A1), (6), and (7) (and the dominated convergence theorem) imply that

$$\rho_{p,q} = \lim_{n \to \infty} \rho_{p,q}^{(n)} = \sum_{m=0}^{\infty} a_{p,m} a_{q,m} m! \, (2^{\beta-1} - 1)^m = \text{E}[|B_i^{\beta/2} - B_{i-1}^{\beta/2}|^p |B_{i+1}^{\beta/2} - B_i^{\beta/2}|^q], \tag{8}$$

where $B^{\beta/2}$ is the fractional Brownian motion with Hurst parameter $\beta/2$. Consequently, Theorem 2 yields the uniform convergence

$$V(G; p, q)_t^n \xrightarrow{\text{ucp}} \rho_{p,q} t.$$

Next, we present the weak limit of the properly normalized sequence $V(G; p, q)_t^n$. Note that the central limit theorem for bipower variation is valid under the same assumptions that are required to show the corresponding result for the power variation case (see [7]).

Theorem 3. *Assume that conditions (A1)–(A3) hold and that $0 < \beta < \frac{3}{2}$. Then we obtain the weak convergence (in the space $\mathfrak{D}([0, T])^2$ equipped with the local uniform topology)*

$$\left(G_t, \sqrt{n}\left(\frac{V(G; p, q)_t^n}{\rho_{p,q}^{(n)}} - t\right)\right) \xrightarrow{\text{w}} \left(G_t, \frac{\sigma_{p,q}}{\rho_{p,q}} W_t\right), \tag{9}$$

where W is a Brownian motion that is defined on an extension of the filtered probability space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ *and is independent of* $\mathcal{F}$, *and* $\sigma_{p,q}^2$ *is given by*

$$\sigma_{p,q}^2 = \lim_{n\to\infty} n \operatorname{var}(V(B^{\beta/2}; p, q)_1^n), \tag{10}$$

where $B^{\beta/2}$ *is the fractional Brownian motion with Hurst parameter* $\beta/2$.

Proof. See Appendix A.

Remark 3. In Theorem 3 the constant $\rho_{p,q}^{(n)}$ cannot be replaced by its limit $\rho_{p,q}$ defined in (8). This is due to the fact that the bias, $\sqrt{n}(\rho_{p,q}^{(n)} - \rho_{p,q})$, can, in general, converge to ∞.

Remark 4. The finiteness of $\sigma_{p,q}^2$ (for $0 < \beta < \frac{3}{2}$) is shown in (19) in Appendix A. Note that, owing to assumption (A1), the behaviour of the function R near 0 is similar to that of the fractional Brownian motion with Hurst parameter $\beta/2$. This is reflected in (10).

Remark 5. As for the power variation case (see [7]), we expect the (pointwise) limit of $\sqrt{n}(V(G; p, q)_t^n/\rho_{p,q}^{(n)} - t)$ to be an element of the second Wiener chaos if $\frac{3}{2} < \beta < 2$ (in particular, it is not normal). When $\beta = \frac{3}{2}$, both types of limits may appear. See [16] for a detailed discussion of different cases.

The proof of Theorem 3 relies on the methods developed in the previous section. In the first step we apply the multiplication formula (1) to obtain the chaos decomposition of the sequence $\sqrt{n}(V(G; p, q)_t^n/\rho_{p,q}^{(n)} - t)$. Then we show the convergence of finite-dimensional distributions of the sequence given in (9). Finally, we prove the tightness condition.

Note that the weak convergence in (9) is equivalent to the stable convergence (in $\mathcal{D}([0, T])$)

$$\sqrt{n}\left(\frac{V(G; p, q)_t^n}{\rho_{p,q}^{(n)}} - t\right) \xrightarrow{\mathcal{F}^G - st} \frac{\sigma_{p,q}}{\rho_{p,q}} W_t,$$

where $\mathcal{F}^G$ denotes the σ-algebra generated by process G (see [1], [19, p. 512], or [27] for more details on stable convergence) and $Y^n \xrightarrow{\mathcal{F}^G - st} Y$ (Y^n and Y are stochastic processes) means that Y^n converges $\mathcal{F}^G$-stably in law to Y. The latter result is crucial for proving a functional central limit theorem for the bipower variation of integral processes, which is presented below.

4. Extensions to integral processes

In this section we extend the limit theorems of the previous section to integral processes

$$Z_t = \int_0^t u_s \, \mathrm{d}G_s \tag{11}$$

defined on the same probability space as G, where the stochastic integral is the pathwise Riemann–Stieltjes integral. Assumption (A1) implies that G has finite r-variation for any $r > 2/\beta$ and, hence, by [34], the integral in (11) is well defined for any stochastic process u of finite q-variation with $q < 1/(1 - \beta/2)$.

Example 1. Let us present some concrete examples of the process defined in (11).

(i) Consider two Hurst parameters $H, H^* \in (0, 1)$ with $H + H^* > 1$. If $u_s = f(B_s^{H^*})$ for some locally Lipschitz function f then the integral

$$Z_t = \int_0^t u_s \, \mathrm{d}B_s^H$$

is well defined in the pathwise Riemann–Stieltjes sense (note that $\beta = 2H$).

(ii) In [13] the authors used a process of the form

$$w_t^{H^*} = \int_0^t \frac{(t-s)^{H^*-1/2}}{\Gamma(H^*+1/2)} \, dW_s, \qquad \tfrac{1}{2} < H^* < 1,$$

where W is a Brownian motion, to model the log-volatility of asset prices. This process is a truncated version of the fractional Brownian motion B^{H^*}. In fact, they considered the log-volatility, x, as the solution of the stochastic differential equation

$$dx_t = -kx_t \, dt + \sigma \, dw_t^{H^*}, \qquad t \geq 0,$$

with $x(0) = 0$ and $k > 0$. Then the asset price, Z, can be written as

$$Z_t = \int_0^t \exp(x_s) \, dW_s^*, \qquad t \geq 0,$$

where W^* is another Brownian motion, possibly correlated with W. In this example, $\beta = 1$ and the process $u_t = \exp(x_t)$ is $(H^* - \varepsilon)$-Hölder continuous for any $0 < \varepsilon < H^*$. So the process Z can be defined as a pathwise integral. These models try to capture the long-memory effect observed in real data on volatility.

In this example we can also replace W^* by a fractional Brownian motion B^H as long as $H + H^* > 1$. These kinds of processes appear when asset prices, not only the volatility, are modelled by diffusions driven by a fractional Brownian motion. See [15] for an alternative model, using a fractional Brownian motion, to the well-known Black–Scholes model. However, the use of the fractional Brownian motion for modelling prices in finance is controversial, because these processes are not semimartingales. Then, without any additional assumption on the trading operations, these models imply the possibility of arbitrage. An interesting discussion on this point can be found in [11].

Now we state the law of large numbers for the integral process which is valid under the same assumptions as in the power variation case.

Theorem 4. *Assume that conditions (A1)–(A3) hold. Suppose that $u = \{u_t, \, t \in [0, T]\}$ is a stochastic process with finite r-variation, where $r < 1/(1 - \beta/2)$. Set*

$$Z_t = \int_0^t u_s \, dG_s.$$

Then, for $p, q > 0$, we obtain

$$V(Z; p, q)_t^n \xrightarrow{\text{ucp}} \rho_{p,q} \int_0^t |u_s|^{p+q} \, ds \quad \text{as } n \to \infty.$$

Proof. See Appendix A.

Next we provide the weak limit theorem of the properly normalized bipower variation.

Theorem 5. *Assume that conditions (A1)–(A3) hold. Suppose that $u = \{u_t, \, t \in [0, T]\}$ is an $\mathcal{F}^G$-measurable stochastic process with finite r-variation, where $r < 1/(1 - \beta/2)$, and is Hölder continuous of the order a with $a > \max(1/(2(p \wedge 1)), 1/(2(q \wedge 1)))$. Then we obtain, for $Z_t = \int_0^t u_s \, dG_s$ and $p, q > 0$,*

$$\left(G_t, \sqrt{n} \left(\frac{V(Z; p, q)_t^n}{\rho_{p,q}^{(n)}} - \int_0^t |u_s|^{p+q} \, ds \right) \right) \xrightarrow{\text{w}} \left(G_t, \frac{\sigma_{p,q}}{\rho_{p,q}} \int_0^t |u_s|^{p+q} \, dW_s \right) \quad \text{as } n \to \infty,$$

where the convergence is in $\mathcal{D}([0, T])^2$ equipped with the local uniform topology, and W is a Brownian motion defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ and is independent of $\mathcal{F}$.

Proof. See Appendix A.

Remark 6. Theorem 3 holds for the processes introduced in Example 1 if the measurability condition is satisfied, i.e. $H + H^* > 1$ and $H^* > \max(1/(2(p \wedge 1)), 1/(2(q \wedge 1)))$.

Combining Theorems 4 and 5 we can derive a standard central limit theorem for the bipower variation.

Corollary 1. *Under the assumption of Theorem 5, it holds that*

$$\frac{\sqrt{n}(V(Z; p, q)_t^n/\rho_{p,q}^{(n)} - \int_0^t |u_s|^{p+q}\, \mathrm{d}s)}{\sqrt{(V(Z; 2p, 2q)_t^n/\rho_{2p,2q})(\sigma_{p,q}^2/\rho_{p,q}^2)}} \xrightarrow{\mathrm{D}} N(0, 1).$$

Remark 7. Finally, we demonstrate some applications of the limiting results presented in Theorems 4 and 5.

(i) In practice, the *realised ratio* statistic defined as

$$R_t^n = \frac{\mu_1^{-2} V(Z; 1, 1)_t^n}{V(Z; 2, 0)_t^n},$$

where $\mu_1 = \mathrm{E}\,|U|$, $U \sim N(0, 1)$, turns out to be very informative about the properties of the (unobserved) path of the process Z. When Z is an Itô semimartingale it is well known (see, e.g. [6]) that R_t^n converges in probability and $p - \lim_{n\to\infty} R_t^n \leq 1$. Hence, the statistic R_t^n can be used to test whether the underlying process Z is a semimartingale or not (we refer the reader to [33], where a similar type of statistic is used to solve the same test problem).

When $G = B^H$, $H \in (0, 1)$, R_t^n can be used to estimate the Hurst parameter H. By Theorem 4 we have

$$R_t^n = \frac{\mu_1^{-2} V(Z; 1, 1)_t^n}{V(Z; 2, 0)_t^n} \xrightarrow{\mathrm{ucp}} \mu_1^{-2}\,\mathrm{E}[|B_1^H||B_2^H - B_1^H|] = c_H \arcsin(c_H) + \sqrt{1 - c_H^2}$$

with $c_H = 2^{2H-1} - 1$. The right-hand side is bijective as a function in $H \in (0, 1)$. Consequently, a consistent estimator $\hat{H}$ of the Hurst parameter H can be constructed by using the above convergence.

(ii) Another useful property of the bipower variation is its robustness to certain jump processes. Consider a pure-jump semimartingale Y of the form

$$Y_t = Y_0 + B(h)_t + h * (\mu_t - \nu_t) + (x - h(x)) * \mu_t, \qquad t \geq 0,$$

where $B(h)$ is a predictable process of bounded variation, h is a truncation function that behaves like x at the origin, μ is a jump measure, and ν is its predictable compensator. Let γ be the Blumenthal–Getoor index of Y. When $\gamma < 1$, we additionally assume that $B(h) + (x - h(x)) * \nu = 0$. Then Theorem 4 is robust to Y if

$$\frac{2}{\beta} > \gamma \quad \text{and} \quad \frac{2}{\beta} > \max(p, q),$$

i.e. the statistics $V(Z; p, q)_t^n$ and $V(Z + Y; p, q)_t^n$ have the same limit in probability. This property can be deduced by the same methods as presented in [14]. Under the stronger conditions

$$\frac{1}{\beta} > \gamma \quad \text{and} \quad \frac{2}{\beta} > p + q > \frac{\gamma}{2 - \gamma\beta},$$

Theorem 5 is also robust to the presence of the process Y (see again [14] for more details).

5. Conclusion

In this paper we derived convergence in probability and stable central limit theorems for bipower variation of Gaussian processes with stationary increments and for associated integral processes. The corresponding asymptotic theory for multipower variation can be obtained similarly in a straightforward manner. Extensions of the results presented here to spatial and tempo-spatial settings would be of interest, as would simulation and empirical studies of how well the limit laws work in applications.

Appendix A

In the following we denote all constants which do not depend on n by C.

Let $\mathcal{H}_1$ be the first Wiener chaos associated with the triangular array $(\Delta_j^n G/\tau_n)_{n \geq 1, \, 1 \leq j \leq [nt]}$, i.e. the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables

$$(\Delta_j^n G/\tau_n)_{n \geq 1, \, 1 \leq j \leq [nt]}.$$

Note that $\mathcal{H}_1$ can be seen as a separable Hilbert space with a scalar product induced by the covariance function of the process $(\Delta_j^n G/\tau_n)_{n \geq 1, \, 1 \leq j \leq [nt]}$. This means that we can apply the theory of Section 2 to the canonical Hilbert space $\mathbb{H} = \mathcal{H}_1$. Denote by $\mathcal{H}_m$ the mth Wiener chaos associated with the triangular array $(\Delta_j^n G/\tau_n)_{n \geq 1, \, 1 \leq j \leq [nt]}$, and denote by I_m the corresponding linear isometry between the symmetric tensor product $\mathcal{H}_1^{\odot m}$ (equipped with the norm $\sqrt{m!} \, \| \cdot \|_{\mathcal{H}_1^{\otimes m}}$) and the mth Wiener chaos.

First, we present the chaos decomposition for the sequence $V(G; p, q)_t^n - \rho_{p,q}^{(n)} t$.

Lemma 1. *For any $t > 0$, we obtain the decomposition*

$$V(G; p, q)_t^n - \rho_{p,q}^{(n)} t = \sum_{m=2}^{\infty} I_m\left(\frac{1}{n} \sum_{i=1}^{[nt]} f_i^m \right) + O(n^{-1}), \tag{12}$$

where the kernels $f_i^m \in \mathcal{H}_1^{\odot m}$ are given by

$$f_i^m = \sum_{h=0}^{m} s_{h,m}^{(n)} \left(\frac{\Delta_i^n G}{\tau_n} \right)^{\otimes h} \tilde{\otimes} \left(\frac{\Delta_{i+1}^n G}{\tau_n} \right)^{\otimes m-h}$$

(for simplicity we suppress the dependency of f_i^m on n) with

$$s_{h,m}^{(n)} = \sum_{l=0}^{\infty} a_{p,l+h} a_{q,l+m-h} l! \binom{l+h}{l} \binom{l+m-h}{l} r_n^l(1).$$

Moreover, it holds that

$$
|s_{h,m}^{(n)}| \leq \sum_{l=0}^{\infty} |a_{p,l+h}| |a_{q,l+m-h}| l! \binom{l+h}{l} \binom{l+m-h}{l} |r_n^l(1)|
$$

$$
\leq C \binom{m}{h} \left(\frac{1}{|r_n(1)|(1-|r_n(1)|)} \right)^m, \tag{13}
$$

where the constant C does not depend on n, m, or h.

   Proof. Using the multiplication formula (1) and the linearity of the mapping I_m, we obtain the representation

$$
V(G; p, q)_t^n = \frac{1}{n} \sum_{i=1}^{[nt]} \sum_{m_1,m_2=0}^{\infty} a_{p,m_1} a_{q,m_2} \left(\sum_{l=0}^{m_1 \wedge m_2} l! \binom{m_1}{l} \binom{m_2}{l} \right.
$$

$$
\left. \times I_{m_1+m_2-2l} \left(\left(\frac{\Delta_i^n G}{\tau_n} \right)^{\otimes m_1} \otimes_l \left(\frac{\Delta_{i+1}^n G}{\tau_n} \right)^{\otimes m_2} \right) \right)
$$

$$
= \frac{1}{n} \sum_{i=1}^{[nt]} \sum_{m_1,m_2=0}^{\infty} a_{p,m_1} a_{q,m_2} \left(\sum_{l=0}^{m_1 \wedge m_2} l! \binom{m_1}{l} \binom{m_2}{l} r_n^l(1) \right.
$$

$$
\left. \times I_{m_1+m_2-2l} \left(\left(\frac{\Delta_i^n G}{\tau_n} \right)^{\otimes m_1-l} \otimes \left(\frac{\Delta_{i+1}^n G}{\tau_n} \right)^{\otimes m_2-l} \right) \right)
$$

$$
= \sum_{m=0}^{\infty} I_m \left(\frac{1}{n} \sum_{i=1}^{[nt]} f_i^m \right).
$$

Note that $a_{p,2m+1} = 0$ for all $m \geq 0$ and $p \geq 0$, because the H_{2m+1} are odd functions. This implies the identity

$$
V(G; p, q)_t^n - \rho_{p,q}^{(n)} t = \sum_{m=2}^{\infty} I_m \left(\frac{1}{n} \sum_{i=1}^{[nt]} f_i^m \right) + O(n^{-1}).
$$

Now, observe the identity

$$
\mathrm{var}\left(\left| \frac{\Delta_i^n G}{\tau_n} \right|^p \right) = \sum_{l=2}^{\infty} a_{p,l}^2 l! < \infty.
$$

From this we deduce that $a_{p,l}^2 \leq C/l!$ (for any fixed $p \geq 0$). Recall that $|r_n(1)| < 1$, since r_n is a correlation function of a process with stationary increments. Consequently, we obtain the inequality

$$
\sum_{l=0}^{\infty} |a_{p,l+h}| |a_{q,l+m-h}| l! \binom{l+h}{1} \binom{l+m-h}{1} |r_n^l(1)|
$$

$$
\leq \frac{C}{h!(m-h)!} \sum_{l=0}^{\infty} (l+1) \cdots (l+m) |r_n(1)|^l
$$

$$
= C \binom{m}{h} \left(\frac{1}{|r_n(1)|(1-|r_n(1)|)} \right)^m,
$$

which completes the proof.

Next, we present a lemma which has been shown in [7].

Lemma 2. *Suppose that conditions (A1)–(A3) hold. Let $\epsilon > 0$ with $\epsilon < 2 - \beta$. Define the sequence $r(j)$ by*

$$r(j) = (j-1)^{\beta+\epsilon-2}, \qquad j \geq 2, \tag{14}$$

and $r(0) = r(1) = 1$. Then we obtain the following assertions.

(i) *It holds that*

$$\frac{1}{n}\sum_{j=1}^{n} r^2(j) \to 0.$$

　If, moreover, $\beta + \epsilon - 2 < -\frac{1}{2}$, it holds that

$$\sum_{j=1}^{\infty} r^2(j) < \infty.$$

(ii) *For any $0 < \epsilon < 2 - \beta$ from (14), there exists a natural number $n_0(\epsilon)$ such that*

$$|r_n(j)| \leq Cr(j), \qquad j \geq 0,$$

　for all $n \geq n_0(\epsilon)$.

(iii) *Set $\rho(0) = 1$ and $\rho(j) = \frac{1}{2}((j-1)^\beta - 2j^\beta + (j+1)^\beta)$ for $j \geq 1$. Then it holds that*

$$r_n(j) \to \rho(j) \quad \text{for any } j \geq 0.$$

Now we introduce two independent variables, $X_i^n(1)$ and $X_i^n(2) \sim N(0, 1)$, that are given by

$$X_i^n(1) = \frac{\Delta_i^n G}{\tau_n}, \qquad X_i^n(2) = a_n \frac{\Delta_i^n G}{\tau_n} + b_n \frac{\Delta_{i+1}^n G}{\tau_n}$$

with $b_n = (1 - r_n^2(1))^{-1/2}$ and $a_n = -(1/r_n^2(1) - 1)^{-1/2}$. Note that a_n and b_n converge, because $r_n(1) \to \rho(1)$ (see Lemma 2(iii)), and, consequently, they are bounded. It is clear that f_i^m can be represented as

$$f_i^m = \sum_{k_l \in \{1,2\}} c_{k_1,\ldots,k_m}^n X_i^n(k_1) \otimes \cdots \otimes X_i^n(k_m)$$

for some constants $c_{k_1,\ldots,k_m}^n$. Note that all summands are orthogonal. We obtain

$$\|f_1^m\|_{\mathcal{H}_1^{\otimes m}}^2 = \sum_{k_l \in \{1,2\}} |c_{k_1,\ldots,k_m}^n|^2 =: c_m^n.$$

Let c_m, $c_{k_1,\ldots,k_m}$, and $\tilde{f}_i^m$ be analogues of c_m^n, $c_{k_1,\ldots,k_m}^n$, and f_i^m, respectively, which correspond to $G = B^{\beta/2}$, where $B^{\beta/2}$ denotes a fractional Brownian motion with Hurst parameter $\beta/2$ (note that c_m, $c_{k_1,\ldots,k_m}$, and $\langle \tilde{f}_i^m, \tilde{f}_j^m \rangle_{\mathcal{H}_1^{\otimes m}}$ do not depend on n, because $B^{\beta/2}$ is self-similar). Owing to (13) and the convergence $r_n(j) \to \rho(j)$, we deduce that

$$c_{k_1,\ldots,k_m}^n \to c_{k_1,\ldots,k_m}, \qquad c_m^n \to c_m, \quad \text{and} \quad \langle f_1^m, f_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}} \to \langle \tilde{f}_1^m, \tilde{f}_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}},$$

for all m and k. Furthermore, we obtain (since $r_n(1) \to \rho(1)$)

$$\text{var}\left(\left|\frac{\Delta_i^n G}{\tau_n}\right|^p \left|\frac{\Delta_{i+1}^n G}{\tau_n}\right|^q\right) = \sum_{m=2}^{\infty} c_m^n m! < \infty, \tag{15}$$

$$\text{var}\left(\left|\frac{\Delta_i^n G}{\tau_n}\right|^p \left|\frac{\Delta_{i+1}^n G}{\tau_n}\right|^q\right) \to \text{var}(|B_1^{\beta/2}|^p |B_2^{\beta/2} - B_1^{\beta/2}|^q) = \sum_{m=2}^{\infty} c_m m!. \tag{16}$$

Proof of Theorem 2. We first show the pointwise convergence $V(G; p, q)_t^n - \rho_{p,q}^{(n)} t \xrightarrow{\text{P}} 0$. Using expansion (12) and the stationarity of the increments of G, we obtain the identity

$$\text{var}(V(G; p, q)_t^n - \rho_{p,q}^{(n)} t) = \sum_{m=2}^{\infty} m! \left(\frac{[nt]}{n^2} \|f_1^m\|_{\mathcal{H}_1^{\otimes m}}^2 + \frac{2}{n^2} \sum_{k=1}^{[nt]-1} ([nt] - k) \langle f_1^m, f_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}}\right).$$

On the other hand, we have (since a_n and b_n are bounded, $|r_n(j)| \le Cr(j)$ by Lemma 2(ii) and $r(j)$ is decreasing)

$$|\langle f_1^m, f_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}}| = \sum_{h_l \in \{1,2\}, g_l \in \{1,2\}} c_{h_1,\dots,h_m}^n c_{g_1,\dots,g_m}^n \prod_{l=1}^m \langle X_i^n(h_l), X_{i+k}^n(g_l) \rangle_{\mathcal{H}_1}$$

$$\le c_m^n (Cr(k-1))^m. \tag{17}$$

Since $\text{E}[|X_i^n(1)|^2] = \text{E}[|X_i^n(2)|^2] = 1$, we also have

$$|\langle f_1^m, f_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}}| \le c_m^n.$$

Now, there exists a Q with $Cr(k-1) < 1$ for all $k \ge Q$. It holds that

$$\frac{1}{n} \sum_{k=1}^{[nt]-1} ([nt] - k) \langle f_1^m, f_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}} \le Q c_m^n + \frac{1}{n}\left| \sum_{k=Q}^{[nt]-1} ([nt] - k) \langle f_1^m, f_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}} \right|$$

$$\le c_m^n \left(Q + \sum_{k=Q}^{[nt]-1} (Cr(k-1))^2\right)$$

$$\le C c_m^n \sum_{k=1}^{[nt]-1} r^2(k). \tag{18}$$

Consequently, we deduce that

$$\text{var}(V(G; p, q)_t^n - \rho_{p,q}^{(n)} t) \le \frac{C}{n} \sum_{m=2}^{\infty} c_m^n m! \sum_{k=1}^{[nt]-1} r^2(k).$$

By (15), (16), and Lemma 2(i), we obtain the pointwise convergence

$$\frac{V(G; p, q)_t^n}{\rho_{p,q}^{(n)}} \xrightarrow{\text{P}} t.$$

The ucp convergence follows immediately, because $V(G; p, q)_t^n / \rho_{p,q}^{(n)}$ is increasing in t and the limit process $g(t) = t$ is continuous.

Proof of Theorem 3. We divide the proof of Theorem 3 into two steps. In the first step we prove the convergence of the finite-dimensional distribution of the sequence

$$\left(G_t, \sqrt{n}\left(\frac{V(G;p,q)_t^n}{\rho_{p,q}^{(n)}} - t \right) \right).$$

Then we prove the tightness of this sequence.

Step 1. Define the vector $Y_n = (Y_n^1, \ldots, Y_n^d)^\top$ by

$$Y_n^k = \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \left(\left| \frac{\Delta_i^n G}{\tau_n} \right|^p \left| \frac{\Delta_{i+1}^n G}{\tau_n} \right|^q - \rho_{p,q}^{(n)} \right),$$

where $(a_k, b_k]$, $k = 1, \ldots, d$, are disjoint intervals contained in $[0, T]$. Clearly, it suffices to prove that

$$(G_{b_k} - G_{a_k}, Y_n^k)_{1 \le k \le d} \xrightarrow{\mathrm{D}} (G_{b_k} - G_{a_k}, \sigma_{p,q}(W_{b_k} - W_{a_k}))_{1 \le k \le d},$$

where $\sigma_{p,q}$ is given by (10) (because $\rho_{p,q}^{(n)} \to \rho_{p,q}$, where $\rho_{p,q}$ is given in (8)).

By Lemma 1 we obtain the representation

$$Y_n^k = \sum_{m=2}^\infty I_m\left(\frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m \right) + O(n^{-1/2}).$$

Since

$$\mathrm{E}[(G_{b_k} - G_{a_k}) Y_n^l] = 0 \quad \text{for any } 1 \le k, l \le d,$$

it is sufficient to check the following conditions.

(i) For any $m \ge 1$ and $k = 1, \ldots, d$, the limit

$$\lim_{n \to \infty} m! \left\| \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m \right\|_{\mathcal{H}_1^{\otimes m}}^2 = \sigma_{p,q}^2(m, k)$$

exists, $\sum_{m=2}^\infty \sigma_{p,q}^2(m, k) = \sigma_{p,q}^2(b_k - a_k) < \infty$, and

$$\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{m=N+1}^\infty m! \left\| \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m \right\|_{\mathcal{H}_1^{\otimes m}}^2 = 0.$$

(ii) For any $m \ge 1$ and $k \ne h$,

$$\lim_{n \to \infty} \left\langle \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m, \frac{1}{\sqrt{n}} \sum_{i=[na_h]+1}^{[nb_h]} f_i^m \right\rangle_{\mathcal{H}_1^{\otimes m}} = 0.$$

(iii) For any $m \ge 1$, $k = 1, \ldots, d$, and $1 \le p \le m-1$, we have

$$\lim_{n \to \infty} \left\| \left(\frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m \right) \otimes_p \left(\frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m \right) \right\|_{\mathcal{H}_1^{\otimes 2(m-p)}} = 0.$$

Under conditions (i)–(iii), we then obtain (by Theorem 1) the central limit theorem

$$Y_n \xrightarrow{\mathrm{D}} N_d(0, \sigma_{p,q}^2 \operatorname{diag}(b_1 - a_1, \dots, b_d - a_d)),$$

where $\sigma_{p,q}^2$ is given by (10). Since the increments of the process G are stationary, we will prove part (i) and (iii) only for $k = 1$, $a_1 = 0$, and $b_1 = 1$.

(i) Recall (18):

$$m! \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i^m \right\|_{\mathcal{H}_1^{\otimes m}}^2 \leq Cm! \, c_m^n \sum_{k=1}^{n-1} r^2(k).$$

Since $c_m^n \to c_m$ and $\sum_{k=1}^{\infty} r^2(k) < \infty$ (by Lemma 2(ii) with $0 < \epsilon < \frac{3}{2} - \beta$ in the definition of r (see (14))), we deduce by the dominated convergence theorem that

$$\sigma_{p,q}^2(m, 1) = m! \left(\| \tilde{f}_1^m \|_{\mathcal{H}_1^{\otimes m}}^2 + 2 \sum_{k=1}^{\infty} \langle \tilde{f}_1^m, \tilde{f}_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}} \right),$$

where the kernels $\tilde{f}_i^m$ correspond to the case in which $G = B^{\beta/2}$. Clearly,

$$\lim_{n \to \infty} n \operatorname{var}(V(B^{\beta/2}; p, q)_1^n) = \sum_{m=2}^{\infty} \sigma_{p,q}^2(m, 1) = \sigma_{p,q}^2 < \infty, \tag{19}$$

because

$$\sum_{m=2}^{\infty} \sigma_{p,q}^2(m, 1) \leq C \sum_{m=2}^{\infty} c_m m! \sum_{k=1}^{\infty} r^2(k) < \infty$$

owing to Lemma 2(i) and (16). On the other hand, we have

$$\sum_{m=N+1}^{\infty} m! \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i^m \right\|_{\mathcal{H}_1^{\otimes m}}^2 \leq C \sum_{m=N+1}^{\infty} c_m^n m! \sum_{k=1}^{\infty} r^2(k) \tag{20}$$

and

$$\lim_{n \to \infty} \sum_{m=N+1}^{\infty} c_m^n m! = \sum_{m=N+1}^{\infty} c_m m!$$

for any fixed N by (16) and $c_m^n \to c_m$. Since $\sum_{m=2}^{\infty} c_m m! < \infty$, we obtain

$$\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{m=N+1}^{\infty} m! \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i^m \right\|_{\mathcal{H}_1^{\otimes m}}^2 = 0.$$

(ii) For any $1 \leq k, h \leq d$ with $b_k \leq a_h$, we have

$$\left\langle \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m, \frac{1}{\sqrt{n}} \sum_{i=[na_h]+1}^{[nb_h]} f_i^m \right\rangle_{\mathcal{H}_1^{\otimes m}} = \frac{m!}{n} \sum_{j=[na_k]+1}^{[nb_k]} \sum_{i=[na_h]+1}^{[nb_h]} \langle f_1^m, f_{1+i-j}^m \rangle_{\mathcal{H}_1^{\otimes m}}.$$

Assume, without loss of generality, that $a_k = 0$, $b_k = a_h = 1$, and $b_h = 2$ (the case in which $b_k < a_h$ is much easier). By Lemma 2(ii) with $0 < \epsilon < \frac{3}{2} - \beta$ in the definition of r (see (14)), we obtain the approximation (by (17))

$$\left| \left\langle \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m, \frac{1}{\sqrt{n}} \sum_{i=[na_h]+1}^{[nb_h]} f_i^m \right\rangle_{\mathcal{H}_1^{\otimes m}} \right| \leq C_m \left(\frac{1}{n} \sum_{j=1}^{n} j r^m(j) + \sum_{j=1}^{n-1} r^m(n+j) \right)$$

for some constant $C_m > 0$. It follows that $r^m(j) \leq (j-1)^{-1-\delta}$ for some $\delta > 0$ and all $m, j \geq 2$. Hence, we obtain

$$\left\langle \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f_i^m, \frac{1}{\sqrt{n}} \sum_{i=[na_h]+1}^{[nb_h]} f_i^m \right\rangle_{\mathcal{H}_1^{\otimes m}} \to 0 \quad \text{as } n \to \infty.$$

(iii) Fix $1 \leq p \leq m-1$. Now, by Lemma 2(ii) (with $0 < \epsilon < \frac{3}{2} - \beta$ in the definition of r) and (17), we deduce the inequality

$$\left\| \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i^m \right) \tilde{\otimes}_p \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i^m \right) \right\|_{\mathcal{H}_1^{\otimes 2(m-p)}}$$

$$\leq \frac{C_m}{n^2} \sum_{1 \leq j,l,h,k \leq n} r^p(|j-l|-1) r^p(|h-k|-1)$$

$$\times \left| \left\langle \left(\frac{\Delta_j^n G}{\tau_n} \right)^{\otimes(m-p)} \tilde{\otimes} \left(\frac{\Delta_l^n G}{\tau_n} \right)^{\otimes(m-p)}, \left(\frac{\Delta_h^n G}{\tau_n} \right)^{\otimes(m-p)} \tilde{\otimes} \left(\frac{\Delta_k^n G}{\tau_n} \right)^{\otimes(m-p)} \right\rangle_{\mathcal{H}_1^{\otimes 2(m-p)}} \right|,$$

where '$\sim$' denotes the symmetrization and $r(-1) := 1$. Again applying Lemma 2(ii) and (17), we see that it suffices to prove that

$$n^{-2} \sum_{1 \leq j,l,h,k \leq n} r^p(|j-l|-1) r^p(|h-k|-1) r^\alpha(|j-h|-1) r^{m-p-\alpha}(|l-h|-1) r^{m-p-\alpha}$$

$$\times (|j-k|-1) r^\alpha(|l-k|-1)$$

$$\to 0,$$

where $0 \leq \alpha \leq m-p$. The latter term is smaller than

$$n^{-1} \sum_{-1 \leq j,l,k \leq n-2} r^p(|j-l|) r^p(k) r^\alpha(j) r^{m-p-\alpha}(l) r^{m-p-\alpha}(|j-k|) r^\alpha(|l-k|).$$

Without any loss of generality, we can assume that $p = m - p = 1$ and $\alpha = 0$ or $\alpha = 1$. For $\alpha = 0$ and any $0 < \varepsilon < 1$, we obtain

$$n^{-1} \sum_{-1 \leq j \leq n-2} \left(\sum_{-1 \leq l \leq n-2} r(|j-l|) r(l) \right)^2$$

$$\leq n^{-1} \sum_{-1 \leq j \leq [n\varepsilon]} \left(\sum_{-1 \leq l \leq n-2} r(|j-l|) r(l) \right)^2$$

$$+ 2n^{-1} \sum_{[n\varepsilon] < j \leq n-2} \left(\sum_{-1 \leq l \leq [n\varepsilon/2]} r(|j-l|) r(l) \right)^2$$

$$+ 2n^{-1} \sum_{[n\varepsilon] < j \leq n-2} \left(\sum_{[n\varepsilon/2] < l \leq n-1} r(|j-l|) r(l) \right)^2$$

$$\leq 2\varepsilon \left(\sum_{-1 \leq l < n-2} r^2(l) \right)^2 + 6 \sum_{-1 \leq l < n-2} r^2(l) \sum_{[n\varepsilon/2] < l < \infty} r^2(l),$$

which converges to $2\varepsilon(\sum_{-1\le l<\infty} r^2(l))^2$ as $n\to\infty$ by Lemma 2(i). The desired result follows by letting ε tend to 0.

Step 2. Clearly, it suffices to show the tightness of the sequence $\sqrt{n}(V(G;p,q)^n_t - \rho^{(n)}_{p,q}t)$. Set

$$\sqrt{n}(V(G;p,q)^n_t - \rho^{(n)}_{p,q}t) = \sum_{m=2}^{\infty} I_m\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} f_i^m\right) + O(n^{-1/2}) =: Z_t^n + O(n^{-1/2})$$

(where the approximation holds locally uniformly in t) and

$$Z_t^{n,N} = \sum_{m=2}^{N} I_m\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} f_i^m\right).$$

In step 1 we have proved that conditions (i)–(iii) of Theorem 1 are satisfied. Then, by (3) and the Cauchy–Schwarz inequality, we obtain the approximation

$$
\begin{aligned}
\mathrm{P}(|Z_t^{n,N} - &Z_{t_1}^{n,N}| \ge \lambda,\ |Z_{t_2}^{n,N} - Z_t^{n,N}| \ge \lambda)\\
&\le \frac{\mathrm{E}^{1/2}[|Z_t^{n,N} - Z_{t_1}^{n,N}|^4]\,\mathrm{E}^{1/2}[|Z_{t_2}^{n,N} - Z_t^{n,N}|^4]}{\lambda^4}\\
&\le C\frac{\sigma_{p,q}^4([nt]-[nt_1])([nt_2]-[nt])}{\lambda^4}\\
&\le C\frac{\sigma_{p,q}^4(t_2-t_1)^2}{\lambda^4}
\end{aligned}
$$

for any $t_1 \le t \le t_2$ and $\lambda > 0$ (and the constant C does not depend on n). On the other hand, (15) and (20) imply that

$$\lim_{N\to\infty}\mathrm{E}[|Z_t^n - Z_t^{n,N}|^2] = 0$$

for any n and any t. Using this, we conclude that

$$\mathrm{P}(|Z_t^n - Z_{t_1}^n| \ge \lambda,\ |Z_{t_2}^n - Z_t^n| \ge \lambda) \le C\frac{\sigma_{p,q}^4(t_2-t_1)^2}{\lambda^4}$$

for any $t_1 \le t \le t_2$ and $\lambda > 0$, from which we deduce the tightness of the sequence Z_t^n by Theorem 15.6 of [12]. This completes the proof of Theorem 3.

Proof of Theorem 4. The assertion of Theorem 4 follows from Theorem 2 by similar methods as presented in [7] (see the proof of Theorem 2 therein).

Proof of Theorem 5. The assertion of Theorem 5 follows from Theorem 3 by similar methods as presented in [7] (see the proof of Theorem 6 therein).

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