THE PENTAGON AS A SUBSTRUCTURE LATTICE OF MODELS OF PEANO ARITHMETIC

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Abstract. Wilkie proved in 1977 that every countable model $\mathcal{M}$ of Peano Arithmetic has an elementary end extension $\mathcal{N}$ such that the interstructure lattice $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is the pentagon lattice $\mathbf{N}_5$. This theorem implies that every countable nonstandard $\mathcal{M}$ has an elementary cofinal extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$. It is proved here that whenever $\mathcal{M} \prec \mathcal{N} \models \mathsf{PA}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$, then $\mathcal{N}$ must be either an end or a cofinal extension of $\mathcal{M}$. In contrast, there are $\mathcal{M}^* \prec \mathcal{N}^* \models \mathsf{PA}^*$ such that $\mathrm{Lt}(\mathcal{N}^*/\mathcal{M}^*) \cong \mathbf{N}_5$ and $\mathcal{N}^*$ is neither an end nor a cofinal extension of $\mathcal{M}^*$.

Throughout, the (possibly adorned) script letters $\mathcal{M}, \mathcal{N}, \mathcal{K}$ denote models of Peano Arithmetic (PA) having universes denoted by the (similarly adorned) roman letters M, N, K, respectively. When we write $\mathcal{M} \prec \mathcal{N}$, we allow the possibility that $\mathcal{M} = \mathcal{N}$. As usual, we write $\mathcal{M} \prec_{\mathsf{end}} \mathcal{N}$ if $\mathcal{N}$ is an *end* elementary extension of $\mathcal{M}$ (that is, $a < b$ whenever $a \in M$ and $b \in N \backslash M$), and we write $\mathcal{M} \prec_{\mathsf{cf}} \mathcal{N}$ if $\mathcal{N}$ is a *cofinal* (necessarily elementary) extension of $\mathcal{M}$ (that is, for every $b \in N$ there is $a \in M$ such that $b < a$). If the elementary extension is neither end nor cofinal, then we say that it is *mixed* and write $\mathcal{M} \prec_{\mathsf{mix}} \mathcal{N}$.

For a model $\mathcal{N}$, its *substructure lattice* $\mathrm{Lt}(\mathcal{N})$ is the lattice of all those $\mathcal{K} \prec \mathcal{N}$ ordered by $\prec$. More generally, if $\mathcal{M} \prec \mathcal{N}$, then the *interstructure lattice* $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is the sublattice of $\mathrm{Lt}(\mathcal{N})$ consisting of those $\mathcal{K}$ in $\mathrm{Lt}(\mathcal{N})$ such that $\mathcal{M} \prec \mathcal{K}$. The question of which finite lattices can be substructure (or, equivalently, interstructure, by Corollary 1.2) lattices is discussed in [1, Chapter 4]. It is still unknown whether there are any finite lattices that are not substructure lattices; however, many lattices are known to be, among which are all the finite distributive lattices. In fact [1, Corollary 4.3.8], for any $\mathcal{M}$ and any finite distributive lattice D, there is $\mathcal{N} \succ_{\mathsf{end}} \mathcal{M}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong D$.

Recall that a lattice is distributive iff it embeds neither the pentagon lattice $\mathbf{N}_5$ nor the diamond lattice $\mathbf{M}_3$, both of which are depicted in Figure 1. A lattice is modular iff it does not embed $\mathbf{N}_5$.

Paris [2] gave, historically, the first example of a substructure lattice that is not distributive. The following theorem of Wilkie [4] gives the first example of a substructure lattice that is not modular.

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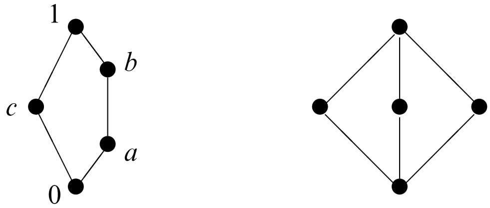

FIGURE 1. Lattices $\mathbf{N}_5$ and $\mathbf{M}_3$.

THEOREM 1. *For every countable* $\mathcal{M}$ *there is* $\mathcal{N} \succ_{\mathsf{end}} \mathcal{M}$ *such that* $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.

Incidentally, as proved in [1, Theorem 4.6.5], for every $\mathcal{M}_0$ there is $\mathcal{M} \succ_{\mathsf{end}} \mathcal{M}_0$ for which no $\mathcal{N} \succ_{\mathsf{end}} \mathcal{M}$ is such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$. Theorem 1 has the following corollary. (See Theorem 1.1 for the reason.)

COROLLARY 2. *For every countable and nonstandard* $\mathcal{M}$ *there is* $\mathcal{N} \succ_{\mathsf{cf}} \mathcal{M}$ *such that* $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.

It is still unresolved if, for every nonstandard $\mathcal{M}$, there is $\mathcal{N} \succ_{\mathsf{cf}} \mathcal{M}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$. A positive answer would immediately yield a positive answer to the question [1, Question 2, Chapter 12] if every uncountable model has a minimal cofinal extension.

Theorem 1 and Corollary 2 together suggest the question of whether the pentagon lattice can be realized by an elementary mixed extension. It was impetuously stated in [1, p. 123] that $\mathbf{N}_5$ does have such a representation. There was no published proof at that time, but there was an outline of a proof that later was seen to be flawed.[1] Mea culpa! In fact, there are no such extensions. That is the content of the next theorem, which is our first main result.

THEOREM 3. *If* $\mathcal{M} \prec_{\mathsf{mix}} \mathcal{N}$, *then* $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \ncong \mathbf{N}_5$.

Despite Theorem 3, there is a sliver of truth to the claim in [1], as will now be explained. Let $\mathcal{L}$ be one of the usual finite languages in which PA is formulated; to be definitive, let $\mathcal{L} = \{+, \times, \leq, 0, 1\}$. If $\mathcal{M}$ is a model and $X \subseteq M$, then $\mathcal{L}(X)$ is the language that, in addition to $\mathcal{L}$, has constant symbols denoting elements of X; in particular, $\mathcal{L} = \mathcal{L}(\varnothing)$. Let $\mathcal{L}^*$ be the language obtained from $\mathcal{L}$ by adjoining to it the denumerably many new and distinct unary relation symbols $U_0, U_1, U_2, \ldots$. Thus, $\mathcal{L}^* = \mathcal{L} \cup \{U_i : i < \omega\}$. Let PA^* be the $\mathcal{L}^*$-theory of those structures $\mathcal{M}^* = (\mathcal{M}, U_0, U_1, U_2, \ldots)$, where $\mathcal{M} \models \mathsf{PA}$ and $\mathcal{M}^*$ satisfies the induction scheme for all $\mathcal{L}^*(M)$-formulas, where $\mathcal{L}^*(M) = \mathcal{L}(M) \cup \mathcal{L}^*$. (From now on, $\mathcal{M}^*, \mathcal{N}^*, \ldots$ always denote models of PA^* that are expansions of $\mathcal{M}, \mathcal{N}, \ldots$, respectively.) We can think of PA^* as a subtheory of PA by identifying PA with $\mathsf{PA}^* \cup \{U_i = \varnothing : i < \omega\}$. Many concepts, such as interstructure lattices, that concern models of PA extend in an obvious and natural way to models of PA^*. Also, many results about models

[1] A referee has asked for a description of the flaw, which, unfortunately, has disappeared from my memory.

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of PA, together with their proofs, extend in a straightforward manner to models of PA^*. Almost all results in [1] do. Theorem 1 and Corollary 2 also do. In other words, Theorem 1* and Corollary 2* are valid, where we are adjoining * to indicate that PA^* rather than just PA is being considered. But Theorem 3 has the unusual feature that it does not. The next theorem, our second new result, indicates why.

THEOREM 4. *Every countable, recursively saturated $\mathcal{M}$ has an expansion $\mathcal{M}^*$ for which there is $\mathcal{N}^* \succ_{\mathsf{mix}} \mathcal{M}^*$ such that* $\mathrm{Lt}(\mathcal{N}^*/\mathcal{M}^*) \cong \mathbf{N}_5$.

As far as notation and terminology go, we generally follow what is standard or what can be found in [1].

There are four numbered sections following this introduction. Section 1 contains some preliminary material much of which is a rehash. Theorem 3 is proved in Section 2. Section 3 is almost purely combinatorial in nature and prepares the way for the proof of Theorem 4, which is then presented in Section 4.

§1. Ranked lattices and their representations. This section, comprising three subsections, culminates with a description of how to obtain elementary extensions realizing a given finite ranked lattice. Section 1.1 repeats some material from [1, Chapter 4]. Section 1.2 extends Section 1.1 and puts a new perspective on it. Finally, Section 1.3 extends Section 1.2 from lattices to ranked lattices.

1.1. Representations of lattices. For any set A, let $\mathrm{Eq}(A)$ be the lattice of equivalence relations on A, ordered in such a way that if $\Theta_1, \Theta_2 \in \mathrm{Eq}(A)$, then $\Theta_1 \leq \Theta_2$ iff $\Theta_1 \subseteq \Theta_2$ (that is, Θ_1 *refines* Θ_2). We let $\mathbb{0}_A$ be the *discrete* equivalence relation on A (that is, $\mathbb{0}_A$ is the equality relation on A) and $\mathbb{1}_A$ be the *trivial* equivalence relation (that is, $\mathbb{1}_A = A \times A$). Thus, for any $\Theta_1, \Theta_2 \in \mathrm{Eq}(A)$, we have that

$$\mathbb{0}_A \leq \Theta_1 \leq \mathbb{1}_A$$

and

$$\Theta_1 \wedge \Theta_2 = \Theta_1 \cap \Theta_2.$$

If $\Theta \in \mathrm{Eq}(A)$ and $B \subseteq A$, then $\Theta \cap B^2 \in \mathrm{Eq}(B)$. If f is a function with domain A, then f *induces* $\Theta \in \mathrm{Eq}(A)$ if whenever $a, b \in A$, then $\langle a, b\rangle \in \Theta$ iff $f(a) = f(b)$.

Let L be a finite lattice. A *representation* of L is a one-to-one function $\alpha : L \longrightarrow \mathrm{Eq}(A)$ such that

$$\alpha(0_L) = \mathbb{1}_A,$$
$$\alpha(1_L) = \mathbb{0}_A,$$

and

$$\alpha(r \vee s) = \alpha(r) \wedge \alpha(s)$$

for each $r, s \in L$. (It is not required that $\alpha(r \wedge s) = \alpha(r) \vee \alpha(s)$.) We say that α is *finite* if A is a finite set. If $B \subseteq A$, then $\alpha|B : L \longrightarrow \mathrm{Eq}(B)$ is such that $(\alpha|B)(r) = \alpha(r) \cap B^2$ for each $r \in L$. The representation $\beta : L \longrightarrow \mathrm{Eq}(B)$ is *isomorphic* to α (in symbols: $\alpha \cong \beta$) if there is a bijection $f : A \longrightarrow B$ such that for any $x, y \in A$ and $r \in L$, $\langle x, y\rangle \in \alpha(r)$ iff $\langle f(x), f(y)\rangle \in \beta(r)$. If such is the case, then we say that *f demonstrates* that $\alpha \cong \beta$. If $\alpha : L \longrightarrow \mathrm{Eq}(A)$ is a representation and $\Theta \in \mathrm{Eq}(B)$, then Θ is *canonical* (*for* α) if $B \subseteq A$ and $\Theta = \alpha(r) \cap B^2$ for some $r \in L$.

Suppose that $\alpha : L \longrightarrow \mathrm{Eq}(A)$ is a representation of the finite lattice L, and $\mathcal{B}$ is a set of representations of L. Then α *arrows* $\mathcal{B}$ (in symbols: $\alpha \longrightarrow \mathcal{B}$) if whenever $\Theta \in \mathrm{Eq}(A)$, then there is $B \subseteq A$ such that $\Theta \cap B^2$ is canonical for α and $\alpha|B \cong \beta$ for some $\beta \in \mathcal{B}$. We usually write $\alpha \longrightarrow \beta$ instead of $\alpha \longrightarrow \{\beta\}$.

We next define, by recursion on $n < \omega$, when the representation $\alpha : L \longrightarrow \mathrm{Eq}(A)$ of the finite lattice L has the *n-canonical partition property* (or, briefly, is *n-CPP*). First, α is 0-CPP if for every $r \in L$, there do not exist exactly 2 $\alpha(r)$-classes; next, α is $(n+1)$-CPP if there is a set $\mathcal{B}$ of n-CPP representations of L such that $\alpha \longrightarrow \mathcal{B}$.

Given $\mathcal{M}$, we say that a representation α of a finite lattice L is an *$\mathcal{M}$-representation* if it is $\mathcal{M}$-definable. Also, if $A \in \mathrm{Def}(\mathcal{M})$, then we let $\mathrm{Eq}^{\mathcal{M}}(A)$ be the set of those $\Theta \in \mathrm{Eq}(A)$ that are definable in $\mathcal{M}$. All the definitions in this subsection up to this point make sense when interpreted in a model $\mathcal{M}$ and are applied just to $\mathcal{M}$-representations. In particular, it makes sense to refer to an $\mathcal{M}$-finite $\mathcal{M}$-representation α as being a-CPP for $a \in M$. Thus, for every finite lattice L, there is a Σ_1 formula $cpp_L(x)$ such that for any $\mathcal{M}$ and $a \in M$, $\mathcal{M} \models cpp_L(a)$ iff there is an $\mathcal{M}$-finite $\mathcal{M}$-representation of L that $\mathcal{M}$ thinks is a-CPP. The following theorem can be found in [1, Chapter 4] or [3].

Theorem 1.1. *Let L be a finite lattice and $\mathcal{M}$ be a nonstandard, countable model. The following are equivalent*:

(1) *There are* $\mathcal{N}_0 \succ \mathcal{M}_0 \equiv \mathcal{M}$ *such that* $\mathrm{Lt}(\mathcal{N}_0/\mathcal{M}_0) \cong L$.
(2) *For every* $n < \omega$, $\mathcal{M} \models cpp_L(n)$.
(3) *There is* $\mathcal{N} \succ_{\mathsf{cf}} \mathcal{M}$ *such that* $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

Notice that Corollary 2 follows from Theorem 1 and (1) $\Longrightarrow$ (3) of the previous theorem.

Corollary 1.2. *If L is a finite lattice, then the following are equivalent*:

(1) *There is* $\mathcal{N}$ *such that* $\mathrm{Lt}(\mathcal{N}) \cong L$.
(2) *There are* $\mathcal{M} \prec \mathcal{N}$ *such that* $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

Proof. Obviously, (1) $\Longrightarrow$ (2) by letting $\mathcal{M}$ be the prime elementary submodel of $\mathcal{N}$. The converse (2) $\Longrightarrow$ (1) follows from Theorem 1.1 as long as $\mathcal{M}$ is not a model of True Arithmetic (TA) and so its prime elementary submodel is nonstandard. If $\mathcal{M}$ is a model of TA, then by (1) $\Longrightarrow$ (2) of Theorem 1.1, $\mathcal{M} \models cpp_L(n)$ for all $n < \omega$. Since TA is undecidable, there is a prime, nonstandard $\mathcal{M}_0$ such that $\mathcal{M}_0 \models cpp_L(n)$ for all $n < \omega$. Then by (1) $\Longrightarrow$ (2) of Theorem 1.1, there is $\mathcal{N}_0 \succ_{\mathsf{cf}} \mathcal{M}_0$ such that $\mathrm{Lt}(\mathcal{N}_0) = \mathrm{Lt}(\mathcal{N}_0/\mathcal{M}_0) \cong L$. ⊣

1.2. Correct sets of representations. This subsection consists of a definition followed by a theorem generalizing Theorem 1.1.

Definition 1.3. Let $\mathcal{M}$ be a model and L be a finite lattice. We say that $\mathcal{C}$ is an *$\mathcal{M}$-correct set of representations of L* if each of the following holds.

(1) $\mathcal{C}$ is a nonempty set of 0-CPP $\mathcal{M}$-representations of L.
(2) Whenever $\alpha : L \longrightarrow \mathrm{Eq}(A)$ is in $\mathcal{C}$ and $\Theta \in \mathrm{Eq}^{\mathcal{M}}(A)$, then there is a $B \subseteq A$ such that $\alpha|B \in \mathcal{C}$ and $\Theta \cap B^2$ is canonical for α.

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Here is an example. Suppose that $\mathcal{M}$ is nonstandard and that $\mathcal{M} \models cpp_L(n)$ for every $n < \omega$. Let $\mathcal{C}$ be the set of those $\mathcal{M}$-finite $\mathcal{M}$-representations α of L such that, for some nonstandard $n \in M$, $\mathcal{M}$ thinks that α is n-CPP. Then, $\mathcal{C}$ is $\mathcal{M}$-correct. With this example, we see that the following theorem generalizes a good portion of Theorem 1.1. It is a consequence of Theorem 1.1 when $\mathcal{M}$ is countable and nonstandard.

THEOREM 1.4. *Suppose that $\mathcal{M}$ is a model and L is a finite lattice.*

(1) *If there is $\mathcal{N} \succ \mathcal{M}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$, then there is an $\mathcal{M}$-correct set of representations of L.*
(2) *If $\mathcal{M}$ is countable and there is an $\mathcal{M}$-correct set of representations of L, then there is $\mathcal{N} \succ \mathcal{M}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.*

PROOF. (1) Suppose that $\mathcal{N} \succ \mathcal{M}$ and that $F : L \longrightarrow \mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is an isomorphism. Let $f : L \longrightarrow N$ be such that for $r \in L$, $f(r)$ generates $F(r)$ over $\mathcal{M}$. Let $a = f(1_L)$.

For each pair of elements $r, s \in L$, let $g_{r,s} : N \longrightarrow N$ and $h_{r,s} : N^2 \longrightarrow N$ be functions that are $\mathcal{N}$-definable using parameters only from M such that

- $g_{r,s}(f(r \vee s)) = f(r)$,
- $h_{r,s}(f(r), f(s)) = f(r \vee s)$.

The functions $g_{r,s}$ exist since $f(r) \in F(r \vee s)$; the functions $h_{r,s}$ exist for a similar reason. Let $g_r = g_{r,1}$, so that $g_r(a) = f(r)$. In particular, $g_1(a) = a$. The two equalities above become

- $g_{r,s}(g_{r\vee s}(a)) = g_r(a)$,
- $h_{r,s}(g_r(a), g_s(a)) = g_{r\vee s}(a)$.

For each $X \in \mathrm{Def}(\mathcal{M})$, let $\alpha_X : L \longrightarrow \mathrm{Eq}(X)$ be such that whenever $r \in L$, then $\alpha_X(r)$ is the equivalence relation in $\mathrm{Eq}(X)$ induced by $g_r \restriction X$. Let B be the set of all $x \in M$ such that

- $g_{r,s}(g_{r\vee s}(x)) = g_r(x)$,
- $h_{r,s}(g_r(x), g_s(x)) = g_{r\vee s}(x)$,
- $g_1(x) = x$.

Clearly, $B \in \mathrm{Def}(\mathcal{M})$ and $a \in B^{\mathcal{N}}$. We claim that α_B is an $\mathcal{M}$-representation of L. But even more is true. If $X \subseteq B$, $X \in \mathrm{Def}(\mathcal{M})$ and $a \in X^{\mathcal{N}}$, then $\alpha_X = \alpha_B | X$.

We now claim that each such α_X is an $\mathcal{M}$-representation of L.

First, α_X is one-to-one. For, suppose that $r, s \in L$ and $\alpha_X(r) = \alpha_X(s)$. Then, $g_r \restriction X$ and $g_s \restriction X$ induce the same equivalence relations on X. It follows that there are $\mathcal{M}$-definable functions $e_0, e_1 : M \longrightarrow M$ such that for all $x \in X$, $e_0(g_r(x)) = g_s(x)$ and $e_1(g_s(x)) = g_r(x)$. But then $e_0^{\mathcal{N}}(g_r(a)) = g_s(a)$ and $e_1^{\mathcal{N}}(g_s(a)) = g_r(a)$, implying that $F(r) = F(s)$ and, therefore, $r = s$.

Next, to prove that each α_X is a representation of L, it is enough to show that α_B is.

For all $x \in B$, $g_0(x) = f(0)$ and $g_1(x) = x$, so $\alpha_B(0)$ is trivial and $\alpha_B(1)$ is discrete. Finally, we show that if $r, s \in L$, then $\alpha_B(r \vee s) = \alpha_B(r) \wedge \alpha_B(s)$. To do so, we let $x, y \in X$, and then show that $\langle x, y\rangle \in \alpha_B(r \vee s) \Longleftrightarrow \langle x, y\rangle \in \alpha_B(r) \cap \alpha_B(s)$.

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$$\begin{aligned}\langle x, y\rangle \in \alpha_B(r \vee s) &\Rightarrow g_{r\vee s}(x) = g_{r\vee s}(y)\\ &\Rightarrow g_{r,s}(g_{r\vee s}(x)) = g_{r,s}(g_{r\vee s}(y))\\ &\Rightarrow g_r(x) = g_r(y)\\ &\Rightarrow \langle x, y\rangle \in \alpha_B(r).\end{aligned}$$

Similarly, $\langle x, y\rangle \in \alpha_B(r \vee s) \Rightarrow \langle x, y\rangle \in \alpha_B(s)$. Conversely,

$$\begin{aligned}\langle x, y\rangle \in \alpha_B(r) \cap \alpha_B(s) &\Rightarrow g_r(x) = g_r(y) \,\&\, g_s(x) = g_s(y)\\ &\Rightarrow h_{r,s}(g_r(x), g_s(x)) = h_{r,s}(g_r(y), g_s(y))\\ &\Rightarrow g_{r\vee s}(x) = g_{r\vee s}(y)\\ &\Rightarrow \langle x, y\rangle \in \alpha_B(r \vee s).\end{aligned}$$

Having that each α_X is a representation of L, we easily see that it is 0-CPP. For if X is partitioned into $Y, Z \in \mathrm{Def}(\mathcal{M})$, then either $a \in Y^{\mathcal{N}}$ or $a \in Z^{\mathcal{N}}$, but nodoubling both.

Now let $\mathcal{C}$ be the set of all such α_X; that is,

$$\mathcal{C} = \{\alpha_X : X \subseteq B, X \in \mathrm{Def}(\mathcal{M}), a \in X^{\mathcal{N}}\}.$$

We have just seen that $\mathcal{C}$ is a nonempty set of 0-CPP $\mathcal{M}$-representations of L, so that (1) of Definition 1.3 is verified. We prove (2) of Definition 1.3. Consider $\alpha_X \in \mathcal{C}$. Let $\Theta \in \mathrm{Eq}(X)$ be $\mathcal{M}$-definable. Define $m : X \longrightarrow X$ so that if $x \in X$, then $m(x) = \min([x]_\Theta)$. Let $r \in L$ be such that $m^{\mathcal{N}}(a)$ generates $F(r)$ over $\mathcal{M}$. There are functions $e_0, e_1 : N \longrightarrow N$ that are $\mathcal{N}$-definable but using parameters only from M such that $e_0(m^{\mathcal{N}}(a)) = g_r(a)$ and $e_1(g_r(a)) = m^{\mathcal{N}}(a)$. Let $Y = \{x \in X : e_0(m^{\mathcal{N}}(x)) = g_r(x)$ and $e_1(g_r(x)) = m^{\mathcal{N}}(x)\}$. Then $m\restriction Y$ induces $\alpha_Y(r)$. This completes the proof of (1).

(2) Since $\mathcal{C} \neq \varnothing$, let $\alpha : L \longrightarrow \mathrm{Eq}(A)$ be in $\mathcal{C}$. Let $\Theta_0, \Theta_1, \Theta_2, \dots$ enumerate all $\mathcal{M}$-definable equivalence relations on M. By recursion, obtain a sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ of sets in $\mathrm{Def}(\mathcal{M})$ as follows. Let $X_0 = A$. Suppose that we have X_n and that $\alpha|X_n \in \mathcal{C}$. Let $X_{n+1} \subseteq X_n$ be such that $\alpha|X_{n+1} \in \mathcal{C}$ and $\Theta_n \cap X_{n+1}^2$ is canonical for α. The X_n's generate a complete type over $\mathcal{M}$ (using that each $\alpha|X_n$ is 0-CPP). Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ generated by an element a realizing this type.

For each $r \in L$, let $t_r : M \longrightarrow M$ be $\mathcal{M}$-definable such that whenever $x \in X_0$, then $t_r(x) = \min([x]_{\alpha(r)})$. Define the function F on L so that if $r \in L$, then $F(r)$ is the elementary substructure of $\mathcal{N}$ generated by $t_r^{\mathcal{N}}(a)$ over $\mathcal{M}$. One easily checks that $F : L \longrightarrow \mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is an isomorphism. $\dashv$

1.3. Ranked lattices. To refine the notions of end/cofinal/mixed extensions, we appeal to rankings of lattices [1, Definition 4.2.6]. Suppose that L is a finite lattice. A function $\rho : L \longrightarrow L$ is a *ranking* of L if for each $r, s \in L$:

(1) $\rho(r) \geq r$,
(2) $\rho(\rho(r)) = \rho(r)$,
(3) $\rho(r) \leq \rho(s)$ or $\rho(s) \leq \rho(r)$,
(4) $\rho(r \vee s) = \rho(r) \vee \rho(s)$.

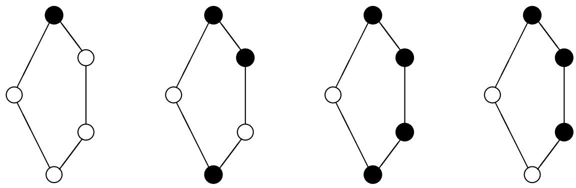

FIGURE 2. Four ranked pentagon lattices.

A ranking ρ of L uniquely determines, and is uniquely determined by, its *rankset* $\{\rho(r) : r \in L\}$. If L is finite and $R \subseteq L$, then R is a rankset iff R is linearly ordered and $1_L \in R$. If ρ is a ranking of L, then (L, ρ) is a *ranked lattice*.

If $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is finite, then let $\rho : \mathrm{Lt}(\mathcal{N}/\mathcal{M}) \longrightarrow \mathrm{Lt}(\mathcal{N}/\mathcal{M})$ be such that if $\mathcal{K} \in \mathrm{Lt}(\mathcal{N}/\mathcal{M})$, then $\rho(\mathcal{K})$ is uniquely defined by

$$\mathcal{K} \prec_{\mathsf{cf}} \rho(\mathcal{K}) \prec_{\mathsf{end}} \mathcal{N}.$$

One easily verifies that ρ is a ranking of $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$. We let $\mathrm{Ltr}(\mathcal{N}/\mathcal{M})$ be the ranked lattice $(\mathrm{Lt}(\mathcal{N}/\mathcal{M}), \rho)$.

Suppose that ρ is a ranking of the finite lattice L. Then ρ is an *end* ranking if $\rho(0_L) = 0_L$, a *cofinal* ranking if $\rho(0_L) = 1_L$ and a *mixed* ranking if $0_L < \rho(0_L) < 1_L$. Obviously, L has a unique cofinal ranking. If ρ is an end, cofinal, or mixed ranking, then (L, ρ) is, respectively, an *end*, *cofinal*, or *mixed* ranked lattice. These definitions are appropriate: if $\mathrm{Ltr}(\mathcal{N}/\mathcal{M})$ is an end, cofinal, or mixed ranked lattice, then $\mathcal{N}$ is, respectively, an end, cofinal, or mixed extension of $\mathcal{M}$.

Of the 10 rankings of $\mathbf{N}_5$, four are depicted in Figure 2 by letting $\bullet$ denote those points in the rankset and $\circ$ those that are not. Of all the ranked pentagons, the four in Figure 2 are the most important for us because of the following.

Henceforth, we use the labeling of $\mathbf{N}_5$ as given in Figure 1.

PROPOSITION 1.5. *If* $\mathcal{M} \prec \mathcal{N}$ *and* $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (\mathbf{N}_5, \rho)$, *then* $\rho = v_i$ *for some* $i \leq 3$.

PROOF. We first show that $\rho(c) = 1$. If $\rho(c) \neq 1$, then $\rho(c) = c$. We apply the Gaifman Condition [1, Proposition 4.2.12] by letting $x = a$, $y = b$, and $z = c$, to get the contradiction that $a = b$.

If $\rho(0) = 1$, then $\rho = v_0$. So, assume that $\rho(0) < 1$. Since $\rho(c) = 1$ and $c \wedge b = 0$, it follows from the Blass Condition [1, Proposition 4.2.7] that $\rho(b) = b$. Finally, $\rho(0) \neq b$ by [1, Theorem 4.6.1]. Thus, $\rho(0) \in \{0, a\}$, so it must be that $\rho \in \{v_1, v_2, v_3\}$. ⊣

We make some comments about this proposition. First, Proposition 1.5* is also valid. Theorem 1 can now be restated as: For all countable $\mathcal{M}$ there are $i \in \{1, 2\}$ and $\mathcal{N} \succ \mathcal{M}$ such that $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (\mathbf{N}_5, v_i)$. In fact, Wilkie's proof of Theorem 1 yields that $i = 1$. A similar proof shows that for every countable $\mathcal{M}$ there is $\mathcal{N} \succ_{\mathsf{end}} \mathcal{M}$

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such that $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (\mathbf{N}_5, v_2)$. Since v_3 is the only mixed ranking of the four in Figure 2, then in Theorem 4 we get $\mathcal{N}^*$ such that $\mathrm{Ltr}(\mathcal{N}^*/\mathcal{M}^*) \cong (\mathbf{N}_5, v_3)$.

The next order of business is to generalize Definition 1.3 and Theorem 1.4 from lattices to ranked lattices.

First, we need some terminology. Suppose that $\mathcal{M}$ is a model, $A \in \mathrm{Def}(\mathcal{M})$, and $\Theta \in \mathrm{Eq}^{\mathcal{M}}(A)$. We say that a set $\mathcal{E}$ of Θ-classes is *$\mathcal{M}$-bounded* if there is a bounded $I \in \mathrm{Def}(\mathcal{M})$ such that $I \cap X \neq \varnothing$ for each $X \in \mathcal{E}$.

If (L, ρ) is a finite ranked lattice, then a representation α of L is a *representation of* (L, ρ) if whenever $r \leq s \in L$, then $s \leq \rho(r)$ iff every $\alpha(r)$-class is the union of a finite set of $\alpha(s)$-classes. This definition should help motivate the next definition.

Definition 1.6. Let $\mathcal{M}$ be a model and (L, ρ) a finite ranked lattice.

(1) A representation $\alpha : L \longrightarrow \mathrm{Eq}(A)$ is an *$\mathcal{M}$-representation* of (L, ρ) if α is an $\mathcal{M}$-representation of L and whenever $r \leq s \in L$, then $s \leq \rho(r)$ iff every $\alpha(r)$-class is the union of an $\mathcal{M}$-bounded set of $\alpha(s)$-classes.

(2) We say that $\mathcal{C}$ is an *$\mathcal{M}$-correct set of representations of* (L, ρ) if $\mathcal{C}$ is an $\mathcal{M}$-correct set of representations of L and each $\alpha \in \mathcal{C}$ is an $\mathcal{M}$-representation of (L, ρ).

We next generalize Theorem 1.4 from lattices to ranked lattices.

Theorem 1.7. *Suppose that $\mathcal{M}$ is a model and (L, ρ) is a finite ranked lattice.*

(1) *If there is $\mathcal{N} \succ \mathcal{M}$ such that* $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (L, \rho)$, *then there is an $\mathcal{M}$-correct set of representations of (L, ρ).*

(2) *If $\mathcal{M}$ is countable and there is an $\mathcal{M}$-correct set of representations of (L, ρ), then there is $\mathcal{N} \succ \mathcal{M}$ such that* $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (L, \rho)$.

Proof. (1) Obtain $\mathcal{C}$ as in the proof of Theorem 1.4(1), so that $\mathcal{C}$ is an $\mathcal{M}$-correct set of representations of L. If $\alpha : L \longrightarrow \mathrm{Eq}(A)$ is in $\mathcal{C}$, $r \leq s$ but not $s \leq \rho(r)$, then there is some $\alpha(r)$-class that is not the union of an $\mathcal{M}$-bounded set of $\alpha(s)$-classes. (For, otherwise, there would be an $\mathcal{M}$-definable function $b : M \longrightarrow M$ such that $b(f(r)) \geq f(s)$.) However, it could be that $r \leq s \leq \rho(r)$ and some $\alpha(r)$-class is not the union of an $\mathcal{M}$-bounded set of $\alpha(s)$-classes. Let $\mathcal{C}_0$ be the set of those $\alpha \in \mathcal{C}$ that are $\mathcal{M}$-representations of (L, ρ). We will show that this $\mathcal{C}_0$ is an $\mathcal{M}$-correct set of representations of (L, ρ). To see this, it suffices to show that for each $\alpha : L \longrightarrow \mathrm{Eq}(A)$ in $\mathcal{C}$, there is $B \subseteq A$ such that $\alpha | B \in \mathcal{C}_0$.

Suppose that we have $\alpha : L \longrightarrow \mathrm{Eq}(A)$ in $\mathcal{C}$ and that $r \leq s \leq \rho(r)$. Partition A into two sets A_0, A_1, so that A_0 is the union of those $\alpha(r)$-classes that are the union of an $\mathcal{M}$-bounded set of $\alpha(s)$-classes. Since $\mathcal{C}$ is $\mathcal{M}$-correct, then either $\alpha | A_0 \in \mathcal{C}$ or $\alpha | A_1 \in \mathcal{C}$. By what was previously said, the latter option is impossible, so we have that $\alpha | A_0 \in \mathcal{C}$. Repeating this for all such $r, s \in L$, finally yields $B \subseteq A$ as required. This completes the proof of (1).

(2) Let $\mathcal{C}$ be an $\mathcal{M}$-correct set of representations of (L, ρ). Then $\mathcal{C}$ is an $\mathcal{M}$-correct set of representations of L, so we can obtain $\mathcal{N} \succ \mathcal{M}$ as in the proof of Theorem 1.4(2). Then $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

We use the notation from the proof of Theorem 1.4(2). Thus, $F : L \longrightarrow \mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is an isomorphism and $F(r)$ is generated by $t_r(a)$ over $\mathcal{M}$. We prove that F is also an isomorphism of the ranked lattices (L, ρ) and $\mathrm{Ltr}(\mathcal{N}/\mathcal{M})$. It suffices to prove: whenever $r < s \in L$, then $s \leq \rho(r)$ iff $F(r) \prec_{\mathsf{cf}} F(s)$. So, let $r < s \in L$.

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($\Longrightarrow$): Suppose that $s \leq \rho(r)$. Consider $\alpha \in \mathcal{C}$. Every $\alpha(r)$-class is the union of an $\mathcal{M}$-bounded set of $\alpha(s)$-classes. Let $g : M \longrightarrow M$ be an $\mathcal{M}$-definable function such that if $x \in X$, then $g(x) = \max\{t_s(y) : \langle x, y\rangle \in \alpha(r)\}$. Clearly, $g(x)$ is well defined for $x \in X$, so there is such an $\mathcal{M}$-definable g. Thus, $g(t_r(x)) \geq t_s(x)$ for all $x \in X$, so that $g^{\mathcal{N}}(t_r^{\mathcal{N}}(a)) \geq t_s^{\mathcal{N}}(a)$. Therefore, $F(r) \prec_{\mathsf{cf}} F(s)$.

($\Longleftarrow$): Suppose that $F(r) \prec_{\mathsf{cf}} F(s)$. Then there is an $\mathcal{M}$-definable $g : M \longrightarrow M$ such that $g^{\mathcal{N}}(t_r^{\mathcal{N}}(a)) \geq t_s^{\mathcal{N}}(a)$. There is X_i such that $g(t_r(x)) \geq t_s(x)$ for all $x \in X_i$. Let $\alpha_i = \alpha | X_i \in \mathcal{C}$. Thus, each $\alpha_i(r)$-class is the union of an $\mathcal{M}$-bounded set of $\alpha_i(s)$-classes. Then $s \leq \rho(r)$. ⊣

Wilkie's proof of Theorem 1 made implicit use of Theorem 1.7(2).

§2. Proving Theorem 3. This section is devoted to proving Theorem 3.

With the idea of obtaining a contradiction, assume that $\mathcal{M} \prec_{\mathsf{mix}} \mathcal{N}$ and that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$. Proposition 1.4 implies that $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (\mathbf{N}_5, \nu_3)$. Following Theorem 1.7(1), we let $\mathcal{C}$ be an $\mathcal{M}$-correct set of representations of $(\mathbf{N}_5, \nu_3)$. In the course of this proof, we will see that $\mathcal{C}$ must have certain properties. We will also see that there are other properties that $\mathcal{C}$ possibly could have, and we will then assume that $\mathcal{C}$ does have these properties.

Since $\mathcal{C} \neq \varnothing$, fix some $\alpha \in \mathcal{C}$. Thus, $\alpha : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(A)$. We can assume:

(C1) *For every $\beta \in \mathcal{C}$, there is $B \subseteq A$ such that $\beta = \alpha | B$.*

Since $a \vee c = 1$, then $\alpha(a) \cap \alpha(c) = \mathbb{0}_A$; therefore, whenever X is an $\alpha(a)$-class and Z is an $\alpha(c)$-class, then $|X \cap Z| \leq 1$. Since $0 < a = \nu_3(0)$, then, according to Definition 1.6, the set of $\alpha(a)$-classes is $\mathcal{M}$-bounded; let $n + 1 \in M$ be the number of $\alpha(a)$-classes according to $\mathcal{M}$. Then, we can assume:

(C2) $\alpha : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(A)$, where $A = [0, n] \times M$ and n is nonstandard, is such that if $\langle i, j\rangle, \langle i', j'\rangle \in A$, then

$$\big\langle \langle i, j\rangle, \langle i', j'\rangle \big\rangle \in \alpha(a) \text{ iff } i = i',$$

$$\big\langle \langle i, j\rangle, \langle i', j'\rangle \big\rangle \in \alpha(c) \text{ iff } j = j'.$$

At first, it may look as if we can only assume that $A \subseteq [0, n] \times M$. But it is always possible to enlarge the set A so as to get $[0, n] \times M$.

For just this proof, let us say that the $\mathcal{M}$-representation β of $\mathbf{N}_5$ is *rectangular* if $|X \cap Z| = 1$ for each $\beta(a)$-class X and $\beta(c)$-class Z. We see from (C2) that α is rectangular. We can even assume:

(C3) *Every $\beta \in \mathcal{C}$ is a rectangular representation.*

To see why, let $\mathcal{C}_0$ be the set of those rectangular $\mathcal{M}$-representations β, where $B \subseteq A$ and $\beta = \alpha | B$, for which there is $A_0 \subseteq B$ such that $\alpha | A_0 \in \mathcal{C}$. To prove that this $\mathcal{C}_0$ is an $\mathcal{M}$-correct set of representations of $(\mathbf{N}_5, \nu_3)$, it suffices to prove that if $A_1 \subseteq A$ and $\alpha | A_1 \in \mathcal{C}$, then there is $B \subseteq A_1$ such that $\alpha | B \in \mathcal{C}_0$. To prove this, consider some $\alpha_1 = \alpha | A_1 \in \mathcal{C}$. Define $\Theta \in \mathrm{Eq}(A_1)$ so that if $y, z \in A_1$, then $\langle y, z\rangle \in \Theta$ iff the following holds for each $\alpha_1(a)$-class X: there is $u \in X$ such that $\langle u, y\rangle \in \alpha_1(c)$ iff there is $v \in X$ such that $\langle v, z\rangle \in \alpha_1(c)$. Clearly, $\alpha_1(c) \subseteq \Theta \in \mathrm{Eq}(A_1)$. Since $\mathcal{C}$ is $\mathcal{M}$-correct, there are $A_0 \subseteq A_1$ and $r \in \{0, c\}$ such that $\alpha | A_0 \in \mathcal{C}$ and $\alpha_1(r) \cap A_0^2 = \Theta \cap A_0^2$. The number of Θ-classes is at most 2^{n+1}, so it must be that $r = 0$. Let

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B be the union of those $\alpha_1(c)$-classes that have a nonempty intersection with A_0. Then $A_0 \subseteq B \subseteq A_1$ and $\beta = \alpha|B \in \mathcal{C}_0$. This proves that $\mathcal{C}_0$ is an $\mathcal{M}$-correct set of rectangular representations of $(\mathbf{N}_5, v_3)$, so we can assume (C3).

Moreover, we can also assume:

(C4) *If* $I \subseteq I' \subseteq [0, n]$, $J \subseteq J' \subseteq M$, *and* $\alpha|(I \times J) \in \mathcal{C}$, *then* $\alpha|(I' \times J') \in \mathcal{C}$.

Working in $\mathcal{M}$, let $\langle B_k : k \in M\rangle$ be a one-to-one enumeration of all $\alpha(b)$-classes. Thus, we let $\psi(u, v)$ be an $\mathcal{L}(M)$-formula such that

$$\mathcal{M} \models \forall u, v[\psi(u, v) \leftrightarrow v \in B_u].$$

We also let $q : [0, n] \times M \longrightarrow M$ be such that if $\langle i, j\rangle \in A$, then $q(i, j) = k$, where $\langle i, j\rangle \in B_k$.

If $\beta : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(X)$ is in $\mathcal{C}$ and $p \in M$, then there is $X' \subseteq X$ such that $\beta|X' \in \mathcal{C}$ and if $B_k \cap X' \neq \varnothing$, then $k > p$. To see why, just consider $\Theta \in \mathrm{Eq}(X)$ such that $\bigcup\{B_k \cap X : k > p\}$ is a Θ-class and then apply Definitions 1.3 and 1.6.

For each $j \in M$, there is a (unique) permutation π_j of $[0, n]$ defined by the following condition: if $i, i' \leq n$, then $\pi_j(i) \leq \pi_j(i')$ iff $q(i, j) \leq q(I', j)$. Using these permutations, we define $\Psi \in \mathrm{Eq}(A)$ so that $\big\langle\langle i, j\rangle, \langle i', j'\rangle\big\rangle \in \Psi$ iff $\pi_j = \pi_{j'}$. Clearly, $\beta(c) \subseteq \Psi$ and the set of Ψ-classes is $\mathcal{M}$-bounded as $\mathcal{M}$ thinks that there are no more than $(n + 1)!$ Ψ-classes. Thus, there are $I \times J \subseteq [0, n] \times M = A$ and π such that $\alpha|(I \times J) \in \mathcal{C}$ and $\pi_j = \pi$ whenever $\langle i, j\rangle \in I \times J$. Without loss of generality, we assume that $J = M$ and that π is the identity permutation. Thus, we can assume:

(C5) $A = I \times J = [0, n] \times M$, and if $i, i' \leq n$ and $j \in M$, then

$$i \leq i' \text{ iff } q(i, j) \leq q(i', j).$$

We now have that $\mathcal{C}$ is an $\mathcal{M}$-correct set of representations of $(\mathbf{N}_5, v_3)$ satisfying (C1)–(C5).

With (C5) in mind, we make a couple of definitions concerning a $\beta \in \mathcal{C}$, where $\beta : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(I \times J)$. Suppose that X and Y are $\beta(b)$-classes. We say that X is *below* Y if there are $i, i' \in I$ and $j \in J$ such that $\langle i, j\rangle \in X$, $\langle i', j\rangle \in Y$, and $i < i'$. If X is below Y, then Y is *above* X. Thus, (C5) says: If B_k is below $B_{k'}$ (as $\alpha(b)$-classes), then $k < k'$. The following is a consequence of (C5):

(C6) For each $\alpha(b)$-class, the set of $\alpha(b)$-classes below it is $\mathcal{M}$-bounded.

We next show that it can be assumed that $\alpha : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(A)$ has a very specific form. Recall that if $d \in M$, then M^d is the set (of codes) of all definable sequences of length d. For a given $d \in M$, with $d > 0$, we will think of $M^d = M$. Also, $t \in M^d$ and $i < d$, then t_i is the i-th element of the sequence coded by t. If $t \in M^d$ and $e \leq d$, then $t{\restriction}e$ is the (code of) the sequence of length e whose i-th element is the same as the i-th element of t.

We can assume the following, which improves on (C2):

(C7) $\alpha : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(A)$, where $A = [0, n] \times M^n$, such that, in addition to (C2), we have if $\langle i, t\rangle, \langle i', t'\rangle \in A$, then

$$\big\langle\langle i, t\rangle, \langle i', t'\rangle\big\rangle \in \alpha(b) \text{ iff } i = i' \text{ and } t{\restriction}i = t'{\restriction}i'.$$

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Let α and A be, for now, as in (C2). Let G be a function on A such that if $\langle i, j\rangle \in A$, then

$$G(\langle i, j\rangle) = \{q(i', j) : i' \leq i\}.$$

Then G is definable in $\mathcal{M}$. Let $\Theta \in \mathrm{Eq}(A)$ be induced by G. We easily see that $\Theta \subseteq \alpha(b)$. Thus, there is a definable $X \subseteq A$ such that $X = I \times J$, $\alpha{\restriction}X \in \mathcal{C}$ and $\alpha(r) \cap X^2 = \Theta \cap X^2$ for some $r \in \mathbf{N}_5$, with $r \geq b$. From (C6) we get that $r = b$. We then see that if $j \in J$, $i, i' \in I$ and $i < i'$, then $B_{q(i,j)} \cap X \supseteq B_{q(i',j)}$. It then follows that we can also assume (C7).

A consequence of (C7) is that α and q are defined by Σ_0 $\mathcal{L}(\{n\})$-formulas.

We saw in (C3) that every $\beta \in \mathcal{C}$ is rectangular. We also can assume the following:

(C8) If $\beta = \alpha|(I \times J) \in \mathcal{C}$, then $\{\min(I)\} \times J$ is a $\beta(b)$-class.

To see why, suppose that $\beta = \alpha|(I \times J) \in \mathcal{C}$ and $\{\min(I)\} \times J$ is not a $\beta(b)$-class. Let $\Theta \in \mathrm{Eq}(I \times J)$ be such that and if $\langle i, j\rangle, \langle i', j'\rangle \in I \times J$, then $\big\langle\langle i, j\rangle, \langle i', j'\rangle\big\rangle \in \Theta$ iff $\big\langle\langle \min(I), j\rangle, \langle \min(I), j'\rangle\big\rangle \in \beta(b)$. Clearly, $\Theta \in \mathrm{Eq}^{\mathcal{M}}(I \times J)$. Let $X \subseteq I \times J$ be such that $\beta|X \in \mathcal{C}$ and for some $r \in \mathbf{N}_5$, $\Theta \cap X^2 = \beta(r) \cap X^2$. It must be that $r = 0$, so there is a $\beta(b)$-class B such that $X \subseteq B \times J$. Then, we can assume that $\beta \notin \mathcal{C}$ and that $\beta|(B \times J) \in \mathcal{C}$.

We will refer to the $\beta(b)$-class from (C8) as the *root* of β.

Corollary 2.1. *For each $\beta \in \mathcal{C}$, there is a $\beta(b)$-class X having an $\mathcal{M}$-unbounded set of $\beta(b)$-classes above it.*

Proof. In fact, the root is such $\beta(b)$-class. ⊣

Corollary 2.2. *If $\beta : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(X)$ is in $\mathcal{C}$ and $p \in M$, then there is $Y \subseteq X$ such that $\beta|Y \in \mathcal{C}$ and if $\langle i, j\rangle \in Y$, then $q(i, j) > p$.*

Proof. Consider $\Theta \in \mathrm{Eq}(X)$ such that for $\langle i, j\rangle, \langle i', j'\rangle \in X$, then $\big\langle\langle i, j\rangle, \langle i', j'\rangle\big\rangle \in \Theta$ iff $q(i, j) > p \iff q(i', j') > p$. Then apply Corollary 2.1. ⊣

Consider some $\beta : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(X)$ in $\mathcal{C}$, where $X = I \times J \subseteq [0, n] \times M$. For each $m < \omega$, we say that β is *m-thick* if whenever $f : M \longrightarrow M$ is definable, then there are $i_0, i_1, \dots, i_m \in I$ and $j \in J$ such that $f(q(i_k, j)) < q(i_{k+1}, j)$ for each $k < m$.

Lemma 2.3. *If $m < \omega$, then every $\beta \in \mathcal{C}$ is m-thick.*

Proof. Notice that every $\beta \in \mathcal{C}$ is vacuously 0-thick.

First, we show that every $\beta : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(B)$ in $\mathcal{C}$ is 1-thick. By Corollary 2.1, there is a $\beta(b)$-class $B \cap B_{k_0}$ having an $\mathcal{M}$-unbounded set of $\beta(b)$-classes above it. Thus, for any $\mathcal{M}$-definable $f : M \longrightarrow M$, there is a $\beta(b)$-class $B \cap B_{k_1}$ above $B \cap B_{k_0}$ such that $f(k_0) < k_1$. Let i_0, i_1, j be such that $\langle i_0, j\rangle \in B \cap B_{k_0}$ and $\langle i_1, j\rangle \in B \cap B_{k_1}$. Then $f(q(i_0, j)) = k_0 < k_1 = q(i_1, j)$.

Next, we assume that $1 < m < \omega$. We will prove that every $\beta \in \mathcal{C}$ is m-thick. Actually, we will prove something even stronger:

> If $\beta : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(B)$ is in $\mathcal{C}$ and $f : M \longrightarrow M$ is definable, then there is $I \times J \subseteq B$ such that $\beta|(I \times J) \in \mathcal{C}$ and whenever $i, i' \in I$, $j \in J$ and $i < i'$, then $f(q(i, j)) < q(i', j)$.

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To prove this, suppose that $\beta : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(B)$ is in $\mathcal{C}$ and $f : M \longrightarrow M$ is definable. We assume that $B = I_0 \times J_0$ and, without loss, that whenever $i < i' \in M$, then, then $\max(i', f(i)) < f(i')$.

We will obtain I and J in two steps.

In the first step, for each $j \in J_0$, let $R_j \subseteq I_0^2$ be such that if $\langle i, i' \rangle \in I_0^2$, then

$$\langle i, i' \rangle \in R_j \text{ iff } f(q(i, j)) < q(i', j).$$

Let $\Theta_0 \in \mathrm{Eq}(B)$ be such that if $i, i' \in I_0$ and $j, j' \in J_0$, then

$$\big\langle \langle i, j \rangle, \langle i', j' \rangle \big\rangle \in \Theta_0 \text{ iff } R_j = R_{j'}.$$

Obviously, $\beta(c) \subseteq \Theta_0 \in \mathrm{Eq}^{\mathcal{M}}(B)$ and the set of Θ_0-classes is $\mathcal{M}$-bounded. Thus, there is $B_1 = I_1 \times J_1 \subseteq B$ such that $\beta|B_1 \in \mathcal{C}$ and $\Theta_0 \cap B_1^2$ is trivial. Notice that if $j, j' \in J_1$, then $R_j \cap I_1^2 = R_{j'} \cap I_1^2$. For some (or all) $j \in J_1$, let $R = R_j \cap I_1^2$.

For the second step, define $\Theta_1 \in \mathrm{Eq}(B_1)$ so that if $i, i' \in I_1$ and $j, j' \in J_1$, then

$$\big\langle \langle i, j \rangle, \langle i', j' \rangle \big\rangle \in \Theta_1 \text{ iff } \{k \in I_1 : \langle i, k \rangle \in R\} = \{k \in I_1 : \langle i', k \rangle \in R\}.$$

Obviously, $(\beta|B_1)(a) \subseteq \Theta_1 \in \mathrm{Eq}^{\mathcal{M}}(B_1)$. Thus, there is $B_2 = I \times J \subseteq B_1$ such that $\beta|B_2 \in \mathcal{C}$ and $\Theta_1 \cap B_2^2 \in \{\beta(a) \cap B_2^2, \beta(0) \cap B_2^2\}$.

We show that $\Theta_1 \cap B_2^2 \neq \beta(0) \cap B_2^2$. Assume to the contrary that $\Theta_1 \cap B_2^2 = \beta(0) \cap B_2^2$. Then, whenever $i, i' \in I$ and $i < i'$, then $f(q(i, j)) \geq f(q(i', j))$. But then $\beta|((I \setminus \max(I)) \times J)$ is in $\mathcal{C}$ but is not 1-thick, which is a contradiction.

Therefore, $\Theta_1 \cap B_2^2 = \beta(0) \cap B_2^2$. It then follows that $I \times J$ has the desired property. $\dashv$

Lemma 2.3 implies a strengthening of itself via Corollary 2.2.

Corollary 2.4. *Suppose that $m < \omega$, $p \in M$, $\beta : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(X)$ in $\mathcal{C}$, where $X = I \times J$, and $f : M \longrightarrow M$ is definable. Then there are $i_0, i_1, \dots, i_m \in I$ and $j \in J$ such that $q(i_0, j) > p$ and $f(q(i_k, j)) < q(i_{k+1}, j)$ for each $k < m$.*

We need some more notation and terminology. Recall that a cut K (of $\mathcal{M}$) is a subset of M such that $0 \in K \neq M$ and that $x + 1 \in K$ whenever $x \leq y \in K$. If $m < \omega$, then the cut K is Σ_m-*closed* iff whenever $\varphi(x)$ is a Σ_m $\mathcal{L}(K)$-formula and $\mathcal{M} \models \exists x \varphi(x)$, then there is $d \in K$ such that $\mathcal{M} \models \varphi(d)$. If K is a Σ_0-closed cut and φ is an $\mathcal{L}(K)$-formula, then $\ulcorner \varphi \urcorner$, the Gödel number of φ, is in K.

Also, recall that $\alpha : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(A)$, $A = [0, n] \times M$ and $\langle B_k : k \in M \rangle$ is a definable, one-to-one enumeration of the $\alpha(b)$-classes (as defined a few lines after (C3)).

We work in $\mathcal{M}$. For each $k \leq n$, let Λ_k be the set of all prenex $\mathcal{L}(M)$-sentences σ having length at most n and having the form

$$\sigma = \mathsf{Q}_1 x_1 \mathsf{Q}_2 x_2 \cdots \mathsf{Q}_i x_i \cdots \mathsf{Q}_\ell x_\ell \varphi(\overline{x}), \tag{$*$}$$

where $\ell < k$, each Q_i is either $\exists$ or $\forall$ and $\varphi(\overline{x})$ is a Σ_0 formula. If $\ell < k \leq n$, then $\Lambda_\ell \subseteq \Lambda_k$. Let $\Lambda = \Lambda_n = \bigcup_{k \leq n} \Lambda_k$. If $\ell < n$, $t = \langle t_0, t_1, \dots, t_\ell \rangle \in [0, n]^{\ell+1}$, $j \in M$, and $\sigma \in \Lambda$ as in $(*)$, let

$$\sigma^{(t,j)} = \mathsf{Q}_1 x_1 \leq q(t_1, j) \mathsf{Q}_2 x_2 \leq q(t_2, j) \cdots \mathsf{Q}_\ell x_\ell \leq q(t_\ell, j) \varphi(\overline{x}). \tag{$**$}$$

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There is an $\mathcal{L}(\{n\})$-formula, call it $\varphi_0(x)$, such that for every standard $\sigma \in \Lambda$ and each t and j as in $(**)$,

$$\mathcal{M} \models \varphi_0(\ulcorner \sigma^{(t,j)} \urcorner) \leftrightarrow \sigma^{(t,j)}. \tag{1}$$

Let $g : M \longrightarrow M$ be an increasing, $\mathcal{L}(\{n\})$-definable function that is sufficiently fast-growing in the following sense: whenever $a \in M$, then there is a Σ_0-closed cut K such that $\{n, a\} \subseteq K \subseteq [0, g(a)]$. (There is such a g since n is nonstandard.) Notice that $g(0) > n^n$; in particular, $g(0) > n^k$ for every $k < \omega$.

Continue working in $\mathcal{M}$. Define $F : A \longrightarrow M$ so that if $\langle i, j \rangle \in A$, then $F(\langle i, j \rangle)$ is the set of all pairs $\langle t, \sigma \rangle$ such that for some $\ell < n$ and $t \in [0, n]^{\ell+1}$, $t_0 = i$, σ is an $\mathcal{L}([0, g(q(i, j)])$ in Λ (as in $(*)$) and that $\sigma^{(t,j)}$ is true. Let $\Theta \in \mathrm{Eq}(A)$ be induced by F. It is clear that if $\big\langle \langle i, j \rangle, \langle i', j' \rangle \big\rangle \notin \alpha(b)$, then $F(\langle i, j \rangle) \neq F(\langle i, j \rangle)$ so that $\big\langle \langle i, j \rangle, \langle i', j' \rangle \big\rangle \notin \Theta$. Thus, $\Theta \subseteq \alpha(b)$. On the other hand, if B_k is an $\alpha(b)$-class, then the set of possible $F(\langle i, j \rangle)$ for $\langle i, j \rangle \in B_k$ is $\mathcal{M}$-bounded. Thus, there is $B = I \times J \subseteq A$ such that $\alpha|B$ is in $\mathcal{C}$ and $\alpha(b) \cap B^2 = \Theta \cap B^2$. Let $\beta = \alpha|B$.

By Lemma 2.3, there are $j \in J$, a sufficiently large $\ell < \omega$, $t = \langle t_0, t_1, \dots, t_\ell \rangle \in I^{\ell+1}$ and a sufficiently fast-growing, definable $f : M \longrightarrow M$ such that $f(q(t_k, j)) < q(t_{k+1}, j)$. By "sufficiently," we mean that for every standard Σ_ℓ $\mathcal{L}([0, g(q(t_0, j))])$-sentence in Λ, we have that

$$\mathcal{M} \models \sigma \leftrightarrow \sigma^{(t,j)}. \tag{2}$$

Thus, it follows from (1) and (2) that for each such σ, that

$$\mathcal{M} \models \varphi_0(\ulcorner \sigma^{(t,j)} \urcorner) \leftrightarrow \sigma. \tag{3}$$

Now let K be the smallest Σ_0-closed cut such that $n, q(t_0, j) \in K$. Thus we have that $K \subseteq [0, g(q(t_0, j))]$. From (3), we get that there is a Σ_ℓ $\mathcal{L}(K)$-formula $\varphi(x)$ such that for every $\Sigma_{\ell-1}$ $\mathcal{L}(K)$-sentence σ,

$$\mathcal{M} \models \varphi(\ulcorner \sigma \urcorner) \leftrightarrow \sigma. \tag{4}$$

But the existence of $\varphi(x)$ in (4) contradicts the following version of Tarski's Theorem on the undefinability of truth, which is an immediate consequence of Gödel's Diagonalization Lemma.

Theorem 2.5. *Suppose that* $1 \le m < \omega$, $K \subseteq M$ *is a* Σ_0*-closed cut, and* $\varphi(u)$ *is an* $\mathcal{L}(K)$*-formula such that for each* Σ_m $\mathcal{L}(K)$*-sentence* σ, $\mathcal{M} \models \varphi(\ulcorner \sigma \urcorner) \leftrightarrow \sigma$. *Then* $\varphi(u)$ *is not a* Π_m *formula.*

This contradiction completes the proof of Theorem 3.

§3. Representations of $\mathbf{N}_5$. For almost all of this section, we ignore PA and concentrate just on representations of $\mathbf{N}_5$. Only in the first and last paragraphs is PA considered.

Caveat lector: In the next definition, and throughout this paper, ω^n is not an ordinal but is the set of n-tuples of natural numbers. If $s \in \omega^n$ and $i < n$, then s_i is the i-th element of s. Also, remember that if $n < \omega$, then $n = \{0, 1, \dots, n-1\}$. If $s \in \omega^n$ and $i < m \le n$, then $s{\restriction}m \in \omega^m$ and $(s{\restriction}m)_i = s_i$.

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DEFINITION 3.1. For $n < \omega$, let $A_n = (n+2) \times \omega^{n+1}$ and then define $\alpha_n : \mathbf{N}_5 \longrightarrow \mathrm{Eq}(A_n)$ so that $\alpha_n(0)$ is trivial, $\alpha_n(1)$ is discrete, and whenever $i, j \leq n+1$ and $s, t \in \omega^{n+1}$, then

- $\big\langle \langle i, s \rangle, \langle j, t \rangle \big\rangle \in \alpha_n(a)$ iff $i = j$;
- $\big\langle \langle i, s \rangle, \langle j, t \rangle \big\rangle \in \alpha_n(b)$ iff $i = j$ and $s \restriction i = t \restriction j$;
- $\big\langle \langle i, s \rangle, \langle j, t \rangle \big\rangle \in \alpha_n(c)$ iff $s = t$.

It is clear that each α_n is a representation of $\mathbf{N}_5$ and that, if $n \geq 1$, then α_n is 0-CPP. (The representation α_0 is not 0-CPP because there are exactly 2 $\alpha_0(a)$-classes.) If $n < \omega$, then $\alpha_n(b) \cap (\{0\} \times \omega^{n+1})^2$ is trivial whereas $\alpha_n(b) \cap (\{n+1\} \times \omega^{n+1})^2$ is discrete.

LEMMA 3.2. *Suppose that $m \leq n < \omega$ and that $I \subseteq n+2$ is such that $|I| = m+2$. Then there is $D \subseteq \omega^{n+1}$ such that $\alpha_m \cong \alpha_n|(I \times D)$.*

PROOF. Suppose that m, n, I are as given. Let $i_{m+1} = \max(I)$ and $I \backslash \{i_{m+1}\} = \{i_0, i_1, \dots, i_m\}$, where $i_0 < i_1 < \cdots < i_m$. We consider separately the two cases: $i_{m+1} < n+1$ and $i_{m+1} = n+1$.

First, suppose that $i_{m+1} < n+1$.

$$D = \{t \in \omega^{n+1} : t_i = 0 \text{ whenever } i = i_{m+1} \text{ or } i \in (n+1) \backslash I\}.$$

We show that $\alpha_m \cong \alpha_n|(I \times D)$. Let $h : (m+2) \times \omega^{m+1} \longrightarrow (n+2) \times \omega^{n+1}$ be such that if $\langle j, s \rangle \in (m+2) \times \omega^{m+1}$, then $h(\langle j, s \rangle) = \langle i_j, t \rangle$, where $t \in \omega^{n+1}$ and

$$t_k = \begin{cases} s_j, & \text{if } j \leq m \text{ and } i_j = k, \\ 0, & \text{otherwise,} \end{cases}$$

for all $k \leq n$. One easily verifies that $h : (m+2) \times \omega^{m+1} \longrightarrow I \times D$ is a bijection and that whenever $\langle j, s \rangle, \langle j', s' \rangle \in (m+2) \times \omega^{m+1}$ and $r \in \mathbf{N}_5$, then

$$\big\langle \langle j, s \rangle, \langle j', s' \rangle \big\rangle \in \alpha_m(r) \Longleftrightarrow \big\langle \langle h(\langle j, s \rangle), h(\langle j', s' \rangle) \big\rangle \in \alpha_n(r).$$

Thus, $\alpha_m \cong \alpha_n|(I \times D)$.

Next, suppose that $i_{m+1} = n+1$. In this case, let

$$D = \{t \in \omega^{n+1} : t_i = 0 \text{ whenever } i \in (n+1) \backslash I\}.$$

Showing that $\alpha_m \cong \alpha_n|(I \times D)$ is much like in the first case. ⊣

Our primary goal in this section is to prove the following theorem, which will be given a more precise formulation in Theorem 3.9.

THEOREM 3.3. *If $m < \omega$, then there is $n < \omega$ such that $\alpha_n \longrightarrow \alpha_m$.*

To prove this theorem, we will take a detour and visit some other lattices and their representations. These lattices are introduced in Definition 3.4 and their representations in Definition 3.5.

DEFINITION 3.4. Suppose that $1 \leq m \leq n < \omega$. Let $G_{m,n}$ be the set consisting of all pairs $\langle \theta, f \rangle$, where $\theta \in \mathrm{Eq}(n+1)$ and $f : n+1 \longrightarrow m+1$ are such that if $i, j \leq n$ and $\langle i, j \rangle \in \theta$, then $f(i) = f(j)$. Let $\trianglelefteq$ be the partial ordering of $G_{m,n}$ such that if $\langle \theta, f \rangle, \langle \psi, g \rangle \in G_{m,n}$, then

$$\langle \theta, f \rangle \trianglelefteq \langle \psi, g \rangle \text{ iff } \theta \supseteq \psi \text{ and } f(i) \leq g(i) \text{ for all } i \leq n.$$

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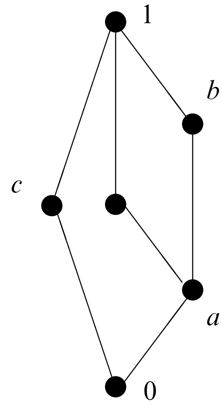

FIGURE 3. Embedding $\mathbf{N}_5$ into G_1.

Clearly, $\trianglelefteq$ really is a partial ordering. It should be observed that $G_{m,n}$ with $\trianglelefteq$, as in Definition 3.4, is a lattice in which

$$0_{G_{m,n}} = \langle \mathbb{1}_{n+1}, 0 \rangle,$$

$$1_{G_{m,n}} = \langle \mathbb{0}_{n+1}, m \rangle,$$

$$\langle \theta, f \rangle \vee \langle \psi, g \rangle = \langle \theta \cap \psi, \sup(f, g) \rangle,$$

where $\sup(f, g) = h$ iff $h(i) = \max(f(i), g(i))$ for all $i \leq n$. In the above equalities, we are identifying $k \leq m$ with the function that is constantly k on $n + 1$. We will continue to do so.

Our real concern is with the lattices $G_n = G_{n,n}$. The more general $G_{m,n}$ are introduced in order to be able to do an inductive proof. One of the reasons for introducing the lattices G_n is that there is an embedding $e_n : \mathbf{N}_5 \longrightarrow G_n$ defined by:

$$\begin{aligned} e_n(0) &= 0_{G_n}, \\ e_n(a) &= \langle \mathbb{0}_{n+1}, 0 \rangle, \\ e_n(b) &= \langle \mathbb{0}_{n+1}, \mathsf{id}_{n+1} \rangle, \\ e_n(c) &= \langle \mathbb{1}_{n+1}, n \rangle, \\ e_n(1) &= 1_{G_n}. \end{aligned}$$

As usual, id_X is the identity function on X.

It is routine to verify that each e_n is an embedding. Figure 3 depicts the lattice G_1 with $\mathbf{N}_5$ embedded in it. If $r \in \mathbf{N}_5$, then $e_1(r)$ is labeled with r. The unlabeled point is $\langle \mathbb{0}_2, 1 - \mathsf{id}_2 \rangle$, where $1 - \mathsf{id}_2$ is the function $f : 2 \longrightarrow 2$ such that $f(0) = 1$ and $f(1) = 0$.

Next, we define representations of the $G_{m,n}$.

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DEFINITION 3.5. Suppose that $1 \leq m \leq n < \omega$. Let $\gamma_{m,n} : G_{m,n} \longrightarrow \mathrm{Eq}\big((n+1) \times \omega^m\big)$ be such that if $\langle \theta, f \rangle \in G_{m,n}$ and $\langle i, s \rangle, \langle j, t \rangle \in (n+1) \times \omega^m$, then $\big\langle \langle i, s \rangle, \langle j, t \rangle \big\rangle \in \gamma_{m,n}(\langle \theta, f \rangle)$ iff $\langle i, j \rangle \in \theta$ and $s \restriction f(i) = t \restriction f(j)$.

Observe that $\gamma_{m,n}$ is indeed a representation of $G_{m,n}$. However, no $\gamma_{m,n}$ is 0-CPP since if θ has exactly 2 equivalence classes, then $\gamma_{m,n}(\langle \theta, 0 \rangle)$ has exactly 2 equivalence classes. In fact, if E is a θ-class, then $E \times \omega^m$ is a $\gamma_{m,n}(\langle \theta, 0 \rangle)$-class. Thus, the number of $\gamma_{m,n}(\langle \theta, 0 \rangle)$-classes is equal to the number of θ-classes. On the other hand, if $f : n+1 \longrightarrow m+1$ is not constantly 0, then there are infinitely many $\gamma_{m,n}(\langle \theta, f \rangle)$-classes; specifically, if $f(i) > 0$ and $s_0 \neq t_0$, where $s, t \in \omega^m$, then $\big\langle \langle i, s \rangle, \langle i, t \rangle \big\rangle \notin \gamma_{m,n}(\langle \theta, f \rangle)$.

Let $\gamma_n = \gamma_{n,n}$. Note that for $n < \omega$, both of the representations γ_{n+1} and α_n are into $\mathrm{Eq}\big((n+2) \times \omega^{n+1}\big)$. In fact, even more is true.

LEMMA 3.6. *If $n < \omega$, then $\alpha_n = \gamma_{n+1} \circ e_{n+1}$.*

PROOF. The routine proof is left to the reader. ⊣

We next come to the main result about the representations $\gamma_{m,n}$.

LEMMA 3.7. *If $1 \leq m \leq n < \omega$, then $\gamma_{m,n} \longrightarrow \gamma_{m,n}$.*

PROOF. What this lemma says is that if $1 \leq m \leq n < \omega$ and $\Theta \in \mathrm{Eq}\big((n+1) \times \omega^m\big)$, then there is $X \subseteq (n+1) \times \omega^m$ and $r \in G_{m,n}$ such that $\gamma_{m,n}|X \cong \gamma_{m,n}$ and $\Theta \cap X^2 = \gamma_{m,n}(r) \cap X^2$. Observe that if $X \subseteq (n+1) \times \omega^m$ and $\gamma_{m,n}|X \cong \gamma_{m,n}$, then there is $D \subseteq \omega^m$ such that $X = (n+1) \times D$. This follows from the fact that each $\gamma_{m,n}(\langle \mathbb{1}_{n+1}, m \rangle)$-class has the form $(n+1) \times \{s\}$ for some $s \in \omega^m$.

The proof of the lemma is by induction on $m \geq 1$ with $n \geq m$ being fixed. The basis step is for $m = 1$ and the inductive step for $m > 1$. Both steps start out the same way. So for now, consider $n \geq m \geq 1$ and let $\Theta \in \mathrm{Eq}\big((n+1) \times \omega^m\big)$.

Consider an arbitrary $s \in \omega^{m-1}$. (Of course, if $m = 1$, then $s = \varnothing$ is the only choice.) With the idea of invoking Infinite Ramsey's Theorem for pairs, we define $F_s : [\omega]^2 \longrightarrow \mathrm{Eq}\big(\{0,1\} \times (n+1)\big)$ so that whenever $\{k, \ell\} \in [\omega]^2$, $k < \ell, e, e' \in \{0,1\}$ and $i, j \leq n$, then $\big\langle \langle e, i \rangle, \langle e', j \rangle \big\rangle \in F_s(\{k, \ell\})$ iff there are $t, t' \in \omega^m$ such that $t_{m-1} = k$, $t'_{m-1} = \ell$, $t \restriction (m-1) = t' \restriction (m-1) = s$ and one of the following:

- $e = e' = 0$ and $\big\langle \langle i, t \rangle, \langle j, t \rangle \big\rangle \in \Theta$;
- $e = 0$, $e' = 1$ and $\big\langle \langle i, t \rangle, \langle j, t' \rangle \big\rangle \in \Theta$;
- $e = 1$, $e' = 0$ and $\big\langle \langle i, t' \rangle, \langle j, t \rangle \big\rangle \in \Theta$;
- $e = e' = 1$ and $\big\langle \langle i, t' \rangle, \langle j, t' \rangle \big\rangle \in \Theta$.

Now we apply Ramsey to get an infinite $H_s \subseteq \omega$ such that $F_s \restriction [H_s]^2$ is constant. Let

$$Y_s = \{t \in \omega^m : t \supseteq s \text{ and } t_{m-1} \in H_s\}.$$

Each of the following is true for each $s \in \omega^{m-1}$:

(1) If $i \leq n$, then $\Theta \cap (\{i\} \times Y_s)^2$ is either trivial or discrete.
(2) If $i, j \leq n$, $\Theta \cap (\{i\} \times Y_s)^2$ is trivial, $\Theta \cap (\{j\} \times Y_s)^2$ is discrete, and $t, t' \in Y_s$, then $\big\langle \langle i, t \rangle, \langle j, t' \rangle \big\rangle \notin \Theta$.

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(3) If $i < j \le n$ and both $\Theta \cap (\{i\} \times Y_s)^2$ and $\Theta \cap (\{j\} \times Y_s)^2$ are discrete, then one of the following:
(3a) if $t, t' \in Y_s$, then $\big\langle \langle i, t\rangle, \langle j, t'\rangle \big\rangle \notin \Theta$;
(3b) if $t, t' \in Y_s$, then $\big\langle \langle i, t\rangle, \langle j, t'\rangle \big\rangle \in \Theta$ iff $t = t'$.

A consequence of (1)–(3) is:

(4) If $i, j \le n$ and $t, t' \in Y_s$, then

$$\big\langle \langle i, t\rangle, \langle j, t\rangle \big\rangle \in \Theta \Longleftrightarrow \big\langle \langle i, t'\rangle, \langle j, t'\rangle \big\rangle \in \Theta.$$

Let $T_s = \{i \le n : \Theta \cap (\{i\} \times Y_s)^2 \text{ is trivial}\}$. Because of (1), $(n+1)\backslash T_s = \{i \le n : \Theta \cap (\{i\} \times Y_s)^2 \text{ is discrete}\}$. With (4) in mind, we can let $\theta_s \in \mathrm{Eq}(n+1)$ be such that

$$\theta_s = \{\langle i, j\rangle \in (n+1) \times (n+1) : \big\langle \langle i, t\rangle, \langle j, t\rangle \big\rangle \in \Theta\}$$

for each $t \in Y_s$. Clearly, T_s is the (possibly empty) union of some θ_s-classes.

Let

$$D_0 = \bigcup \{ Y_s : s \in \omega^{m-1}\}$$

and

$$X_0 = (n+1) \times D_0.$$

It is readily seen that $\gamma_{m,n} | X_0 \cong \gamma_{m,n}$.

Basis step $m = 1$: Since $m = 1$, it must be that $s = \varnothing$. Thus, Y_s is an infinite subset of ω^1 and $X_0 = (n+1) \times Y_\varnothing$. We have already noted that $\gamma_{1,n} | X_0 \cong \gamma_{1,n}$. To complete this step, we need to show that there is $r_0 \in G_{1,n}$ such that $\Theta \cap X_0^2 = \gamma_{1,n}(r_0) \cap X_0^2$.

Let $f : n+1 \longrightarrow 2$ be such that $f(i) = 0$ iff $i \in T_\varnothing$. Then we can take $r_0 = \langle \theta_\varnothing, f\rangle$. One easily verifies that $\Theta \cap X_0^2 = \gamma_{1,n}(r_0) \cap X_0^2$.

Inductive step $m > 1$: Thus, we are assuming $\gamma_{m-1,n} \longrightarrow \gamma_{m-1,n}$. We already have infinite Y_s for each $s \in \omega^{m-1}$ and that (1)–(4) hold. Also, we have $D_0 \subseteq \omega^m$ and $X_0 = (n+1) \times D_0$ and that $\gamma_{m,n} | X_0 \cong \gamma_{m,n}$. Without loss of generality, we assume, for each $s \in \omega^{m-1}$, that $H_s = \omega$ and then $Y_s = \{t \in \omega^m : t \supseteq s\}$. Thus, $D_0 = \omega^m$ and $X_0 = (n+1) \times \omega^m$.

Let $\Theta_1 \in \mathrm{Eq}\left((n+1) \times \omega^{m-1}\right)$ be such that if $i, j \le n$ and $s, s' \in \omega^{m-1}$, then $\big\langle \langle i, s\rangle, \langle j, s'\rangle \big\rangle \in \Theta_1$ iff $T_s = T_{s'}$ and $\theta_s = \theta_{s'}$. By the inductive hypothesis, there are $r_1 \in G_{m-1,n}$, $D_1 \subseteq \omega^{m-1}$, and $X_1 = (n+1) \times D_1$ such that $\gamma_{m-1,n} | X_1 \cong \gamma_{m-1,n}$ and $\Theta_1 \cap X_1^2 = \gamma_{m-1,n}(r_1) \cap X_1^2$. Since there are only finitely many Θ_1-classes, it must be that $r_1 = \langle \psi, 0\rangle$ for some $\psi \in \mathrm{Eq}(n+1)$. Thus, we have, for $i, j \le n$ and $s, s' \in D_1$, that

$$\begin{aligned} \big\langle \langle i, s\rangle, \langle j, s'\rangle \big\rangle \in \Theta_1 &\Longleftrightarrow T_s = T_{s'} \text{ and } \theta_s = \theta_{s'} \\ &\Longleftrightarrow \langle i, j\rangle \in \psi. \end{aligned}$$

This implies that ψ is trivial and that there are T and θ such that $T_s = T$ and $\theta_s = \theta$ whenever $s \in D_1$.

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Without loss of generality, we will assume that $D_1 = \omega^{m-1}$ so that $X_1 = (n+1) \times \omega^{m-1}$. Notice that (1)–(4) remain true and, in addition, the following hold:

(5) If $i \leq n$, then $\Theta \cap (\{i\} \times Y_s)^2$ is trivial iff $i \in T$.
(6) If $i, j \leq n$ and $t \in \omega^m$, then $\langle i, j\rangle \in \theta$ iff $\big\langle\langle i, t\rangle, \langle j, t\rangle\big\rangle \in \Theta$.

Let $\Theta_2 \in \mathrm{Eq}\big((n+1) \times \omega^{m-1}\big)$ be such that if $i, j \leq n$ and $s, s' \in \omega^{m-1}$, then $\big\langle\langle i, s\rangle, \langle j, s'\rangle\big\rangle \in \Theta_2$ iff one of the following:

- $i, j \notin T$, $\langle i, j\rangle \in \theta$ and $s = s'$;
- $i, j \in T$ and for some (or, equivalently, all) $t \in Y_s$ and $t' \in Y_{s'}$, $\big\langle\langle i, t\rangle, \langle j, t'\rangle\big\rangle \in \Theta$.

One easily verifies that, indeed, $\Theta_2 \in \mathrm{Eq}\big((n+1) \times \omega^{m-1}\big)$. By the inductive hypothesis, there are $D_2 \subseteq \omega^{m-1}$, $X_2 = (n+1) \times D_2$ and $r_2 \in G_{m-1,n}$ such that $\gamma_{m-1,n}|X_2 \cong \gamma_{m-1,n}$ and $\Theta_2 \cap X_2^2 = \gamma_{m-1,n}(r_2) \cap X_2^2$. Let $r_2 = \langle \varphi, f'\rangle$. It must be that $\varphi = \theta$.

Now let $D_3 = \{t \in \omega^m : t{\restriction}(m-1) \in D_2\}$ and $X_3 = (n+1) \times D_3$. Then $\gamma_{m,n}|X_3 \cong \gamma_{m,n}$. Let $r_3 = \langle \theta, f\rangle \in G_{m,n}$, where $f(i) = f'(i)$ if $i \in T$ and $f(i) = m$ if $i \notin T$. Although it is not clear if $\Theta \cap X_3^2 = \gamma_{m,n}(r_3) \cap X_3^2$, we do have

$$\Theta \cap (T \times D_3)^2 = \gamma_{m,n}(r_3) \cap (T \times D_3)^2.$$

Without loss of generality, let $D_3 = \omega^m$, so that $X_3 = (n+1) \times \omega^m$. Thus, we have, in addition to (1)–(6), that:

(7) $\Theta \cap (T \times \omega^m)^2 = \gamma_{m,n}(r_3) \cap (T \times \omega^m)^2$.

To complete this inductive step,1 we proceed with what might be called a "thinning" of ω^m. The object is to get $D \subseteq \omega^m$ and $X = (n+1) \times D$ such that $\gamma_{m,n}|X \cong \gamma_{m,n}$ and $\gamma_{m,n}(r_3) \cap X^2 = \Theta \cap X^2$.

Let $\langle s^k : k < \omega\rangle$ be a one-to-one enumeration of ω^m. By recursion on k, choose $t^k \in \omega^m$ so that:

(T1) $t^k \notin \{t^0, t^1, \dots, t^{k-1}\}$,
(T2) $t^k{\restriction}(m-1) = s^k{\restriction}(m-1)$,
(T3) Θ and $\gamma_{m,n}(r_3)$ agree on $(n+1) \times \{t^0, t^1, \dots, t^k\}$.

Clearly, $t^0 = s^0$. If $k > 0$, then there are only finitely many $t \in \omega$ such that (T2) holds but (T3) fails, so it is always possible to get t^k.

Let $r = r_3$, $D = \{t^k : k < \omega\}$, and $X = (n+1) \times D$. Then, X and r are as required. ⊣

Corollary 3.8. *If* $1 \leq n < \omega$, *then* $\gamma_n \longrightarrow \gamma_n$.

Proof. Let $n = m$ in Lemma 3.7. ⊣

Let $R : \omega \longrightarrow \omega$ be the Ramsey function such that if $m < \omega$, then $R(m)$ is the least $k < \omega$ such that whenever $\chi : [k]^2 \longrightarrow 35$, then there is $I \subseteq k$ such that $|I| = m$ and χ is constant on $[I]^2$. (It seems to be right that 35 is large enough for the following proof to work. But if it isn't, replace it with something that is.)

Theorem 3.9. *Suppose that* $m < 4R(m+2)^2 \leq n < \omega$. *Then,* $\alpha_n \longrightarrow \alpha_m$.

Proof. Let m, n be as given. Suppose that $\Theta \in \mathrm{Eq}((n+2) \times \omega^{n+1})$. By Lemma 3.6 and Corollary 3.8, we can assume that $\Theta = \gamma_n(\langle \theta, f\rangle)$, where $\langle \theta, f\rangle \in$

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G_{n+1}. Thus, $\theta \in \mathrm{Eq}(n+2)$ and $f : n+1 \longrightarrow n+1$. Let $J \subseteq n+2$ be such that $|J| = 2R(m+2)$ and that $\theta \cap J^2$ is either trivial or discrete. We consider each of these possibilities.

trivial: Since $\theta \cap J^2$ is trivial, the function $f{\restriction}J$ is constant. Let $r_0 \leq n+1$ be such that $f(j) = r_0$ for all $j \in J$. Let $I \subseteq J$ be such that $|I| = m+2$ and either $r_0 \leq \min(I)$ or $\max(I) < r_0$. In either case, invoke Lemma 3.2 to get $D \subseteq \omega^{n+1}$ such that $\alpha_n|(I \times D) \cong \alpha_m$. Let $X = I \times D$.

If $r_0 \leq \min(I)$, then $\Theta \cap X^2 = \alpha_n(0) \cap X^2$. If $\max(I) < r_0$, then $\Theta \cap X^2 = \alpha_n(c)$.

discrete: Let $\chi : [J]^2 \longrightarrow 35$ be such that if $i, i', j, j' \in J$, $i < j$, $i' < j'$ and $\chi(\{i,j\}) = \chi(\{i',j'\})$, then $\{\langle i,i'\rangle, \langle j,j'\rangle, \langle f(i), f(i')\rangle, \langle f(j), f(j')\rangle\}$ is an order-preserving function. Let $I \subseteq J$ be such that $|I| = m+2$ and χ is constant on $[I]^2$. There are 3 possibilities: (1) $f(i) < j$ for all $i, j \in I$; (2) $f(i) = i$ for all $i \in I$; (3) $f(i) > j$ for all $i, j \in I$. In any case, invoke Lemma 3.2 to get $D \subseteq \omega^{n+1}$ such that $\alpha_n|(I \times D) \cong \alpha_m$. Let $X = I \times D$. If (1), then $\Theta \cap X^2 = \alpha_n(a) \cap X^2$; if (2), then $\Theta \cap X^2 = \alpha_n(b) \cap X^2$; and if (3), then $\Theta \cap X^2 = \alpha_n(1) \cap X^2$.

This completes the proof of the theorem. ⊣

A careful inspection of the previous proofs shows that they can be carried out in ACA_0. (Keep in mind that $(\mathcal{M}, \mathrm{Def}(\mathcal{M})) \models \mathsf{ACA}_0$ for every $\mathcal{M}$, where $\mathrm{Def}(\mathcal{M})$ is the set of definable subsets of M.) For example, we see from the proof of Lemma 3.7 that if $1 \leq m \leq n < \omega$, then $\mathsf{ACA}_0 \vdash \gamma_{m,n} \longrightarrow \gamma_{m,n}$. The function R defined just before Theorem 3.9 can defined in any model of PA, and then also ACA_0. Thus, we get the following theorem.

THEOREM 3.10. *If $m < \omega$ and $n = 4R(m+2)^2$, then*

$$\mathsf{ACA}_0 \vdash \alpha_n \longrightarrow \alpha_m.$$

§4. Proving Theorem 4.

This section is devoted to a proof of Theorem 4. The definitions of $\mathcal{L}^*$ and the $\mathcal{L}^*$-theory PA^* are given in the introduction. For each $\mathcal{M}^*$, observe that $(\mathcal{M}, \mathrm{Def}(\mathcal{M}^*)) \models \mathsf{ACA}_0$. For any countable, recursively saturated $\mathcal{M}$ we will obtain $\mathcal{M}^*$ and $\mathcal{N}^*$ as in that theorem. First, we isolate a certain class of models of PA^* that have such extensions.

DEFINITION 4.1. A model $\mathcal{M}^*$ is *recursively supersaturated* if $(\mathcal{M}, \mathrm{Def}(\mathcal{M}^*))$ (qua a two-sorted, first-order model of second-order arithmetic) is recursively saturated.

PROPOSITION 4.2. *Every countable, recursively saturated $\mathcal{M}$ can be expanded to a recursively supersaturated $\mathcal{M}^*$.*

PROOF. Let $T = \mathrm{Th}(\mathcal{M}) + \mathsf{ACA}_0$. Then $T \in \mathrm{SSy}(\mathcal{M})$, so it has a countable $\mathrm{SSy}(\mathcal{M})$-saturated model $(\mathcal{N}, \mathfrak{X})$. But then $\mathcal{N} \equiv \mathcal{M}$, $\mathrm{SSy}(\mathcal{N}) = \mathrm{SSy}(\mathcal{M})$, and $\mathcal{N}$ is countable and recursively saturated, so $\mathcal{N} \cong \mathcal{M}$. We can then let $\mathcal{N} = \mathcal{M}$. Since $\mathfrak{X}$ is countable, we can let $\mathfrak{X} = \{U_0, U_1, U_2, \ldots\}$, and then let $\mathcal{M}^* = (\mathcal{M}, U_0, U_1, U_2, \ldots)$. ⊣

Having Proposition 4.2, we see that the following theorem implies Theorem 4.

THEOREM 4.3. *If $\mathcal{M}^*$ is countable and recursively supersaturated, then there is $\mathcal{N}^* \succ \mathcal{M}^*$ such that* $\mathrm{Ltr}(\mathcal{N}^*/\mathcal{M}^*) \cong (\mathbf{N}_5, v_3)$.

PROOF. For $n \in M$, let $\alpha_n^{\mathcal{M}^*} : \mathbf{N}_5 \longrightarrow \mathrm{Eq}\left((n+2) \times M^{n+1}\right)$ be the function obtained by interpreting Definition 3.1 within $\mathcal{M}^*$. Then, $\alpha_n^{\mathcal{M}^*}$ is an $\mathcal{M}^*$-representation of $\mathbf{N}_5$. Let $\mathcal{C}$ be the set of those $\mathcal{M}^*$-representations α such that for some nonstandard $n \in M^*$, $\mathcal{M}^* \models \alpha \cong \alpha_n$. It is consequence of Theorem 3.10 and the recursive supersaturation of $\mathcal{M}^*$ that $\mathcal{C}$ is an $\mathcal{M}^*$-correct set (see Definition 1.6*) of representations of $(\mathcal{N}_5, v_3)$. Since $\mathcal{M}^*$ is countable, Theorem 1.7*(2) can be applied, yielding $\mathcal{N}^* \succ \mathcal{M}^*$ such that $\mathrm{Ltr}(\mathcal{N}^*/\mathcal{M}^*) \cong (\mathbf{N}_5, v_3)$. $\dashv$

Theorem 4 now follows from Theorem 1.7*.

REFERENCES

[1] R. KOSSAK and J. H. SCHMERL, ***The Structure of Models of Peano Arithmetic***, Oxford Logic Guides, vol. 50, Oxford University Press, Oxford, 2006.

[2] J. B. PARIS, *Models of arithmetic and the 1–3–1 lattice*. ***Fundamenta Mathematicae***, vol. 95 (1977), pp. 195–199.

[3] J. H. SCHMERL, *Substructure lattices of models of Peano Arithmetic*, ***Logic Colloquium'84***, North-Holland, Amsterdam, 1986, pp. 225–243.

[4] A. J. WILKIE, *On models of arithmetic having nonmodular substructure lattices*. ***Fundamenta Mathematicae***, vol. 95 (1977), pp. 223–237.

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