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ON THE STABLE HOMOTOPY TYPE OF THOM COMPLEXES

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1. Introduction. Let α be a real vector bundle over a finite CW complex X and let $T(\alpha; X)$ be its associated Thom complex. We propose to study the S-type (stable homotopy type) of Thom complexes in the framework of the Atiyah-Adams J-Theory. Therefore we focus our attention on the group $\tilde{J}_{\mathbf{R}}(X)$ which is defined to be the group of orthogonal sphere bundles over X modulo stable fiber homotopy equivalence. A basic link between the S-type of Thom complexes and the functor $\tilde{J}_{\mathbf{R}}$ is given by the following proposition.

(1.1) PROPOSITION (Atiyah [**3**]). *Let α, β be real vector bundles over a finite CW-complex X and suppose $\tilde{J}_{\mathbf{R}}(\alpha) = \tilde{J}_{\mathbf{R}}(\beta)$. Then $T(\alpha, X)$ and $T(\beta, X)$ are of the same S-type.*

The main theme of this paper is to describe under which extra conditions the "converse" of Atiyah's theorem holds. A particular result in this respect can be stated as follows. (We use Adams' [**1**] notation.)

(1.2) THEOREM. *Suppose α, β are real vector bundles over a finite connected CW complex X, both oriented with respect to $K_{\mathbf{R}}^*$ theory and assume that $K_{\mathbf{R}}(X)$ is torsion free. Suppose further that there exists a homotopy equivalence $f : T(\beta, X) \to T(\alpha, X)$ such that*

$$\begin{array}{ccc}
\tilde{H}^*(T(\alpha, X); Q) & \xrightarrow{f^*} & \tilde{H}^*(T(\beta, X); Q) \\
\Phi_\alpha^H \Big\uparrow \cong & & \Phi_\beta^H \Big\uparrow \cong \\
H^*(X; Q) & \xrightarrow[\pm \mathrm{id}]{} & H^*(X; Q)
\end{array}$$

is a commutative diagram, where $\phi_\alpha{}^H, \phi_\beta{}^H$ denote the Thom isomorphisms. Then $\tilde{J}_{\mathbf{R}}(\alpha) = \tilde{J}_{\mathbf{R}}(\beta)$.

Let us point out that the assumption "$K_{\mathbf{R}}(X)$ is torsion free" is crucial. To illustrate this we shall construct (see (2.5)) two real vector bundles α, β over $S^2 \vee S^4$ which give rise to S-equivalent Thom complexes, but whose $\tilde{J}_{\mathbf{R}}$ classes

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are different. We are indebted to our friend U. Suter for drawing our attention to this nice example.

Theorem (1.2) essentially grew out of our work on the Spanier Conjecture; see [**5**]. In fact it synthesizes the basic arguments of that paper.

The proof of (1.2) will be followed by an application (see (3.17)).

2. Proof of the main theorem. Let X be a finite, connected CW-complex and $K_\wedge(X)$ its Grothendrick ring of virtual $\wedge$-bundles, where $\wedge$ stands for either **C** or **R**. By abuse of language we shall denote a $\wedge$-vector bundle, its equivalence class, and even its stable equivalence class by the same (Greek) letter. The basic facts governing the relationship between $K_\wedge(X)$ and $\tilde{J}_\wedge(X)$ can be found in Adams [**1**], Adams-Walker [**2**], and Atiyah [**3**]. In order to avoid confusion between the functor $J_\wedge'$ of [**2**] and $J_\wedge'$ of [**1**]—which are defined differently—we shall denote the functor $J_\wedge'$ of [**2**] by $Bh_\wedge$, referring to the fact that it is defined in terms of the characteristic class $bh_\wedge$.

Now let α, β be $\wedge$-vector bundles over X and suppose that there exists a homotopy-equivalence $f : T(\beta) \to T(\alpha)$ (we suppress X in the notation). Our goal is then to find conditions which imply $\tilde{J}_\wedge(\alpha) = \tilde{J}_\wedge(\beta)$.

An appropriate procedure for doing this is to study the effect of the map induced by f in various cohomology theories. Accordingly we shall henceforth assume α and β are $K_\wedge$-oriented bundles.

Let us fix some further notation. As the bundles α, β are $K_\wedge$-oriented there exist $K_\wedge$-Thom classes U_α, U_β and $H^*(—; \mathbf{Z})$—Thom classes V_α, V_β of α, β respectively, and Thom isomorphisms $\Phi_\alpha{}^K$, $\Phi_\beta{}^K$, $\Phi_\alpha{}^H$, $\Phi_\beta{}^H$ in the corresponding cohomology theories. For example, we evaluate the Thom isomorphism $\Phi_\alpha{}^K : K(X) \to \tilde{K}(T(\alpha))$ by the formula $\Phi_\alpha{}^K(x) = x \cup U_\alpha$ for all $x \in K(X)$.

In [**5**] the concept of an "untwisted homotopy equivalence" $f : T(\beta) \to T(\alpha)$ played a decisive role. This signals that we should concentrate on such specific homotopy equivalences.

(2.1) *Definition.* A homotopy equivalence $f : T(\beta, X) \to T(\alpha, X)$ is said to be untwisted if there exists $\delta = \pm 1$ such that

$$f^*(x \cup V_\alpha) = \delta \cdot x \cup V_\beta \quad \text{for all } x \in H^*(X; Q).$$

It is obvious how an untwisted S-equivalence has to be defined. Note that if f does arise from a fiber homotopy equivalence then it is untwisted!

The proof of theorem (1.2) evolves from the subsequent propositions.

(2.2) PROPOSITION. *Suppose $f : T(\beta, X) \to T(\alpha, X)$ is an untwisted S-equivalence, where α, β are $K_\wedge$-oriented. Then $Bh_\wedge(\alpha) = Bh_\wedge(\beta)$.*

Proof. As a matter of notational convenience define a group homomorphism $F : K_\wedge(X) \to K_\wedge(X)$ by the formula $f^*(x \cup U_\alpha) = F(x) \cup U_\beta$. (Without loss of generality we assume f to be a homotopy equivalence.)

Consider the commutative diagram

$$\begin{array}{ccc} \tilde{K}_\wedge(T(\alpha)) & \xrightarrow{f^*} & \tilde{K}_\wedge(T(\beta)) \\ {\scriptstyle ch_\wedge}\downarrow & & {\scriptstyle ch_\wedge}\downarrow \\ \tilde{H}^*(T(\alpha); Q) & \xrightarrow[f^*]{} & \tilde{H}^*(T(\beta); Q). \end{array}$$

We get on the one hand

$$ch_\wedge f^*(x \cup U_\alpha) = ch_\wedge(F(x) \cup U_\beta) = bh_\wedge(\beta) \cdot ch_\wedge(F(x)) \cup V_\beta$$

by [**1**, p. 156], and on the other hand

$$f^* ch_\wedge(x \cup U_\alpha) = f^*(bh_\wedge(\alpha) ch_\wedge(x) \cup V_\alpha) = \delta bh_\wedge(\alpha) ch_\wedge(x) \cup V_\beta$$

again by [**1**, p. 156].

Therefore $bh_\wedge(\beta) ch_\wedge(F(x)) = bh_\wedge(\alpha) ch_\wedge(\delta x)$ for all $x \in K_\wedge(X)$. If we choose $x \in K_\wedge(X)$ to be invertible we can re-write this as $bh_\wedge(\alpha - \beta) = ch_\wedge(F(x) \cdot \delta \cdot x^{-1})$. But $bh_\wedge$ takes values in the multiplicative group

$$1 + \sum_{s>0} H^s(X; Q)$$

and so we have $F(x)\delta x^{-1} = 1 + y$ for some $y \in \tilde{K}_\wedge(X)$. Hence $bh_\wedge(\alpha - \beta) = ch_\wedge(1 + y)$ and therefore $Bh_\wedge(\alpha - \beta) = 0$. Thus the proof is complete.

In view of (2.2) it is important to know when the group $Bh_\wedge(X)$ is isomorphic to $\tilde{J}_\wedge(X)$. A partial answer to this question is given in the next theorem. By $Bh_\wedge : K_\wedge(X) \to Bh_\wedge(X)$ we denote the "canonical" projection; see [**2**].

(2.3) PROPOSITION. *Suppose ξ is a $K_\wedge$-oriented vector bundle over X, $K_\wedge(X)$ is torsion free and $Bh_\wedge(\xi) = 0$. Then $\tilde{J}_\wedge'(\xi) = 0$.*

Proof. $Bh_\wedge(\xi) = 0$ if, and only if, there exists an element $y \in \tilde{K}_\wedge(X)$ such that $bh_\wedge(\xi) = ch_\wedge(1 + y)$. Making use of the equation

$$bh_\wedge(\xi) \cdot ch_\wedge(\rho_\wedge{}^k(\xi)) = k^n(\psi^k bh_\wedge(\xi))$$

(see [**1**, p. 157]) we get $ch_\wedge((1 + y)\rho_\wedge{}^k(\xi)) = ch_\wedge(k^n \psi_\wedge{}^k(1 + y))$, where $2n$ is the fiber dimension of ξ over $\mathbf{R}$, $\rho_\wedge{}^k(\xi)$ the $\wedge$-cannibalistic class of ξ, and $\psi_\wedge{}^k$ an Adams operation. The chern character $ch_\wedge$ is a monomorphism since there is no torsion in $K_\wedge(X)$; therefore

$$\rho_\wedge{}^k(\xi) = \frac{k^n \psi_\wedge{}^k(1 + y)}{1 + y}$$

for all integers k. If the stable class of ξ is considered this amounts to $\rho_\wedge{}^k(\xi) = \psi_\wedge{}^k(1 + y)/(1 + y)$ as elements of $\tilde{K}_\wedge(X) \otimes \mathbf{Q}_k$, which is equivalent to

$\tilde{J}_\wedge'(\xi) = 0$. Here $\mathbf{Q}_k$ denotes the additive group of rationals of the form p/k^q. Hence the proof is complete.

(2.4) *Remarks.* (i) By virtue of the Adams Conjecture—as proved by Quillen [**7**]—the converse of (2.3) is true in the case $\wedge = \mathbf{R}$. We do not know if the converse is true in case $\wedge = \mathbf{C}$; (Note that $\tilde{J}_{\mathbf{C}}(X) \neq \tilde{J}_{\mathbf{C}}'(X)$ in general; an example is given by $X = S^4$).

(ii) Theorem (1.2) is an immediate consequence of (2.2), (2.3) and the Adams conjecture.

(2.5) *Example.* We now give an example which shows that the assumptions in (1.2) are essential. For this purpose we pick $X = S^2 \vee S^4$. Let ω denote the complex Hopf line bundle over S^2 and let ξ denote the quaternionic (left) Hopf line bundle over S^4. Then $\tilde{K}_{\mathbf{R}}(X) \cong Z_2 \oplus Z$ with generators $\sigma_2 = r(\omega) - 2$, $\sigma_4 = r(\xi) - 4$ and $\tilde{J}_{\mathbf{R}}(X) \cong Z_2 \oplus Z_{24}$ with generators $x_2 = \tilde{J}_{\mathbf{R}}(\sigma_2)$, $x_4 = \tilde{J}_{\mathbf{R}}(\sigma_4)$. Then we choose α, β such that $\tilde{J}_{\mathbf{R}}(\alpha) = (x_2, 0)$ and $\tilde{J}_{\mathbf{R}}(\beta) = (x_2, 12x_4)$.

If γ is any vector bundle over X, then we have

$$T(\gamma, X) = T(\gamma|S^2, S^2) \underset{S^k}{\cup} T(\gamma|S^4, S^4),$$

where k is the fiber dimension of γ and S^k is the sphere carrying the Thom class. But $T(\gamma|S^4, S^4) = S^k \cup_f e^{4+k}$, where the attaching map $f : S^{3+k} \to S^k$ is given by the image of $\gamma|S^4$ via the classical J-homomorphism $J : \pi_3(SO(k)) \to \pi_{3+k}(S^k)$. Similarly for $T(\gamma|S^2, S^2)$. Thus

$$T(\alpha, X) = e^{k+2} \underset{\eta}{\cup} S^k \vee S^{k+4} \quad \text{and} \quad T(\beta, X) = e^{k+2} \underset{\eta}{\cup} S^k \underset{\tau}{\cup} e^{k+4},$$

where $\tau = J(12\sigma_4)$ and η is the stable Hopf map. But $J(12\sigma_4)$ can be represented by η^3 [**9**, p. 190] which is null homotopic as a map $S^{k+3} \to S^k \cup_\eta e^{k+2}$. Therefore $T(\alpha, X) \simeq T(\beta, X)$. In fact this homotopy equivalence can be taken to be untwisted since the top cell of $T(\alpha, X)$ is attached trivially.

Even more is true. Namely: $\tilde{J}_{\mathbf{R}}(\alpha) \neq \tilde{J}_{\mathbf{R}}(e^*\beta)$ for all self equivalences e of X. This is because the induced map e^* on $\tilde{K}_{\mathbf{R}}(X)$ is determined by $e^*(\sigma_2) = \sigma_2$, $e^*(\sigma_4) = \pm\sigma_4$, and therefore $e^*(x_2) = x_2$, $e^*(x_4) = \pm x_4$.

3. Applications of the main theorem. As an application of (1.2) we show how some stunted quaternionic projective spaces can be classified up to S-type. Let ξ be the canonical quaternionic (left) line bundle over $\mathbf{H}P^n$. Then Atiyah [**3**] proved that the stunted quaternionic projective space $\mathbf{H}P^{n+s}/\mathbf{H}P^{s-1}$ is homeomorphic to the Thom complex $T(s\xi, \mathbf{H}P^n)$. Thus the spaces $\mathbf{H}P^{n+s}/\mathbf{H}P^{s-1}$, $\mathbf{H}P^{n+t}/\mathbf{H}P^{t-1}$ are S-equivalent if $\tilde{J}_{\mathbf{R}}(s\xi) = \tilde{J}_{\mathbf{R}}(t\xi)$ as elements of $\tilde{J}_{\mathbf{R}}(\mathbf{H}P^n)$. For some earlier results in this direction see [**6**]. In our approach at the S-type classification we assume that $\mathbf{H}P^{n+s}/\mathbf{H}P^{s-1}$, $\mathbf{H}P^{n+t}/\mathbf{H}P^{t-1}$ are S-equivalent and then prove that $\tilde{J}_{\mathbf{R}}(s\xi) = \tilde{J}_{\mathbf{R}}(t\xi)$ under certain mild hypothesis (see (3.17)).

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Thus, all throughout this section we shall assume that there exists an S-equivalence $f : \mathbf{H}P^{n+s}/\mathbf{H}P^{s-1} \to \mathbf{H}P^{n+t}/\mathbf{H}P^{t-1}$. Letting $a \in H^4\mathbf{H}P^n; \mathbf{Z})$ denote a generator we make the following definition:

(3.1) *Definition.* Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}$ be the sequence of $\pm$ signs defined by $f^*(a^i \cup V_{t\xi}) = \epsilon_{i+1}a^i \cup V_{s\xi}$.

If $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{n+1}$ then f is an untwisted S-equivalence and so by the stable analogue of (1.2) we would have $\tilde{J}_{\mathbf{R}}(s\xi) = \tilde{J}_{\mathbf{R}}(t\xi)$. Our method is to show that the functoriality relationships $f^* \circ \mathfrak{p}^j = \mathfrak{p}^j \circ f^*$, where $\mathfrak{p}^j$ is an arbitrary mod p Steenrod power, automatically imply $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{n+1}$—provided certain conditions hold. By a long but routine calculation the equations $f^* \circ \mathfrak{p}^j(a^{i-1} \cup V_{t\xi}) = \mathfrak{p}^j \circ f^*(a^{i-1} \cup V_{t\xi})$ give the following information (for a similar calculation see [**5**]):

(3.2) Lemma. *For any odd prime p we have the congruences*

$$\binom{2(i-1+s)}{j}\epsilon_i \equiv \binom{2(i-1+t)}{j}\epsilon_{i+\frac{1}{2}(p-1)j} \pmod{p}$$

for all $i, j \geqq 1$ satisfying $i + \frac{1}{2}(p-1)j \leqq n + 1$.

The basic tool in analyzing these congruences is the following lemma which we use repeatedly.

(3.3) Lemma.

$$\binom{A}{B} \equiv \prod_{k \geqq 0} \binom{A_k}{B_k} \pmod{p},$$

where

$$A = \sum_{k \geqq 0} A_k p^k, \quad B = \sum_{k \geqq 0} B_k p^k$$

are the p-adic expansions.

Then we have

(3.4) Theorem. *If $p \leqq 2n - 1$ is an odd prime, then $\epsilon_i = \epsilon_{i+\frac{1}{2}(p-1)}$ for all i such that $1 \leqq i \leqq n + 1 - \frac{1}{2}(p-1)$ and $i - 1 + s \not\equiv 0 \pmod{p}$.*

Proof. Fix an odd prime p and let $2s = \sum a_k p^k$, $2t = \sum b_k p^k$ be the p-adic expansions of $2s$, $2t$ respectively. Assume that $a_k = b_k$ for $k < \nu$; whereas $a_\nu \neq b_\nu$. This is equivalent to assuming $s \equiv t \pmod{p^\nu}$, $s \not\equiv t \pmod{p^{\nu+1}}$.

If $1 + \frac{1}{2}(p-1)p^\nu \leqq n + 1$ then we can substitute $i = 1, j = p^\nu$ into (3.2). By (3.3) the result is $a_\nu \equiv -b_\nu \pmod{p}$, since $a_\nu \not\equiv b_\nu \pmod{p}$. If $p^\nu + \frac{1}{2}(p-1)p^\nu \leqq n$ then we can substitute $i = p^\nu + 1$, $j = p^\nu$ into (3.2). This gives $2 + a_\nu \equiv \pm (2 + b_\nu) \equiv \pm (2 - a_\nu) \pmod{p}$. One can easily check that this gives a contradiction and therefore we must actually have $p^\nu + \frac{1}{2}(p-1)p^\nu > n$.

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We have thus proved that for any odd prime p we have $s \equiv t(\mathrm{mod}\ p^\mu)$, where $\mu = \mu(p)$ is the least integer such that $\frac{1}{2}(p+1)p^\mu > n$. As a consequence (3.2) then gives $\epsilon_i = \epsilon_{i+\frac{1}{2}(p-1)j}$ for all i, j satisfying: $i, j \geqq 1$; $i + \frac{1}{2}(p-1)j \leqq n+1$; $j < p^\mu$ and $\binom{2(i-1+s)}{j} \not\equiv 0(\mathrm{mod}\ p)$. The result now follows by choosing $j = 1$.

(3.5) *Notation.* Instead of writing $\epsilon_i = \epsilon_{i+\frac{1}{2}(p-1)} = \ldots$ we shall use the notation $[i, i + \frac{1}{2}(p-1), \ldots]$.

(3.6) *Example.* If $n \geqq 2$ then we can substitute $p = 3$ into (3.4). There are then three possibilities:

(i) $[1, 2, 3], [4, 5, 6], \ldots$ if $s \equiv 1(\mathrm{mod}\ 3)$;
(ii) $[1, 2], [3, 4, 5], \ldots$ if $s \equiv 2(\mathrm{mod}\ 3)$;
(iii) $[1], [2, 3, 4], \ldots$ if $s \equiv 0\ (3)$.

Therefore the integers $1, 2, \ldots, n+1$ are partitioned into disjoint sets of consecutive integers. Each of these sets contains three elements, except possibly for the first and last sets.

(3.7) THEOREM: $\epsilon_2 = \epsilon_3 = \ldots = \epsilon_n$.

Proof. First note that this theorem has meaning only for $n \geqq 3$. If $n \geqq 3$ then we may use $p = 5$ in (3.4) to relate consecutive sets in the above partition. For example, if $[?, k-2, k-1], [k, k+1, ?]$ are consecutive sets of equal signs then (3.4) for $p = 5$ gives at least one of $\epsilon_{k-2} = \epsilon_k$, $\epsilon_{k-1} = \epsilon_{k+1}$. The question marks are put in because these sets may contain just two elements. The theorem now follows by equating signs from all such consecutive groups.

It should be remarked that one fails to get ϵ_1 (respectively ϵ_{n+1}) by this method if, and only if, $s \equiv 0(\mathrm{mod}\ 3)$ (respectively $s \equiv 1 - n(\mathrm{mod}\ 3)$). To improve on (3.7) we make the following definitions:

(3.8) *Definitions.* We say that s satisfies condition A_n (respectively B_n) if there exists an odd prime $p \leqq 2n-1$ such that $s \not\equiv 0(\mathrm{mod}\ p)$ (respectively $s \not\equiv \frac{1}{2}(p-1) - n(\mathrm{mod}\ p)$).

Then we have

(3.9) THEOREM. $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_n$ *if s satisfies condition A_n and* $\epsilon_2 = \epsilon_3 = \ldots = \epsilon_{n+1}$ *if s satisfies condition B_n.*

Proof. According to (3.4) we can equate ϵ_1 to one of $\epsilon_2, \ldots, \epsilon_n$ if there exists an odd prime $p \leqq 2n-1$ such that $2 \leqq 1 + \frac{1}{2}(p-1) \leqq n$ and $s \not\equiv 0(\mathrm{mod}\ p)$, i.e., if s satisfies condition A_n. By virtue of (3.7) this proves the first part. The second part is similar.

(3.10) COROLLARY. *Suppose* $\mathbf{H}P^{n+s}/\mathbf{H}P^{s-1}$ *and* $\mathbf{H}P^{n+t}/\mathbf{H}P^{t-1}$ *are stably homotopically equivalent and suppose s satisfies conditions A_n and B_n. Then* $\tilde{J}_{\mathbf{R}}(s\xi) = \tilde{J}_{\mathbf{R}}(t\xi)$ *as elements of* $\tilde{J}_{\mathbf{R}}(\mathbf{H}P^n)$.

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It is easy to see that if $n \geqq 2$ then s satisfies at least one of the conditions A_n, B_n and so we have either $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_n$ or $\epsilon_2 = \epsilon_3 = \ldots = \epsilon_{n+1}$. Let $M_k = M_k(\mathbf{H})$ denote the Atiyah-Todd number corresponding to $\mathbf{H}P^{k-1}$, i.e., M_k is the order of ξ as an element of $\tilde{J}_{\mathbf{R}}(\mathbf{H}P^{k-1})$. Then the equality of n consecutive signs is sufficient to give

(3.11) THEOREM. *$s \equiv t (\text{mod } M_n)$ if $n \geqq 2$.*

Proof. First assume $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_n$. Then the stable homotopy equivalence $f : \mathbf{H}P^{n+s}/\mathbf{H}P^{s-1} \rightarrow \mathbf{H}P^{n+t}/\mathbf{H}P^{t-1}$ restricts to an untwisted stable homotopy equivalence $\mathbf{H}P^{n-1+s}/\mathbf{H}P^{s-1} \rightarrow \mathbf{H}P^{n-1+t}/\mathbf{H}P^{t-1}$ and so $\tilde{J}_{\mathbf{R}}(s\xi) = \tilde{J}_{\mathbf{R}}(t\xi)$ as elements of $\tilde{J}_{\mathbf{R}}(\mathbf{H}P^{n-1})$, i.e., $s \equiv t(\text{mod } M_n)$.

Now assume $\epsilon_2 = \epsilon_3 = \ldots = \epsilon_{n+1}$. Consider the stably homotopically commutative diagram

$$\begin{array}{ccc} S^{4s} & \xrightarrow{\epsilon_1} & S^{4t} \\ \downarrow & & \downarrow \\ \mathbf{H}P^{n+s}/\mathbf{H}P^{s-1} & \xrightarrow{f} & \mathbf{H}P^{n+t}/\mathbf{H}P^{t-1}. \end{array}$$

The vertical maps are the cellular inclusions. By taking mapping cones we get a stable homotopy equivalence $g : \mathbf{H}P^{n+s}/\mathbf{H}P^{s} \rightarrow \mathbf{H}P^{n+t}/\mathbf{H}P^{t}$ whose corresponding sequence of signs is just $\epsilon_2, \ldots, \epsilon_{n+1}$. Thus g is untwisted and therefore $\tilde{J}_{\mathbf{R}}((s+1)\xi) = \tilde{J}_{\mathbf{R}}((t+1)\xi)$ as elements of $\tilde{J}_{\mathbf{R}}(\mathbf{H}P^{n-1})$. Again we have $s \equiv t(\text{mod } M_n)$.

To proceed further we need a description of the quaternionic Atiyah-Todd numbers. Sigrist and Suter [**8**] have determined the numbers M_k. Their result is

$\nu_2(M_k) = \max(2k - 1, 2r + \nu_2(r)), \quad 1 \leqq r \leqq k - 1;$
$\nu_p(M_k) = \max(r + \nu_p(r)), \quad 1 \leqq r \leqq [(2k-1)/(p-1)]$ for odd primes $p \leqq 2k - 1;$
$\nu_p(M_k) = 0$ if $p > 2k - 1$.

Here $\nu_p(a)$ denotes the exponent of p in the prime power factorization of a.

(3.12) *Definition.* For any prime p and a fixed integer n let $\tau = \tau(p)$ be the least integer such that $\frac{1}{2}(p-1)p^{\tau} > n$.

As a corollary of (3.11) we then have

(3.13) COROLLARY. *If $n \geqq 2$ and $p \leqq 2n - 1$ is an odd prime then*

$$\binom{2(i-1+s)}{j} \equiv \binom{2(i-1+t)}{j} \quad (\text{mod } p)$$

for all $i, j \geqq 1$ satisfying $i + \frac{1}{2}(p-1)j \leqq n + 1$.

Proof. Notice that if $i, j \geqq 1$ and $i + \frac{1}{2}(p-1)j \leqq n + 1$ then we also have $j < p^{\tau}$. The result will follow if we prove that $\tau \leqq \nu_p(M_n)$, since then the p-adic expansions of $2(i-1+s)$, $2(i-1+t)$ would agree up to at least the $p^{\tau-1}$ term (see (3.3)).

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Let $\sigma = \sigma(p)$ be the largest integer such that

$$p^\sigma \leqq \left[\frac{2n-1}{p-1}\right] = \left[\frac{2n-2}{p-1}\right].$$

Then $\nu_p(M_n) \geqq p^\sigma + \sigma$. But σ and τ are defined by the inequalities

$$\tfrac{1}{2}(p-1)p^{\tau-1} \leqq n < \tfrac{1}{2}(p-1)p^\tau; \qquad \tfrac{1}{2}(p-1)p^\sigma \leqq n-1 < \tfrac{1}{2}(p-1)p^{\sigma+1}$$

and so $\tau - 2 \leqq \sigma \leqq \tau - 1$. Therefore $\nu_p(M_n) \geqq p^\sigma + \tau - 2$ and thus $\nu_p(M_n) \geqq \tau$ if $\sigma > 0$.

To complete the proof we need only consider the case $\sigma = 0$. Then $\tau = 1$ or 2, and since $\nu_p(M_n) \geqq 1$ we need only consider the possibility $\tau = 2$. But $\sigma = \tau - 2$, if, and only if, $\frac{1}{2}(p-1)p^{\tau-1} = n$, and so we have $\frac{1}{2}(p-1)p = n$. But then

$$\nu_p(M_n) = \left[\frac{2n-2}{p-1}\right] = \left[\frac{(p-1)p-2}{p-1}\right] \geqq 2.$$

This completes the proof.

From (3.2) we immediately get:

(3.14) Corollary. *Assume $n \geqq 2$ and $p \leqq 2n-1$ is an odd prime. Then $\epsilon_i = \epsilon_{i+\frac{1}{2}(p-1)j}$ for all $i, j \geqq 1$ satisfying $i + \frac{1}{2}(p-1)j \leqq n+1$ and $\binom{2(i-1+s)}{j} \not\equiv 0 \pmod{p}$.*

(3.15) *Definition.* s satisfies condition C_n (respectively D_n) if there exists an odd prime $p \leqq 2n-1$ such that

$$s \not\equiv 0 \pmod{p^\tau} \text{ (respectively, } 2n+2s+1 \not\equiv 0 \pmod{p^\tau}),$$

where τ is defined in (3.12).

Before stating the strongest result obtainable from our method we need one more lemma.

(3.16) Lemma. *For any prime p we have $\binom{M-pj+j}{j} \equiv 0 \pmod{p}$ for $1 \leqq j \leqq L$ if, and only if, $M+1 \equiv 0 \pmod{p^\sigma}$, where σ is the least integer such that $p^\sigma > L$.*

Proof. Write $M+1 = a_\nu p^\nu + \text{h.o.t.}$, where $0 < a_\nu < p$. First assume $\binom{M-pj+j}{j} \equiv 0 \pmod{p}$ for $1 \leqq j \leqq L$. If we attempt to substitute $j = p^\nu$ into this congruence we get

$$\binom{M-pj+j}{j} \equiv \binom{M+1-p^{\nu+1}+p^\nu-1}{p^\nu}$$
$$\equiv \binom{M+1+p^\nu-1}{p^\nu} \equiv a_\nu \not\equiv 0 \pmod{p}.$$

Therefore $p^\nu > L$ and so $\nu \geqq \sigma$. Now assume $M + 1 \equiv 0 (\mathrm{mod}\, p^\sigma)$. Then $\nu \geqq \sigma$ and for any $j < p^\sigma$ we have

$$\binom{M - pj + j}{j} \equiv \binom{M + 1 - pj + j - 1}{j} \equiv \binom{-pj + j - 1}{j} \equiv 0 \quad (\mathrm{mod}\, p)$$

as $-pj + j - 1 < 0$. This finishes the proof.

Finally we have

(3.17) THEOREM. *Suppose $n \geqq 2$ and $\mathbf{H}P^{n+s}/\mathbf{H}P^{s-1}$, $\mathbf{H}P^{n+t}/\mathbf{H}P^{t-1}$ are stably homotopically equivalent. If s satisfies both condition C_n and condition D_n then $\tilde{J}_{\mathbf{R}}(s\xi) = \tilde{J}_{\mathbf{R}}(t\xi)$ as elements of $\tilde{J}_{\mathbf{R}}(\mathbf{H}P^n)$.*

Proof. To prove this theorem we need only show that $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{n+1}$. We already know that either $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_n$ or $\epsilon_2 = \epsilon_3 = \ldots = \epsilon_{n+1}$ (and perhaps both).

If $\epsilon_2 = \epsilon_3 = \ldots = \epsilon_{n+1}$ then we use (3.14) with $i = 1$. Thus we want to find an odd prime $p \leqq 2n - 1$ and an integer $j \geqq 1$ such that

$$2 \leqq 1 + \tfrac{1}{2}(p - 1)j \leqq n + 1 \quad \text{and} \quad \binom{2s}{j} \not\equiv 0 (\mathrm{mod}\, p).$$

This is equivalent to s satisfying condition C_n.

On the other hand if $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_n$ then we use (3.14) for some i, j such that $i + \frac{1}{2}(p - 1)j = n + 1$. Therefore we need only find an odd prime $p \leqq 2n - 1$ and integers $i, j \geqq 1$ such that $i \leqq n$, $i + \frac{1}{2}(p - 1)j = n + 1$, and $\binom{2(i - 1 + s)}{j} \not\equiv 0 (\mathrm{mod}\, p)$. In other words we must find an odd prime $p \leqq 2n - 1$ and an integer j so that $1 \leqq j \leqq 2n/(p - 1)$ and

$$\binom{2n + 2s - pj + j}{j} \not\equiv 0 (\mathrm{mod}\, p).$$

By (3.16) this is equivalent to s satisfying condition D_n. This concludes the proof.

We could also apply our method to the problem of classifying stunted complex projective spaces $\mathbf{C}P^{n+s}/\mathbf{C}P^{s-1}$ up to stable homotopy type. Recall that the Atiyah-Todd number $M_{k+1}(\mathbf{C})$ is the order of $\tilde{J}_{\mathbf{R}}(\omega)$ in $\tilde{J}_{\mathbf{R}}(\mathbf{C}P^k)$, where ω is the Hopf complex line bundle over $\mathbf{C}P^k$. Feder and Gitler [**4**] have shown:

(3.18) THEOREM. *If n is odd and $\mathbf{C}P^{n+s}/\mathbf{C}P^{s-1}$, $\mathbf{C}P^{n+t}/\mathbf{C}P^{t-1}$ are stably homotopically equivalent, then $s \equiv t (\mathrm{mod}\, M_{n+1}(\mathbf{C}))$.*

Since this is a best possible result we shall not bother doing the number theory associated to the complex case. Therefore we conclude with a conjecture.

(3.19) *Conjecture*. Suppose $\mathbf{F}$ is one of $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ and suppose $(n, \mathbf{F}) \neq (1, \mathbf{H}), (2, \mathbf{C})$. Then $\mathbf{F}P^{n+s}/\mathbf{F}P^{s-1}$ and $\mathbf{F}P^{n+t}/\mathbf{F}P^{t-1}$ are stably homotopically equivalent if, and only if, $s \equiv t (\text{mod } M_{n+1}(\mathbf{F}))$.

By ad hoc homotopy arguments we can show that this conjecture is false for $(n, \mathbf{F}) = (1, \mathbf{H}), (2, \mathbf{C})$. However these counter-examples seem to be only low dimensional anomalies.

References

1. J. F. Adams, *On the groups $J(X)$, II*, Topology *3* (1965), 137–171.
2. J. F. Adams and G. Walker, *On complex stiefel manifolds*, Proc. Cambridge Phil. Soc. *61* (1965), 81–103.
3. M. Atiyah, *Thom complexes*, Proc. London Math. Soc. *11* (1961), 291–310.
4. S. Feder and S. Gitler, *Stable homotopy Types of stunted complex projective spaces* (to appear).
5. R. P. Held and D. Sjerve, *On the Spanier Conjecture*, Comment. Math. Helv. *47* (1972), 75–91.
6. I. James, *Spaces associated with Stiefel manifolds*, Proc. London Math. Soc. *9* (1959), 115–140.
7. D. Quillen, *The Adams Conjecture*, Topology *10* (1971), 67–80.
8. F. Sigrist and U. Suter, *On symplectic Stiefel manifolds* (to appear).
9. H. Toda, *Composition methods in the homotopy groups of spheres*, no. 49 (Princeton University Press, Princeton, N.J.).

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