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TOPOLOGICALLY FREE ACTIONS AND IDEALS IN DISCRETE C^*-DYNAMICAL SYSTEMS

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A C^*-dynamical system is called *topologically free* if the action satisfies a certain natural condition weaker than freeness. It is shown that if a discrete system is topologically free then the ideal structure of the crossed product algebra is related to that of the original algebra. One consequence is that a minimal topologically free discrete system has a simple reduced crossed product. Sharper results are obtained when the algebra is abelian.

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Let (A, G, α) be a C^*-dynamical system with A abelian and G discrete. In [4] it was shown that the reduced crossed-product $A \times_{\alpha r} G$ is simple if G acts minimally and if each automorphism α_t, $t \neq e$, is properly outer (see below for the definition). As a minimal action can be properly outer without being free, this was a significant advance from earlier results of Zeller-Meier and Effros-Hahn (see [9, Corollary 4.5] for naturally occurring examples). Elliott's result applies to actions of discrete groups on a larger class of C^*-algebras than abelian ones. In [8] a general version is proved using a condition (apparently) stronger than proper outerness. In [7] the discrete abelian case is studied. A condition on the ideals of the reduced crossed product is presented which corresponds to the dynamical condition of proper outerness. This is used to give another proof of simplicity of reduced crossed-products.

In this paper we present a somewhat simpler proof than that of [7] of the equivalence of these two conditions. In one direction, that the dynamical condition implies the condition on ideals, the argument is valid for arbitrary discrete systems. In fact, this implication is contained in the proof of [4] and [8] (using the apparently stronger condition of [8]); however, in the general case our argument is much simpler. We remark that no assumptions are made regarding separability, countability, or possession of an identity element. All ideal are assumed to be closed and two-sided.

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Definition 1. Let (A,G,α) be a C^*-dynamical system. We say that the action is *topologically free* if for any $t_1,\dots,t_n \in G\backslash\{e\}$, $\bigcap_{i=1}^n \{x \in \hat{A} \mid t_i x \neq x\}$ is dense in $\hat{A}$ (the spectrum of A).

In [4] an automorphism α of a C^*-algebra A is called *properly outer* if for any non-zero invariant ideal I of A and any inner automorphism β of I, $\|\alpha \mid I - \beta\| = 2$.

Proposition 1. *Let (A,G,α) be a C^*-dynamical system. If the action is topologically free then α_t is properly outer for all $t \neq e$.*

Proof. Suppose α_t is not properly outer for some $t \neq e$. Let $\{0\} \neq I$ be an ideal of A invariant under α_t, and $\beta \in \mathrm{Inn}(I)$ such that $\|\alpha_t \mid I - \beta\| < 2$. By [5], $\beta^{-1} \circ \alpha_t \mid I$ is the exponential of a derivation, and hence is implemented in every faithful representation π of I by a unitary operator in $\pi(I)''$. Applying this to the reduced atomic representation π_r of I, and noting that $\pi_r \beta \pi_r^{-1}$ is also implemented by a unitary operator in $\pi_r(I)''$, we see that $\pi \circ \alpha_t \cong \pi$ for every irreducible representation π of I. But then the open set $\hat{I} \subseteq \{x \in \hat{A} \mid tx = x\}$, and so the action is not topologically free. □

Remarks. (i) If $\hat{A}$ is Hausdorff then $\{x \in \hat{A} \mid tx \neq x\}$ is open for each $t \in G$. Hence the action is topologically free if and only if $\{x \in \hat{A} \mid tx \neq x\}$ is dense in $\hat{A}$ for each $t \in G\backslash\{e\}$. Thus, in this case, the action is topologically free if and only if $\{x \in \hat{A} \mid tx = x\}$ has empty interior for each $t \in G\backslash\{e\}$ (cf. [4, p. 300] and [7, Theorem 4.1A]).

(ii) If A is abelian, then the action is topologically free if and only if α_t is properly outer for each $t \neq e$.

The following lemma generalizes results from [6, p. 389] and [1, p. 304].

Lemma 1. *Let A and B be C^*-algebras with $B \subseteq A$. Let H be a Hilbert space and $\phi: A \to L(H)$ be a norm one completely positive map such that $\phi \mid B$ is multiplicative. Then for $a \in A$ and $b \in B$ we have $\phi(ab) = \phi(a)\phi(b)$ and $\phi(ba) = \phi(b)\phi(a)$.*

Proof. By Stinespring's theorem there exist a representation π of A on a Hilbert space H_1, and a contractive operator $V \in L(H, H_1)$, such that $\phi(x) = V^*\pi(x)V$ for $x \in A$. The multiplicativity of $\phi \mid B$ implies that $V^*\pi(x)(1 - VV^*)\pi(y)V = 0$ for $x, y \in B$, and hence that $(1 - VV^*)\pi(B)V = 0$. Thus for $x \in B$, $\pi(x)V = VV^*\pi(x)V = V\varphi(x)$. Thus $\phi(ab) = V^*\pi(ab)V = V^*\pi(a)\pi(b)V = V^*\pi(a)V\phi(b) = \phi(a)\phi(b)$, and similarly for the second equality. □

The next lemma develops an idea to be found in [3, Proposition 6.1] and [2, Remark 2.8(i)].

Lemma 2. *Let (A,G,α) be a C^*-dynamical system with G discrete. Let π be a non-degenerate representation of A, and let $\phi: A \times_\alpha G \to L(H_\pi)$ be any norm one completely* *positive map extending* π. *If* $t\in G$ *is such that* $\pi\circ\alpha_t^{-1}$ *and* π *are disjoint, then* $\phi(au_t)=0$ *for all* $a\in A$, *where* u_t *is the canonical image of* t *in* $M(A\times_\alpha G)$.

Proof. Fix $t\in G$ such that $\pi\circ\alpha_t^{-1}$ and π are disjoint. Then there is a bounded net $\{x_i\}\subseteq A$ such that $\pi(x_i)\to 1$ and $\pi\circ\alpha_t^{-1}(x_i)\to 0$ in the strong operator topology. If A is nonunital, the nondegeneracy of the representation $\phi|A$ implies that ϕ extends to a unital completely positive map, which we also denote by ϕ, of $\tilde{A}\times_\alpha G$. Then using Lemma 1, we have for $a\in A$,

$$\begin{aligned}\phi(au_t)&=s-\lim\phi(x_i)\phi(au_t)\\&=s-\lim\phi(x_iau_t)\\&=s-\lim\phi(u_t\alpha_t^{-1}(x_ia))\\&=s-\lim\phi(u_t)\phi(\alpha_t^{-1}(x_i))\phi(\alpha_t^{-1}(a))\\&=0.\end{aligned}$$

□

For any C^*-dynamical system (A,G,α) we let I_λ denote the kernel of the canonical surjection from $A\times_\alpha G$ onto $A\times_{\alpha r}G$.

Theorem 1. *Let* (A,G,α) *be a* C^**-dynamical system with* G *discrete. Suppose that the action is topologically free. If* I *is an ideal in* $A\times_\alpha G$ *such that* $I\cap A=\{0\}$ *then* $I\subseteq I_\lambda$.

Proof. Let I be an ideal in $A\times_\alpha G$ with $I\cap A=\{0\}$. Let $E:A\times_\alpha G\to A$ be the canonical conditional expectation. If $I\nsubseteq I_\lambda$ there exists $a\in I$ with $E(a)\neq 0$. Let $b\in C_c(G,A)$ with $\|a-b\|<\|E(a)\|/2$. We have $b=\sum_{s\in F}b_su_s$ where $F\subseteq G$ is finite. Let $X=\bigcap_{t\in F\setminus\{e\}}\{x\in\hat{A}\,|\,tx\neq x\}$. By hypothesis X is dense in $\hat{A}$. For any $[\pi]\in X$ let $\tilde{\pi}$ denote the composition

$$A+I\to\frac{A+I}{I}\cong\frac{A}{A\cap I}\cong A\xrightarrow{\pi}L(H_\pi).$$

Let $\phi:A\times_\alpha G\to L(H_\pi)$ be any completely positive map extending $\tilde{\pi}$ (for example we may take ϕ to be the compression to H_π of an extension to $A\times_\alpha G$ of the irreducible representation $\tilde{\pi}$). Since A contains an approximate identity for $A\times_\alpha G$, $\|\phi\|=1$. Since $[\pi]\in X$ it follows that

$$\begin{aligned}\phi(b)&=\sum_{s\in F}\phi(b_su_s)\\&=\phi(b_e),\quad\text{by Lemma 2,}\end{aligned}$$

$$=\phi(E(b)).$$

Hence $\|\pi(E(b))\|=\|\phi(E(b))\|=\|\phi(b)\|=\|\phi(b-a)\|\leqq\|b-a\|$. This holds for all $[\pi]\in X$, which is dense in $\hat{A}$. Therefore $\|E(b)\|\leqq\|b-a\|$. We derive the contradiction

$$\|E(a)\|\leqq\|E(a-b)\|+\|E(b)\|\leqq 2\|a-b\|<\|E(a)\|. \qquad \square$$

Corollary. *Let (A,G,α) be a C^*-dynamical system with G discrete. If the action is topologically free and minimal (A contains no nontrivial α-invariant ideals) then $A\times_{\alpha r}G$ is simple.*

Remark. It follows from the proof of [**8**, Theorem 2.1] that if A is separable, then topological freeness is a weaker condition than the strong Connes spectrum condition used in [**8**]. Thus for separable algebras, topological freeness lies between Kishimoto's condition and proper outerness. In the following example, topological freeness is quite easy to establish.

Proposition 2. *Let S be an infinite set, let G be a discrete group, and let $G\times S\to S$ be an effective action (the only element of G fixing all points of S is the identity). Let α be the corresponding action of G on $C^*(\mathbf{F}(S))$, where $\mathbf{F}(S)$ is the free group on the elements of S. Then $(C^*(\mathbf{F}(S)),G,\alpha)$ is topologically free, but not free.*

Proof. Let $t_1,\ldots,t_n\in G\backslash\{e\}$, and let $x_j\in S$ be such that $t_jx_j\neq x_j$. Let $\lambda:S\to\{z\in\mathbf{C}\mid |z|=1\}$ be any function such that $\lambda(x_j)\neq\lambda(t_jx_j)$, $1\leqq j\leqq n$. Let π be any irreducible representation of $C^*(\mathbf{F}(S))$. Let $\{P_i\}_{i\in I}$ be an increasing net of finite rank projections in $L(H_\pi)$ tending to 1, and let $\mathscr{F}=\{F\subseteq S\mid |F|<\infty,\ \{x_1,t_1x_1,\ldots,x_n,t_nx_n\}\subseteq F\}$. For any Hilbert space contraction T, let $u(T)$ be a block unitary operator on $H_T\oplus H_T$ whose upper left block is T. Let H be a fixed infinite dimensional separable Hilbert space. For $F\in\mathscr{F}$ and $i\in I$ let $\pi_{F,i}$ be a representation of $C^*(\mathbf{F}(S))$ on $H_i=P_iH_\pi\oplus P_iH_\pi\oplus H$ such that $\pi_{F,i}(s)=u(P_i\pi(s)\mid P_iH_\pi)\oplus\lambda(s)1_H$ for $s\in F$, and such that $\{\pi_{F,i}(s)\mid s\notin F\}$ is an irreducible family of unitaries on H_i. By construction $\pi_{F,i}$ is irreducible, and $\pi_{F,i}$ converges to π in $C^*(\mathbf{F}(S))^\wedge$. For $1\leqq j\leqq n$, the only eigenvalue of infinite order for $\pi_{F,i}(x_j)$ (respectively $\pi_{F,i}\circ\alpha_{t_j}(x_j)$) is $\lambda(x_j)$ (respectively $\lambda(t_jx_j)$). Since $\lambda(x_j)\neq\lambda(t_jx_j)$, it follows that $\pi_{F,i}$ and $\pi_{F,i}\circ\alpha_{t_j}$ are not unitarily equivalent, and hence are disjoint. Thus $(C^*(\mathbf{F}(S)),G,\alpha)$ is topologically free.

It is easily seen that the action in the above example is not free. For example, in the Hilbert space $\mathbf{C}^2$, let $\xi_\theta=(\cos\theta,\sin\theta)$, let P_θ be the rank-one projection $\xi_\theta\otimes\xi_\theta^*$, let R_θ be the unitary operator such that $R_\theta\xi_\phi=\xi_{\theta+\phi}$, and let $T_\theta=2P_\theta-1$. Fix $t\in G\backslash\{e\}$ and $x\in S$ with $tx\neq x$. Let $n=|\{t^ix\mid i\in\mathbf{Z}]|$. Set

$$V=\begin{cases}T_{\pi/3}, & n=2 \text{ or } n=\infty\\ R_{\pi/n}, & 2<n<\infty.\end{cases}$$

Define $\sigma:C^*(\mathbf{F}(S))\to L(\mathbf{C}^2)$ by $\sigma(t^ix)=V^iT_0V^{-i}$ for $i\in\mathbf{Z}$, and $\sigma(s)=1$ for $s\in S\backslash\{t^ix\mid i\in\mathbf{Z}\}$.

Since $\sigma(C^*(\mathbf{F}(S)))$ contains P_0 and P_θ for some $0<\theta<\pi/2$, σ is irreducible. By construction, α_t fixes $[\sigma]$. □

Theorem 2. (*cf.* [7, *Theorem 4.1*]). *Let (A,G,α) be a C^*-dynamical system with A abelian and G discrete. The following are equivalent:*

(i) *The action of G on A is topologically free.*

(ii) *If I is an ideal in $A\times_\alpha G$ with $I\cap A=\{0\}$ then $I\subseteq I_\lambda$.*

Proof. The implication (i)⇒(ii) follows from Theorem 1. The proof of (ii)⇒(i) essentially appears in [7]. For completeness we present the following brief proof, which is more self-contained than that in [7].

For each $x\in\hat{A}$ we define a representation π_x of $A\times_\alpha G$ on $l^2(Gx)$, with standard orthonormal basis $\{\delta_{tx}\mid t\in G\}$, by the (covariant) formulas

$$\pi_x(f)\delta_{tx}=f(tx)\delta_{tx},\quad f\in A\cong C_0(\hat{A}),$$

$$\pi_x(u_r)\delta_{tx}=\delta_{rtx}.$$

(Actually, π_x is the representation of $A\times_\alpha G$ induced from the one-dimensional representation of $A\times G_x$ given by evaluation at x, where G_x is the stability subgroup of G at x.) Let $I=\bigcap\{\ker\pi_x\mid x\in\hat{A}\}$. It is clear that $A\cap I=\{0\}$. Let $s\in G\backslash\{e\}$ and let $f\in A$ with $\operatorname{supp}(f)\subseteq\{x\in\hat{A}\mid sx=x\}$. If $tx\in\operatorname{supp}(f)$ then $stx=tx$, and

$$\pi_x(f-fu_s)\delta_{tx}=f(tx)\delta_{tx}-f(stx)\delta_{stx}=0.$$

If $tx\notin\operatorname{supp}(f)$, then $stx\notin\operatorname{supp}(f)$, and again $\pi_x(f-fu_s)\delta_{tx}=0$. Therefore $f-fu_s\in I$. By hypothesis $f-fu_s\in I_\lambda$, and hence $f=E(f-fu_s)=0$. It follows that $\operatorname{int}\{x\in\hat{A}\mid sx=x\}=\emptyset$, and (i) is established. □

Remark. If we drop the requirement that A be abelian then the implication (ii)⇒(i) may fail. For example, let u and v be unitary operators on a Hilbert space H such that $uvu^{-1}v^{-1}=\omega 1$, where $\omega=e^{2\pi i\theta}$ and $\theta\in(0,1)\backslash\mathbf{Q}$. Then Ad$u$ and Adv determine an action of $\mathbf{Z}\oplus\mathbf{Z}$ on the algebra K of compact operators on H. We will show that the crossed product is simple, and hence that (ii) holds, while it is clear that the action is not topologically free.

Let (π,σ) be a covariant representation of $(K,\mathbf{Z}\oplus\mathbf{Z})$. Then $\pi\cong\mathrm{id}\otimes 1$. Since $\sigma(1,0)(u^*\otimes 1)\in(K\otimes 1)'$ it follows that $\sigma(1,0)=u\otimes u'$ for some unitary operator u'. Similary, $\sigma(0,1)=v\otimes v'$. Since $\mathbf{Z}\oplus\mathbf{Z}$ is abelian we must have $u'v'(u')^{-1}(v')^{-1}=\bar{\omega}1$. Therefore $(\pi\times\sigma)(K\times\mathbf{Z}\oplus\mathbf{Z})=K\otimes A_{1-\theta}$, where $A_{1-\theta}$ is the 'irrational rotation' algebra, which is well-known to be simple.

The following corollary is a more accurate statement than Theorem 4.4 of [7]. In [9] the term *regular* was introduced to describe a C^*-dynamical system for which the full and reduced crossed products are canonically isomorphic (i.e. for which $I_\lambda = \{0\}$). By [**11**], actions of amenable groups are always regular.

Corollary. *Let* (A, G, α) *be a* C^**-dynamical system with* A *abelian and* G *discrete. Then* $A \times_\alpha G$ *is simple if and only if the action is minimal, topologically free, and regular.*

Examples of actions of nonamenable groups for which the equivalent conditions of the above corollary hold appear in [**9**] and [**10**].

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