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# A note on subnormality

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Let $H$ be a subgroup of a finite group $G$ and let $S$ be a set of generators of $H$. We prove that if $G$ is soluble, then $H$ is subnormal in $G$ if and only if there exists an integer $n$ such that for each $g$ in $G$ and $a$ in $S$ the commutator $[g, \underbrace{a, \ldots, a}_{n}]$ lies in $H$. This criterion for subnormality is also valid for soluble groups satisfying the maximal or the minimal condition on subgroups.

Let $G$ be a group. For elements $x$ and $y$ of $G$ we write $[x, y]$ for the commutator $x^{-1}y^{-1}xy$ and define $[x, {}_ny]$ inductively by $[x, {}_0y] = x$ and $[x, {}_{n+1}y] = [[x, {}_ny], y]$ for $n \geq 0$. Let $H$ be a subgroup of $G$ and let $S$ be a set of generators of $H$. If $H$ is subnormal in $G$, then there exists an integer $n \geq 0$ such that for each $g$ in $G$ and $a$ in $S$ the commutator $[g, {}_na]$ lies in $H$. For an arbitrary group $G$ the converse is false. (For an example, see [4], p. 230.) Is the converse true if $G$ is a finite group? This question was raised by Wielandt in a lecture at the Mathematics Institute, Warwick, in early 1973. (See also [6], p. 203.) In this note we give a partial answer to Wielandt's question by proving that the converse is true if $G$ is a finite soluble group. We shall deduce this and a number of other results of a similar kind from a more general theorem on subnormality in soluble groups which are not necessarily finite.

We use standard notation. Let $H$ be a subgroup of a group $G$ and let $S$ be a set of generators of $H$. We shall say that the set $S$

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satisfies the condition $C_n$ (in $G$) if there exists an integer $n \geq 0$ such that for each $g$ in $G$ and $a$ in $S$ the commutator $[g, {}_n a]$ lies in $H$. We write $H$ sn $G$ to indicate that $H$ is a subnormal subgroup of $G$.

THEOREM. *Let $G$ be a soluble group and let $H$ be a subgroup of $G$. Suppose that $H$ is generated by a finite normal subset $S$ (that is, $S^h = S$ for all $h$ in $H$). If $S$ satisfies the condition $C_n$ for some integer $n \geq 0$, then $H$ is subnormal in $G$.*

Proof. We first prove the special case where $S$ consists of a single conjugacy class of $H$. We do this by induction on the derived length $d(G)$ of $G$.

The result is trivially true if $d(G) = 1$ (that is, if $G$ is abelian). So we assume that $d(G) \geq 2$. We also assume that $n \geq 1$ since $n = 0$ implies that $H = G$.

Let $N$ be the last non-trivial term of the derived series of $G$. Clearly $SN/N$ generates $HN/N$ and consists of a single conjugacy class. Since $SN/N$ satisfies the condition $C_n$ in $G/N$ and $d(G/N) = d(G) - 1$, it follows by induction that $HN/N$ sn $G/N$ or $HN$ sn $G$. It is therefore sufficient to prove that $H$ sn $HN$.

Let $M = H \cap N$. Then $M \triangleleft HN$. Since $H$ sn $HN$ if and only if $H/M$ sn $HN/M$, there is no loss of generality if we assume that $M = 1$. Under this assumption we have $H = H/M \cong HN/N$, so that the derived length $d(H)$ of $H$ is at most $d(G) - 1$. (The derived length of $HN$ may well be $d(G)$, but this will not concern us.)

Let $g \in N$ and $a \in S$. Then $[g, {}_i a] \in N$ for all $i \geq 0$. This together with the condition $C_n$ and the assumption $M = 1$ yields $[g, {}_n a] = 1$. Let $m \leq n$ be the smallest positive integer such that $[g, {}_m a] = 1$ for all $g \in N$ and $a \in S$. If $m = 1$, then $[N, H] = 1$, so that $H \triangleleft HN$. We may assume that $m > 1$.

By the choice of $m$ there exist non-identity elements $g \in N$ and $a \in S$ such that $[g, {}_m a] = 1$ but $[g, {}_{m-1} a] \neq 1$. Putting

$x = [g, {}_{m-1}a]$ , we get $[x, a] = 1$ . We show that $x$ centralizes the whole of $H$ .

Let $K = H \cap H^x$ . $K$ is non-trivial since it contains the element $a$ . Suppose that $K$ is a proper subgroup of $H$ . Let $S_1 = \{a^h \mid h \in K\}$ and let $H_1 = \langle S_1 \rangle$ . Since $S_1 \le H^x$ , we have $S_1^{x^{-1}} \le H$ , so that $[x^{-1}, S_1] \le H$ . Since $x$ lies in the normal subgroup $N$ , we have also $[x^{-1}, S_1] \le N$ . Therefore $[x^{-1}, S_1] \le H \cap N = 1$ . Hence $x$ centralizes $S_1$ .

Let $S_1$ and $H_1$ be as above. If the subset $S_i$ and the subgroup $H_i$ have already been defined for $i \ge 1$ , then let $S_{i+1} = \{a^h \mid h \in H_i\}$ and let $H_{i+1} = \langle S_{i+1} \rangle$ . Since $S$ is finite, there exists an integer $r \ge 1$ such that $S_{r+1} = S_r$ . But then $S_r = \{a^h \mid h \in H_r\}$ . Hence $H_r$ is generated by a subset $S_r$ consisting of a single conjugacy class.

Let $h \in H$ and $b \in S_r$ . Then it follows from the condition $C_n$ that $[h^{x^{-1}}, {}_nb] \in H$ . Since $x$ commutes with $b$ , we have

$$[h, {}_nb] = [h^{x^{-1}}, {}_nb]^x \in H^x .$$

But $[h, {}_nb]$ is an element of $H$ . Hence $[h, {}_nb] \in H \cap H^x = K$ . Since $H_1 \triangleleft K$ and $H_{i+1} \triangleleft H_i$ for $i = 1, \ldots, r-1$ , we have

$$[h, {}_{n+r}b] = [[h, {}_nb], {}_rb] \in H_r .$$

Thus as a set of generators of $H_r$ the subset $S_r$ satisfies the condition $C_{n+r}$ in $H$ . Since $d(H)$ is at most $d(G) - 1$ , it follows by induction that $H_r$ sn $H$ . This implies that $S$ lies in a proper normal subgroup of $H$ . This contradiction shows that $K = H$ . But then $H \le H^x$ and consequently $[x^{-1}, H] \leq H \cap N = 1$ , which proves that $x$ centralizes $H$ .

Let $N_1 = C_N(H)$ be the centralizer of $H$ in $N$ . $N_1$ is non-trivial since it contains the element $x$ . In fact, our argument shows that $N_1$ contains the element $[g, {}_{m-1}a]$ for each $g \in N$ and $a \in S$ . Since $N$ is abelian, $N_1$ lies in the centre of $HN$ and is therefore normal in $HN$ . For any $g \in N$ and $a \in S$ , write $\bar{g} = gN_1$ and $\bar{a} = aN_1$ . Then in the factor group $HN/N_1$ we have $[\bar{g}, {}_{m-1}\bar{a}] = 1$ for all $\bar{g} \in N/N_1$ and $\bar{a} \in SN_1/N_1$ . We may therefore repeat the above argument to obtain a subgroup $N_2/N_1$ of $N/N_1$ given by $N_2/N_1 = C_{N/N_1}(HN_1/N_1)$ . If $N_2$ is a proper subgroup of $N$ , then we repeat the argument again with $N_2$ in place of $N_1$ . Continuing in this way we obtain subgroups $N_0 = 1, N_1, N_2, \ldots$ of $N$ with $N_{i+1}/N_i = C_{N/N_i}(HN_i/N_i)$ for $i = 0, 1, 2, \ldots$ . Clearly the process must come to a stop after at most $m$ repetitions. So there exists an integer $1 \leq s \leq m$ such that $N_s = N$ . Since $N_{i+1}/N_i$ lies in the centre of $HN/N_i$ for $i = 0, 1, 2, \ldots$ , it follows that $HN_i/N_i$ is normal in $HN_{i+1}/N_i$ or $HN_i \triangleleft HN_{i+1}$ . Hence $H$ sn $HN$ . This completes the proof of the special case.

We now prove the general case where $S$ is a finite normal subset of $H$ . We use induction on the order $|S|$ of $S$ . Suppose that $S$ is a union of $t \geq 1$ conjugacy classes of $H$ , $D_1, \ldots, D_t$ say. If $t = 1$ , we are back in the special case. So we assume that $t > 1$ . Let $E_i = \langle D_i \rangle$ for $i = 1, \ldots, t$ . Then $E_i \triangleleft H$ , so that $D_i$ satisfies the condition $C_{n+1}$ for each $i$ . Since $D_i$ is a finite normal subset of $E_i$ and $|D_i| < |S|$ , it follows by induction that $E_i$ sn $G$ for each $i$ . A result of Robinson ([5], Lemma 2.2) now gives $H = \langle E_1, \ldots, E_t \rangle$ sn $G$ . This completes the proof of the theorem.

COROLLARY 1. *Let* $G$ *be a soluble group and let* $H$ *be a finite subgroup of* $G$ . *If* $H$ *is generated by a subset* $X$ *which satisfies the* *condition* $C_n$ *for some integer* $n \geq 0$ , *then* $H$ *is subnormal in* $G$ .

Proof. Let $S = \{a^h \mid a \in X \text{ and } h \in H\}$ . Then $S$ is a finite normal subset generating $H$ and satisfying the condition $C_n$ . It follows from the theorem that $H$ is subnormal in $G$ .

Let $G$ be a soluble group satisfying the maximal condition on subgroups and let $H$ be a subgroup of $G$ . Then Kegel has shown in [3] that $H$ is subnormal in $G$ if and only if for each normal subgroup $N$ of $G$ with $G/N$ finite $HN/N$ is subnormal in $G/N$ . This together with Corollary 1 gives the following result.

COROLLARY 2. *Let* $G$ *be a soluble group satisfying the maximal condition on subgroups and let* $H$ *be a subgroup of* $G$ . *If* $H$ *is generated by a subset* $S$ *which satisfies the condition* $C_n$ *for some integer* $n \geq 0$ , *then* $H$ *is subnormal in* $G$ .

There is a corresponding result for soluble groups satisfying the minimal condition on subgroups. But we can prove more. Since it is well known that a soluble group satisfying the minimal condition on subgroups is abelian-by-finite, the corresponding result is contained in the following corollary.

COROLLARY 3. *Let* $G$ *be a soluble abelian-by-finite group and let* $H$ *be a subgroup of* $G$ . *If* $H$ *is generated by a subset* $S$ *which satisfies the condition* $C_n$ *for some integer* $n \geq 0$ , *then* $H$ *is subnormal in* $G$ .

Proof. Let $N$ be an abelian normal subgroup of $G$ such that $G/N$ is finite. Then $HN/N$ is finite. By Corollary 1, $HN/N$ sn $G/N$ or $HN$ sn $G$ . Since $N$ is abelian, $H \cap N$ is normal in $HN$ . Thus $H/H \cap N$ , which is isomorphic to $HN/N$ , is a finite subgroup of $HN/H \cap N$ . By Corollary 1 again, $H/H \cap N$ sn $HN/H \cap N$ or $H$ sn $HN$ . Hence $H$ sn $G$ .

Another easy consequence of our theorem is the following result of Gruenberg ([2], Lemma 12): if $a$ is a bounded left Engel element of a soluble group $G$ that is, there exists an integer $n \geq 0$ such that $[g, {}_n a] = 1$ for all $g \in G$ ), then $a$ sn $G$ . Using this result and a well-known theorem of Baer ([1], §3, Satz 3) on the join of cyclic subnor subnormal subgroups and an argument similar to that of Corollary 3 above, we can prove easily the following result.

**COROLLARY 4.** *Let $G$ be an extension of an abelian group by a finitely generated nilpotent group and let $H$ be a subgroup of $G$. If $H$ is generated by a subset $S$ which satisfies the condition $C_n$ for some integer $n \geq 0$, then $H$ is subnormal in $G$.*

It may be worth remarking that the words "finitely generated" in Corollary 4 cannot be omitted. In fact, for each prime $p$ there exists a metabelian $p$-group with an abelian subgroup $H$ such that $H$ itself satisfies the condition $C_p$ but is not subnormal in $G$ ([4], p. 230).

## References

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