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HEAT KERNEL METHOD FOR THE LEVI-CIVITÁ EQUATION IN DISTRIBUTIONS AND HYPERFUNCTIONS

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Abstract

Let G be a commutative group and $\mathbb{C}$ the field of complex numbers, $\mathbb{R}^+$ the set of positive real numbers and $f, g, h, k : G \times \mathbb{R}^+ \to \mathbb{C}$. In this paper, we first consider the Levi-Civitá functional inequality

$$|f(x+y, t+s) - g(x,t)h(y,s) - k(y,s)| \le \Phi(t,s), \quad x, y \in G, t, s > 0,$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a symmetric decreasing function in the sense that $\Phi(t_2, s_2) \le \Phi(t_1, s_1)$ for all $0 < t_1 \le t_2$ and $0 < s_1 \le s_2$. As an application, we solve the Hyers–Ulam stability problem of the Levi-Civitá functional equation

$$u \circ S - v \otimes w - k \circ \Pi \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \quad [\text{respectively } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})]$$

in the space of Gelfand hyperfunctions, where u, v, w, k are Gelfand hyperfunctions, $S(x,y) = x + y, \Pi(x,y) = y, x, y \in \mathbb{R}^n$, and $\circ, \otimes, \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ and $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$ denote pullback, tensor product and the spaces of bounded distributions and bounded hyperfunctions, respectively.

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1. Introduction

A certain formula or equation is appropriate to model a physical process if a small change in the formula or equation gives rise to a small change in the corresponding result. When this happens we say the formula or equation is stable. In an application, a functional equation like the additive Cauchy functional equation $f(x+y) - f(x) - f(y) = 0$ may not be true for all $x, y \in \mathbb{R}$ but it may be true approximately, that is,

$$f(x+y) - f(x) - f(y) \approx 0$$

for all $x, y \in \mathbb{R}$. This can be stated mathematically as

$$|f(x+y) - f(x) - f(y)| \le \varepsilon$$

for some small positive ε and for all $x, y \in \mathbb{R}$. We would like to know when small changes in a particular equation like the additive Cauchy functional equation have only small effects on its solutions. This is the essence of stability theory.

In 1940, Ulam asked the following question: given a group G, a metric group H with metric $d(\cdot,\cdot)$ and a positive number ϵ, does there exist a $\delta > 0$ such that if $f : G \to H$ satisfies

$$d(f(xy), f(x)f(y)) \leq \varepsilon$$

for all $x, y \in G$, then a homomorphism $\phi : G \to H$ exists with

$$d(f(x), \phi(x)) \leq \delta$$

for all $x \in G$? These kinds of questions form the material for the stability theory of functional equations (see [13]). For Banach spaces, Ulam's problem was solved by Hyers in 1941 [10] with $\delta = \epsilon$ and the additive map

$$\phi(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

In this paper, we consider the functional equation

$$f(x + y) = g(x)h(y) + k(y) \tag{1.1}$$

in the space of Schwartz distributions and Gelfand hyperfunctions. Equation (1.1) is a special case of the Levi-Civitá functional equation

$$f(x + y) = g_1(x)h_1(y) + g_2(x)h_2(y) + \cdots + g_n(x)h_n(y), \tag{1.2}$$

which was studied by Levi-Civitá in [11] under differentiability conditions. The Levi-Civitá functional equation (1.2) was recently studied by Ebanks in [6] on nonabelian groups. The stability of the Levi-Civitá functional equation was investigated by Shulman on locally compact groups in [15]. In [3], we also studied the Ulam–Hyers stability of the functional equation (1.1) on nonunital commutative semigroups G, that is, we investigated the behaviour of $f, g, h, k : G \to \mathbb{C}$ satisfying the functional inequality

$$|f(x + y) - g(x)h(y) - k(y)| \leq M \quad \forall x, y \in G \tag{1.3}$$

for some $M > 0$.

In 1950, Schwartz introduced the theory of distributions in his monograph *Théorie des distributions* (see [14]). In this book, Schwartz systematises the theory of generalised functions, basing it on the theory of linear topological spaces, and obtains many important results. After his elegant theory appeared, many important concepts and results on the classical spaces of functions have been generalised to the space of distributions. For example, the space $L^\infty(\mathbb{R}^n)$ of bounded measurable functions on $\mathbb{R}^n$ has been generalised to the space $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ of *bounded distributions* as a subspace of distributions and later the space $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ was further generalised to the space $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ of *bounded hyperfunctions*. Also, positive functions and positive-definite functions have been generalised to *positive distributions and positive-definite distributions*,

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respectively, and it was shown that every positive distribution is a positive measure [9, page 38] and every positive-definite distribution is the Fourier transform of a positive measure μ such that $\int(1 + |x|)^{-p}\, d\mu < \infty$ for some $p \geq 0$ [8, page 157]. This is the *Bochner–Schwartz theorem* and is a natural generalisation of the famous *Bochner theorem* stating that every positive-definite function is the Fourier transform of a positive finite measure.

As in [1] and [2], using the pullback and the tensor product of distributions, we generalise the functional inequality (1.3) as follows:

$$u \circ S - v \otimes w - k \circ \Pi \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \quad [\text{respectively } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})], \tag{1.4}$$

where $\circ$ and $\otimes$ are the pullback and the tensor product of the generalised functions u, v, w, k (see Section 2), respectively.

The main tool for controlling (1.4) is *the heat kernel method* initiated by Matsuzawa [12], which represents the generalised functions in some class as the initial values of solutions of the heat equation with appropriate growth conditions [5, 12]. Making use of the heat kernel method, we can convert (1.4) to the following classical Hyers–Ulam stability problem: there exist $C > 0$ and $N \geq 0$ (respectively, for every $\epsilon > 0$, there exists $C_\epsilon > 0$) such that

$$|\tilde{u}(x + y, t + s) - \tilde{v}(x, t)\tilde{w}(y, s) - \tilde{k}(y, s)| \leq C\Big(\frac{1}{t} + \frac{1}{s}\Big)^N \quad [\text{respectively } C_\epsilon e^{\epsilon(1/t+1/s)}] \tag{1.5}$$

for all $x, y \in \mathbb{R}^n, t, s > 0$, where $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{k} : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ are solutions of the heat equation whose initial values are u, v, w, k, respectively. In Section 3, we consider the stability problem (1.5) in a more general setting, combined with the heat kernel method [5, 12], to solve the stability problem for (1.4).

2. Bounded distributions and hyperfunctions

We first introduce the spaces $\mathcal{S}'$ of Schwartz tempered distributions and $\mathcal{G}'$ of Gelfand hyperfunctions (see [7–9, 12, 14] for more details of these spaces). We use the notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = (\partial/\partial x_j)$.

Definition 2.1 [14]. We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{\alpha,\,\beta} = \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions and the elements of the dual space $\mathcal{S}'$ are called *tempered distributions*.

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Definition 2.2 [7, 8]. We denote by $\mathcal{G}$ or $\mathcal{G}(\mathbb{R}^n)$ the Gelfand space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n,\ \alpha,\ \beta\in\mathbb{N}_0^n} \frac{|x^\alpha \partial^\beta \varphi(x)|}{h^{|\alpha|}k^{|\beta|}\alpha!^{1/2}\beta!^{1/2}} < \infty$$

for some $h, k > 0$. We say that $\varphi_j \longrightarrow 0$ as $j \to \infty$ if $\|\varphi_j\|_{h,k} \longrightarrow 0$ as $j \to \infty$ for some h, k, and denote by $\mathcal{G}'$ the dual space of $\mathcal{G}$ and call its elements Gelfand hyperfunctions.

As a generalisation of the space L^∞ of bounded measurable functions, Schwartz introduced the space $\mathcal{D}'_{L^\infty}$ of bounded distributions as a subspace of tempered distributions.

Definition 2.3 [14]. We denote by $\mathcal{D}_{L^1}(\mathbb{R}^n)$ the space of smooth functions on $\mathbb{R}^n$ such that $\partial^\alpha\varphi \in L^1(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ equipped with the topology defined by the countable family of seminorms

$$\|\varphi\|_m = \sum_{|\alpha|\le m} \|\partial^\alpha \varphi\|_{L^1}, \quad m \in \mathbb{N}_0.$$

We denote by $\mathcal{D}'_{L^\infty}$ the dual space of $\mathcal{D}_{L^1}$ and call its elements bounded distributions.

Generalising bounded distributions, the space $\mathcal{A}'_{L^\infty}$ of bounded hyperfunctions has been introduced as a subspace of $\mathcal{G}'$.

Definition 2.4 [5]. We denote by $\mathcal{A}_{L^1}$ the space of smooth functions on $\mathbb{R}^n$ satisfying

$$\|\varphi\|_h = \sup_\alpha \frac{\|\partial^\alpha\varphi\|_{L^1}}{h^{|\alpha|}\alpha!} < \infty$$

for some constant $h > 0$. We say that $\varphi_j \to 0$ in $\mathcal{A}_{L^1}$ as $j \to \infty$ if there is a positive constant h such that

$$\sup_\alpha \frac{\|\partial^\alpha\varphi_j\|_{L^1}}{h^{|\alpha|}\alpha!} \to 0 \quad \text{as } j \to \infty.$$

We denote by $\mathcal{A}'_{L^\infty}$ the dual space of $\mathcal{A}_{L^1}$.

It is well known that the following topological inclusions hold:

$$\begin{aligned} &\mathcal{G} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{D}_{L^1}, \qquad && D'_{L^\infty} \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{G}', \\ &\mathcal{G} \hookrightarrow \mathcal{A}_{L^1} \hookrightarrow \mathcal{D}_{L^1}, && \mathcal{D}'_{L^\infty} \hookrightarrow \mathcal{A}'_{L^\infty} \hookrightarrow \mathcal{G}'. \end{aligned}$$

It is known that the space $\mathcal{G}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ which can be extended to an entire function on $\mathbb{C}^n$ satisfying

$$|\varphi(x+iy)| \le C\exp(-a|x|^2 + b|y|^2), \quad x, y \in \mathbb{R}^n \tag{2.1}$$

for some $a, b, C > 0$ (see [7]).

By virtue of [9, page 134, Theorem 6.12], we have the following definition.

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DEFINITION 2.5. Let $u_j \in \mathcal{G}'(\mathbb{R}^{n_j})$ for $j = 1, 2$, with $n_1 \geq n_2$, and let $\lambda : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ be a smooth function such that for each $x \in \mathbb{R}^{n_1}$, the Jacobian matrix $\nabla\lambda(x)$ of λ at x has rank n_2. Then there exists a unique continuous linear map $\lambda^* : \mathcal{G}'(\mathbb{R}^{n_2}) \to \mathcal{G}'(\mathbb{R}^{n_1})$ such that $\lambda^* u = u \circ \lambda$ when u is a continuous function. We call $\lambda^* u$ the pullback of u by λ and denote it by $u \circ \lambda$.

In particular, let $S : \mathbb{R}^{2n} \to \mathbb{R}^n$ be defined by $S(x, y) = x + y$ for $x, y \in \mathbb{R}^n$. In view of the proof of [9, page 134, Theorem 6.12],

$$\langle u \circ S, \varphi(x, y)\rangle = \Big\langle u, \int \varphi(x - y, y)\, dy\Big\rangle.$$

DEFINITION 2.6. Let $u_j \in \mathcal{G}'(\mathbb{R}^{n_j})$ for $j = 1, 2$. Then the tensor product $u_1 \otimes u_2$ of u_1 and u_2 is defined by

$$\langle u_1 \otimes u_2, \varphi(x_1, x_2)\rangle = \langle u_1, \langle u_2, \varphi(x_1, x_2)\rangle\rangle$$

for $\varphi(x_1, x_2) \in \mathcal{G}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and belongs to $\mathcal{G}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

For more details on the pullback and the tensor product of distributions, we refer the reader to [9, Chs V–VI].

3. Stability of the Levi-Civitá functional equation with time variables

Let G be a commutative group, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{C}$ the set of complex numbers and $f, g, h, k : G \times \mathbb{R}^+ \to \mathbb{C}$. In this section, we investigate the behaviour of functions $f, g, h, k : G \times \mathbb{R}^+ \to \mathbb{C}$ satisfying the functional inequality

$$|f(x + y, t + s) - g(x, t)h(y, s) - k(y, s)| \leq \Phi(t, s) \tag{3.1}$$

for all $x, y \in G, t, s > 0$, where $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the conditions

$$\Phi(t_2, s_2) \leq \Phi(t_1, s_1), \tag{3.2}$$

$$\Phi(t, s) = \Phi(s, t) \tag{3.3}$$

for all $0 < t_1 \leq t_2, 0 < s_1 \leq s_2, t, s > 0$.

In the following theorem, we exclude the cases when g or h is constant.

THEOREM 3.1. *Let $f, g, h, k : G \times \mathbb{R}^+ \to \mathbb{C}$ satisfy the functional inequality (3.1). Then either there exist positive constants C_1, C_2, C_3, C_4 and $\delta > 0$ such that*

$$\begin{aligned} &|h(x, t)| \leq C_1\Phi(t, t), \quad |g(x, t)| \leq C_2\Phi(t, t),\\ &|k(x, t)| \leq C_3\Phi(t, t), \quad |f(x, 2t)| \leq C_4\Phi(t, t)^2 \end{aligned} \tag{3.4}$$

for all $x \in G, 0 < t < \delta$, or else

$$\begin{aligned} h(x, t) &= \beta m_1(t)m_2(x),\\ g(x, t) &= \alpha\beta m_1(t)m_2(x) + \gamma,\\ |k(x, t) + \beta\gamma m_1(t)m_2(x) - \mu| &\leq 2\Phi(t, t),\\ |f(x, t) - \alpha\beta^2 m_1(t)m_2(x) - \mu| &\leq 3\Phi\Big(\frac{t}{2}, \frac{t}{2}\Big) \end{aligned} \tag{3.5}$$

for all $x \in G, 0 < t < \delta$, where $\alpha, \beta, \gamma, \mu \in \mathbb{C}$, $m_1 : \mathbb{R}^+ \to \mathbb{C}$ and $m_2 : G \to \mathbb{C}$ are exponential functions.

PROOF. Fix $y_0 \in G$, $s_0 > 0$ and let

$$D(x, y, t, s) = f(x + y, t + s) - g(x, t)h(y, s) - k(y, s).$$

Then

$$\begin{aligned} &D(y, x, s, t) - D(x, y, t, s) - D(y_0, x, s_0, t) + D(x, y_0, t, s_0) \\ &\quad = (h(y, s) - h(y_0, s_0))g(x, t) - (g(y, s) - g(y_0, s_0))h(x, t) + k(y, s) - k(y_0, s_0). \end{aligned}$$

Thus, using the triangle inequality and (3.3),

$$\begin{aligned} &|(h(y, s) - h(y_0, s_0))g(x, t) - (g(y, s) - g(y_0, s_0))h(x, t)| \\ &\quad \le |k(y, s) - k(y_0, s_0)| + |D(y, x, s, t)| + |D(x, y, t, s)| \\ &\qquad + |D(y_0, x, s_0, t)| + D(x, y_0, t, s_0)| \\ &\quad \le |k(y, s) - k(y_0, s_0)| + 2\Phi(t, s) + 2\Phi(t, s_0) \end{aligned} \tag{3.6}$$

for all $x, y \in G, t, s > 0$.

Choosing $y_1 \in G$, $s_1 > 0$ such that $h(y_1, s_1) - h(y_0, s_0) \neq 0$, putting $y = y_1$, $s = s_1$ in (3.6) and dividing the result by $|h(y_1, s_1) - h(y_0, s_0)|$,

$$|g(x, t)| \le b_1|h(x, t)| + c_1(\Phi(t, s_1) + \Phi(t, s_0)) + d_1, \tag{3.7}$$

where

$$b_1 = \left|\frac{g(y_1, s_1) - g(y_0, s_0)}{h(y_1, s_1) - h(y_0, s_0)}\right|, \quad c_1 = \frac{2}{|h(y_1, s_1) - h(y_0, s_0)|}, \quad d_1 = \left|\frac{k(y_1, s_1) - k(y_0, s_0)}{h(y_1, s_1) - h(y_0, s_0)}\right|.$$

From (3.2),

$$\Phi(t, s_1) + \Phi(t, s_0) \le 2\Phi(t, t) \tag{3.8}$$

for all $0 < t < \delta_1 := \min\{s_1, s_0\}$. Thus, from (3.7) and (3.8),

$$|g(x, t)| \le b_1|h(x, t)| + 2c_1\Phi(t, t) + d_1 \tag{3.9}$$

for all $x \in G, 0 < t < \delta_1$. Similarly, there exist $b_2, c_2, d_2 \ge 0$ and $\delta_2 > 0$ such that

$$|h(x, t)| \le b_2|g(x, t)| + 2c_2\Phi(t, t) + d_2 \tag{3.10}$$

for all $x \in G, 0 < t < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and assume that there exists $C_1 > 0$ such that

$$\Phi(t, t)^{-1}|h(x, t)| \le C_1$$

for all $x \in G, 0 < t < \delta$. Then, dividing both sides of (3.9) by $\Phi(t, t)$,

$$\begin{aligned} \Phi(t, t)^{-1}|g(x, t)| &\le b_1\Phi(t, t)^{-1}|h(x, t)| + 2c_1 + d_1\Phi(t, t)^{-1} \\ &\le b_1C_1 + 2c_1 + d_1\Phi(\delta, \delta)^{-1} := C_2 \end{aligned}$$

for all $x \in G, 0 < t < \delta$. Thus, $\Phi(t, t)^{-1}g(x, t)$ is bounded in $G \times (0, \delta)$. Now, using the triangle inequality,

$$\begin{aligned} &|h(y_0, s_0)g(x, t) - g(y_0, s_0)h(x, t) - k(x, t) + k(y_0, s_0)| \\ &\quad = |D(y_0, x, s_0, t) - D(x, y_0, t, s_0)| \le 2\Phi(t, s_0) \le 2\Phi(t, t) \end{aligned} \tag{3.11}$$

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for all $x \in G, 0 < t < \delta$. Dividing both sides of (3.11) by $\Phi(t,t)$ and using the triangle inequality again,

$$\begin{aligned}
&\Phi(t,t)^{-1}|k(x,t)| \\
&\quad \le \Phi(t,t)^{-1}|h(y_0,s_0)g(x,t) - g(y_0,s_0)h(x,t) + k(y_0,s_0)| + 2 \\
&\quad \le \Phi(t,t)^{-1}(|h(y_0,s_0)||g(x,t)| + |g(y_0,s_0)||h(x,t)| + |k(y_0,s_0)|) + 2 \\
&\quad \le C_2|h(y_0,s_0)| + C_1|g(y_0,s_0)| + \Phi(\delta,\delta)^{-1}|k(y_0,s_0)| + 2 := C_3
\end{aligned}$$

for all $x \in G, 0 < t < \delta$. Putting $y = 0$ and $s = t$ in (3.1), dividing the result by $\Phi(t,t)^2$ and using the triangle inequality,

$$\begin{aligned}
\Phi(t,t)^{-2}|f(x,2t)| &\le \Phi(t,t)^{-2}(|g(x,t)||h(0,t)| + |k(0,t)|) + \Phi(t,t) \\
&\le C_1C_2 + (C_3+1)\Phi(t,t)^{-1} \\
&\le C_1C_2 + (C_3+1)\Phi(\delta,\delta)^{-1} := C_4
\end{aligned}$$

for all $x \in G, 0 < t < \delta$. Thus, we obtain (3.4).

Now we assume that $\Phi(t,t)^{-1}h(x,t)$ is unbounded in $G \times (0,\epsilon)$ for all $\epsilon \le \delta$. Then it follows from (3.10) that $\Phi(t,t)^{-1}g(x,t)$ is unbounded in $G \times (0,\epsilon)$ for all $\epsilon \le \delta$. Dividing both sides of (3.6) by $|h(y,s) - h(y_0,s_0)|$,

$$|g(x,t) - \alpha(y,s)h(x,t)| \le |\gamma(y,s)| + \frac{2\Phi(t,s) + 2\Phi(t,s_0)}{|h(y,s) - h(y_0,s_0)|} \tag{3.12}$$

for all $(x,t) \in G \times \mathbb{R}^+$ and $(y,s) \in J := \{(y,s) \in G \times \mathbb{R}^+ : h(y,s) \neq h(y_0,s_0)\}$, where

$$\alpha(y,s) = \frac{g(y,s) - g(y_0,s_0)}{h(y,s) - h(y_0,s_0)} \quad \text{and} \quad \gamma(y,s) = \frac{k(y,s) - k(y_0,s_0)}{h(y,s) - h(y_0,s_0)}.$$

Replacing (y,s) by (y_1,s_1) and (y,s) by (y_2,s_2) in (3.12) gives two inequalities. Using the triangle inequality and these two inequalities,

$$\begin{aligned}
&|\alpha(y_1,s_1) - \alpha(y_2,s_2)||h(x,t)| \\
&\quad \le |\gamma(y_1,s_1)| + |\gamma(y_2,s_2)| + \frac{2\Phi(t,s_1) + 2\Phi(t,s_0)}{|h(y_1,s_1) - h(y_0,s_0)|} \\
&\qquad + \frac{2\Phi(t,s_2) + 2\Phi(t,s_0)}{|h(y_2,s_2) - h(y_0,s_0)|}
\end{aligned} \tag{3.13}$$

for all $(x,t) \in G \times \mathbb{R}^+$. Dividing both sides of (3.13) by $\Phi(t,t)$,

$$\begin{aligned}
&|\alpha(y_1,s_1) - \alpha(y_2,s_2)|\Phi(t,t)^{-1}|h(x,t)| \\
&\quad \le \Phi(t,t)^{-1}|\gamma(y_1,s_1)| + \Phi(t,t)^{-1}|\gamma(y_2,s_2)| \\
&\qquad + \frac{4}{|h(y_1,s_1) - h(y_0,s_0)|} + \frac{4}{|h(y_2,s_2) - h(y_0,s_0)|}
\end{aligned} \tag{3.14}$$

for all $x \in G, 0 < t < \rho := \min\{s_0, s_1, s_2\}$. Since the right-hand side of (3.14) is bounded on $(0,\rho)$ and $\Phi(t,t)^{-1}|h(x,t)|$ is unbounded on $G \times (0,\rho)$,

$$\alpha(y_1,s_1) = \alpha(y_2,s_2).$$

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Thus, $\alpha(y, s) := \alpha$ is independent of $(y, s) \in J := \{(y, s) \in G \times \mathbb{R}^+ : h(y, s) \neq h(y_0, s_0)\}$ and

$$g(y, s) - g(y_0, s_0) = \alpha(h(y, s) - h(y_0, s_0)) \tag{3.15}$$

for all $(y, s) \in J$. Now we show that $g(y, s) - g(y_0, s_0) = 0$ if and only if $h(y, s) - h(y_0, s_0) = 0$. Assume that $g(y_1, s_1) - g(y_0, s_0) = 0$ and $h(y_1, s_1) - h(y_0, s_0) \neq 0$ for some $y_1 \in G, s_1 > 0$. Then, since $b_1 = 0$ in (3.9),

$$\Phi(t, t)^{-1}|g(x, t)| \leq 2c_1 + d_1\Phi(\delta, \delta)^{-1}$$

for all $x \in G, 0 < t < \delta$, which gives a contradiction since $\Phi(t, t)^{-1}|g(x, t)|$ is unbounded on $G \times (0, \delta)$. Thus, $g(y_1, s_1) - g(y_0, s_0) = 0$ implies that $h(y, s) - h(y_0, s_0) = 0$. By changing the role of g and h, we can prove that $h(y_1, s_1) - h(y_0, s_0) = 0$ implies that $g(y_1, s_1) - g(y_0, s_0) = 0$. Thus, it follows from (3.15) that

$$g(y, s) = \alpha h(y, s) + \gamma \tag{3.16}$$

for all $(y, s) \in G \times \mathbb{R}^+$, where $\gamma = g(y_0, s_0) - \alpha h(y_0, s_0)$. Putting (3.16) in (3.1),

$$|f(x + y, t + s) - \alpha h(x, t)h(y, s) - \gamma h(y, s) - k(y, s)| \leq \Phi(t, s). \tag{3.17}$$

Replacing (x, y, t, s) by (y, x, s, t) in (3.17), using the triangle inequality and putting $y = y_0, s = s_0$,

$$|k(x, t) + \gamma h(x, t) - \mu| \leq 2\Phi(t, s_0) \leq 2\Phi(t, t) \tag{3.18}$$

for all $x \in G, 0 < t < \delta$, where $\mu = \gamma k(y_0, s_0) + h(y_0, s_0)$.

Using the triangle inequality with (3.1) and (3.18),

$$|f(x + y, 2t) - \alpha h(x, t)h(y, t) - \mu| \leq 3\Phi(t, t) \tag{3.19}$$

for all $x, y \in G, 0 < t < \delta$.

Now we investigate h. Since $\Phi(t, t)^{-1}g(x, t)$ is unbounded in $G \times (0, \epsilon)$ for all $0 < \epsilon \leq \delta$, we can choose a sequence $(z_n, r_n) \in G \times (0, \delta), n = 1, 2, 3, \ldots,$ such that $r_n \to 0$ and $\Phi(r_n, r_n)|g(z_n, r_n)|^{-1} \to 0$ as $n \to \infty$. Putting $x = z_n, t = r_n$ in (3.1), dividing the result by $|g(z_n, r_n)|$ and using the triangle inequality,

$$\begin{aligned}\left|h(y, s) - \frac{f(z_n + y, r_n + s)}{g(z_n, r_n)}\right| &\leq \frac{\Phi(r_n, s) + |k(y, s)|}{|g(z_n, r_n)|} \\ &\leq \frac{\Phi(r_n, r_n) + |k(y, s)|}{|g(z_n, r_n)|}\end{aligned} \tag{3.20}$$

for all $y \in G, s > 0$ and $r_n < s$. Letting $n \to \infty$ in (3.20),

$$h(y, s) = \lim_{n\to\infty} \frac{f(z_n + y, r_n + s)}{g(z_n, r_n)} \tag{3.21}$$

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for all $y \in G$, $s > 0$. Thus, it follows from (3.1) and (3.21) that

$$\begin{aligned} h(x+y,t+s)h(z,r) &= \lim_{n\to\infty} \frac{f(z_n+x+y, r_n+t+s)h(z,r)}{g(z_n,r_n)} \\ &= \lim_{n\to\infty} \frac{h(x,t)g(z_n+y, r_n+s)h(z,r)+R_1}{g(z_n,r_n)} \\ &= \lim_{n\to\infty} \frac{h(x,t)f(z_n+y, r_n+s+r)+R_1+R_2}{g(z_n,r_n)} \\ &= h(x,t)h(y,s+r) + \lim_{n\to\infty} \frac{R_1+R_2}{g(z_n,r_n)}. \end{aligned} \tag{3.22}$$

Since $|R_1| \le |h(z,r)|(\Phi(r_n+s,t)+|k(x,t)|)$, $|R_2| \le |h(x,t)|(\Phi(r_n+s,r)+|k(z,r)|)$ and $\Phi(r_n,r_n)|g(z_n,r_n)|^{-1} \to 0$ as $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{R_1+R_2}{g(z_n,r_n)}\right| &\le \lim_{n\to\infty} \frac{|R_1|+|R_2|}{|g(z_n,r_n)|} \\ &\le \lim_{n\to\infty} \frac{|h(z,r)|(\Phi(r_n,r_n)+|k(x,t)|)}{|g(z_n,r_n)|} \\ &\qquad + \lim_{n\to\infty} \frac{|h(x,t)|(\Phi(r_n,r_n)+|k(z,r)|)}{|g(z_n,r_n)|} \\ &= 0. \end{aligned} \tag{3.23}$$

Thus, it follows from (3.22) and (3.23) that

$$h(x+y,t+s)h(z,r) = h(x,t)h(y+z,s+r) \tag{3.24}$$

for all $x, y, z \in G$ and $t, s, r > 0$. Assume that $h(0,r) = 0$ for all $r > 0$. Putting $z = 0$ and replacing y by x and t by $s + r$ in (3.24),

$$0 = h(2x, t+s)h(0,r) = h(x, s+r)^2$$

for all $x \in G$, $s, r > 0$, which implies that $h \equiv 0$. Thus, we can choose $r_0 > 0$ such that $h(0, r_0) \neq 0$. Putting $x = y = z = 0$, $t = r_0$, $s = r = r_0/2$ in (3.24),

$$h(0, 3r_0/2)h(0, r_0/2) = h(0, r_0)^2 \neq 0.$$

Putting $z = 0$, $r = r_0/2$ in (3.24), multiplying the result by $h(0, r_0/2)$ and using (3.24) again,

$$\begin{aligned} h(x+y,t+s)h(0,r_0/2)^2 &= h(x,t)h(y,s+r_0/2)h(0,r_0/2) \\ &= h(x,t)h(y,s)h(0,r_0) \end{aligned} \tag{3.25}$$

for all $x, y \in G$ and $t, s > 0$. Dividing both sides of (3.25) by $h(0, r_0/2)^4/h(0, r_0)$,

$$\beta^{-1}h(x+y,t+s) = \beta^{-1}h(x,t) \cdot \beta^{-1}h(y,s) \tag{3.26}$$

for all $x, y \in G$ and $t, s > 0$, where

$$\beta = \frac{h(0,r_0/2)^2}{h(0,r_0)} \neq 0.$$

Thus, it follows from (3.26) that

$$h(x,t) = \beta m(x,t) \tag{3.27}$$

for all $x \in G, t > 0$, where $m : G \times \mathbb{R}^+ \to \mathbb{C}$ is an exponential function. Note that every exponential function $m : G \times \mathbb{R}^+ \to \mathbb{C}$ can be written as

$$m(x,t) = m_1(t)m_2(x), \quad x \in G, t > 0 \tag{3.28}$$

for some exponential functions $m_1 : \mathbb{R}^+ \to \mathbb{C}$ and $m_2 : G \to \mathbb{C}$. Now, from (3.16), (3.18), (3.19) and (3.28), we get (3.5). This completes the proof. □

Remark 3.2. In particular, if $g(x,t) \equiv \gamma$ is a constant function, then replacing (x,y) by (y,x) in (3.1) and using the triangle inequality with (3.1) gives

$$|k(x,t) + \gamma h(x,t) - k(y,s) - \gamma h(y,s)| \le 2\Phi(t,s)$$

for all $x,y \in G, t,s > 0$. Put $y = y_0, s = s_0$ and let $\mu = k(y_0,s_0) - \gamma h(y_0,s_0)$. Then

$$|k(x,t) + \gamma h(x,t) - \mu| \le 2\Phi(t,s_0) \le 2\Phi(t,t) \tag{3.29}$$

for all $x \in G, 0 < t < \delta$. Using the triangle inequality with (3.1) and (3.29),

$$|f(x,t) - \mu| \le 2\Phi(t,s_0) + \Phi(t,t) \le 3\Phi(t,t) \tag{3.30}$$

for all $x \in G, 0 < t < \delta$.

If h is a constant function (without loss of generality, we may assume that $h \equiv 1$), the inequality (3.1) is reduced to the Hyers–Ulam stability of the Pexider functional equation

$$|f(x+y,t+s) - g(x,t) - k(y,s)| \le \Phi(t,s)$$

for all $x,y \in G, t,s > 0$ (see [1]).

4. Stability of the Levi-Civitá equation in distributions and hyperfunctions

As the main result, we consider the stability of

$$u \circ S - v \otimes w - k \circ \Pi \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \quad [\text{respectively } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})], \tag{4.1}$$

where $u,v,w,k \in \mathcal{G}'(\mathbb{R}^n)$, $\mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ and $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$ are the spaces of bounded distributions and bounded hyperfunctions, respectively, $S(x,y) = x + y, \quad x,y \in \mathbb{R}^n$, and $\circ$ and $\otimes$ denote the pullback and the tensor product of generalised functions, respectively.

For the proof of our theorems, we employ the n-dimensional heat kernel $E_t(x)$ given by

$$E_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t), \quad t > 0.$$

In view of (2.1), we see that the heat kernel E_t belongs to the Gelfand space $\mathcal{G}(\mathbb{R}^n)$ for each $t > 0$. Thus, for each $u \in \mathcal{G}'(\mathbb{R}^n)$, the convolution $(u * E_t)(x) := \langle u_y, E_t(x-y)\rangle$ is well defined. We call $(u * E_t)(x)$ *the Gauss transform* of u and denote it by $\tilde{u}(x,t)$. It is well known that the Gauss transform $\tilde{u}(x,t)$ is a smooth solution of the heat equation such that $\tilde{u}(x,t) \to u$ as $t \to 0^+$, that is,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx$$

for all $\varphi \in \mathcal{G}$.

Example 4.1. Let $u(x) = x^\alpha, \alpha \in \mathbb{N}_0^n, v(x) = e^{c\cdot x}, w(x) = a \cdot xe^{c\cdot x}, a = (a_1, a_2, \ldots, a_n), c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$. Then $u, v, w \in \mathcal{G}'(\mathbb{R}^n)$ and simple calculations show that

$$\tilde{u}(x,t) = [\xi^\alpha * E_t(\xi)](x) = \alpha! \sum_{0 \le 2\gamma \le \alpha} \frac{t^{|\gamma|} x^{\alpha - 2\gamma}}{\gamma!(\alpha - 2\gamma)!},$$

$$\tilde{v}(x,t) = [e^{c\cdot\xi} * E_t(\xi)](x) = e^{c\cdot x + (c_1^2 + \cdots + c_n^2)t}$$

and

$$\tilde{w}(x,t) = [a \cdot \xi e^{c\cdot\xi} * E_t(\xi)](x) = (a \cdot x + 2a \cdot ct)e^{c\cdot x + (c_1^2 + \cdots + c_n^2)t}.$$

The proof of [4, Theorem 2.3] works even when $p = \infty$, giving the following result.

Lemma 4.2. *The Gauss transform $\tilde{u}(x,t) := (u * E)(x,t)$ of $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ is a smooth solution of the heat equation $(\Delta - \partial/\partial_t)\tilde{u} = 0$ satisfying:*

(i) *there exist constants $C > 0$, $N \ge 0$ and $\delta > 0$ such that*

$$|\tilde{u}(x,t)| \le Ct^{-N} \quad \forall x \in \mathbb{R}^n, t \in (0,\delta); \tag{4.2}$$

(ii) *$\tilde{u}(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{D}_{L^1}$,*

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx.$$

Conversely, every smooth solution $\tilde{u}(x,t)$ of the heat equation satisfying the estimate (4.2) can be uniquely expressed as $\tilde{u}(x,t) = (u * E)(x,t)$ for some $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$.

Similarly, we can represent bounded hyperfunctions as initial values of solutions of the heat equation. The estimate (4.2) is replaced by the following: there exists $\delta > 0$ such that, for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|\tilde{u}(x,t)| \le C_\epsilon \exp(\epsilon/t) \quad \forall x \in \mathbb{R}^n, t \in (0,\delta).$$

For the proof, we refer the reader to [5, Theorem 3.5].

For the proof of the following structure theorem, we refer the reader to [14, Theorem 25 in Ch. 6] and [5, Theorem 3.4].

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LEMMA 4.3 [5, 12, 14].

(i) *Every* $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ *can be expressed as*

$$u = \sum_{|\alpha| \le p} \partial^\alpha f_\alpha \tag{4.3}$$

for some $p \in \mathbb{N}_0$, *where* f_α *are bounded continuous functions on* $\mathbb{R}^n$. *The equality (4.3) implies that*

$$\langle u, \varphi \rangle = \sum_{|\alpha| \le p} (-1)^{|\alpha|} \int f_\alpha(x) \partial^\alpha \varphi(x)\, dx$$

for all $\varphi \in \mathcal{D}_{L^1}$.

(ii) *Every* $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ *can be expressed as*

$$u = \Big(\sum_{k=0}^{\infty} a_k \Delta^k \Big) g + h, \tag{4.4}$$

where Δ *denotes the Laplacian,* g, h *are bounded continuous functions on* $\mathbb{R}^n$ *and* $a_k, k = 0, 1, 2, \ldots,$ *satisfy the following estimates: for every* $L > 0$, *there exists* $C > 0$ *such that*

$$|a_k| \le CL^k / k!^2$$

for all $k = 0, 1, 2, \ldots.$

The following properties of the heat kernel will be useful.

LEMMA 4.4 [12]. *For each* $t > 0$, $E_t(\cdot)$ *is an entire function and the following estimate holds: there exists* $C > 0$ *such that*

$$|\partial_x^\alpha E_t(x)| \le C^{|\alpha|} t^{-(n+|\alpha|)/2} \alpha!^{1/2} \exp(-|x|^2/8t). \tag{4.5}$$

Also, for each $t, s > 0$,

$$(E_t * E_s)(x) := \int E_t(x-y) E_s(y)\, dy = E_{t+s}(x). \tag{4.6}$$

PROOF. The equality (4.6) is proved by well-known calculus, which we omit. We prove (4.5) for the case $n = 1$. By the Cauchy integral formula,

$$\frac{d^k}{dx^k} E_t(x) = \frac{k!}{2\pi i} \int_{C_r} \frac{E_t(z)}{(z-x)^{k+1}}\, dz, \tag{4.7}$$

where C_r is the circle of radius r with centre at $z = x$. Using (4.7) and the triangle inequality,

$$\begin{aligned} |\partial^k E_t(x)| &\le \frac{k!}{\sqrt{4\pi t} r^k} \sup_{z \in C_r} |\exp(-z^2/4t)| \\ &\le \frac{k!}{\sqrt{4\pi t} r^k} \sup_{0 \le \theta \le 2\pi} \exp\Big(\frac{-(x + r\cos\theta)^2 + r^2 \sin^2\theta}{4t} \Big) \\ &\le \frac{k!}{\sqrt{4\pi t} r^k} \exp\Big(\frac{r^2}{4t}\Big) \exp\Big(-\frac{x^2}{8t}\Big). \end{aligned} \tag{4.8}$$

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The right-hand side of (4.8) attains its minimum at $r = \sqrt{2kt}$. Thus, (4.8) is reduced to

$$|\partial^k E_t(x)| \le \frac{(e/2)^{k/2}}{\sqrt{4\pi}} k!^{1/2} t^{-(1+k)/2} \exp\Big(-\frac{x^2}{8t}\Big).$$

The general case is proved in the same manner. This completes the proof. □

Now we state and prove the main theorem. If w is constant (without loss of generality, we may assume that $w \equiv 1$), then the stability problem of (4.1) is reduced to that of the Pexider equation (see [1] for the result). Thus, we exclude the case when w is constant in the following theorem.

THEOREM 4.5. *Let $u, v, w, k \in \mathcal{G}'(\mathbb{R}^n)$. Then (u, v, w, k) satisfies (4.1) if and only if (u, v, w, k) satisfies one of the following:*

(i) *u, v, w, k are all bounded distributions [respectively bounded hyperfunctions];*

(ii) *there exist $\alpha, \beta, \gamma \in \mathbb{C}, a \in \mathbb{C}^n$ such that*

$$\begin{aligned} w = \beta e^{a\cdot x}, \quad v = \alpha\beta e^{a\cdot x} + \gamma, \\ k = \beta\gamma e^{a\cdot x} + k_0, \quad u = \alpha\beta^2 e^{a\cdot x} + u_0, \end{aligned} \tag{4.9}$$

where $k_0, u_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [respectively $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$];

(iii) *there exists $\gamma \in \mathbb{C}$ such that*

$$v \equiv \gamma, \quad k = -\gamma w + k_0, \quad u = u_0, \tag{4.10}$$

where $k_0, u_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [respectively $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$].

PROOF. We first convert the stability of (4.1) to the following classical functional inequalities: there exist $C > 0$ and $d > 0$ [respectively for every $\epsilon > 0$ there exists $C_\epsilon > 0$] such that

$$\begin{aligned} &|\tilde{u}(x + y, t + s) - \tilde{v}(x, t)\tilde{w}(y, s) - \tilde{k}(y, s)| \\ &\quad \le C\Big(\frac{1}{t} + \frac{1}{s}\Big)^N + d \quad [\text{respectively } C_\epsilon e^{\epsilon(1/t+1/s)}] \end{aligned} \tag{4.11}$$

for all $x, y \in \mathbb{R}^n, t, s > 0$, where $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{k}$ denote the Gauss transforms of u, v, w, k, respectively, given in Lemma 4.2.

Convolving the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels in the left-hand side of (4.1) and using the semigroup property (4.6),

$$\begin{aligned} [(u \circ S) * (E_t(\xi)E_s(\eta))](x, y) &= \Big\langle u_\xi, \int E_t(x - \xi + \eta)E_s(y - \eta)\, d\eta \Big\rangle \\ &= \langle u_\xi, (E_t * E_s)(x + y - \xi)\rangle \\ &= \tilde{u}(x + y, t + s). \end{aligned}$$

Similarly,

$$\begin{aligned} [(v \otimes w) * (E_t(\xi)E_s(\eta))](x, y) = \tilde{u}(x, t)\tilde{v}(y, s), \\ [(k \circ \Pi) * (E_t(\xi)E_s(\eta))](x, y) = \tilde{k}(y, s), \end{aligned}$$

where $\tilde{u}, \tilde{v}, \tilde{v}$ and $\tilde{k}$ are the Gauss transforms of u, v, w and k, respectively.

Let $\tau := u \circ S - v \otimes w - k \circ \Pi$. Then $\tau \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ [respectively $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$]. First, we suppose that $\tau \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$. Using (4.3) and (4.5),

$$\begin{aligned}
|[\tau * (E_t(\xi)E_s(\eta))](x,y)| &\le \sum_{|\alpha|\le p} |[\partial^\alpha f_\alpha * (E_t(\xi)E_s(\eta))](x,y)| \\
&\le \sum_{|\alpha|\le p} |[f_\alpha * \partial^\alpha_{\xi,\eta}(E_t(\xi)E_s(\eta))](x,y)| \\
&\le \sum_{|\alpha|\le p} \|f_\alpha\|_{L^\infty} \|\partial^\alpha_{\xi,\eta}(E_t(\xi)E_s(\eta))\|_{L^1} \\
&\le C_1 \sum_{|\beta|+|\gamma|\le p} \|\partial^\beta_\xi E_t(\xi)\|_{L^1} \|\partial^\gamma_\eta E_s(\eta)\|_{L^1} \\
&\le C_2 \sum_{|\beta|+|\gamma|\le p} t^{-|\beta|/2} s^{-|\gamma|/2} \\
&\le C(1/t + 1/s)^N + d,
\end{aligned}$$

where $N = p/2$ and the constants C and d depend only on p. Secondly, we suppose that $\tau \in \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$. Then, using (4.5),

$$\begin{aligned}
\|\Delta^k(E_t(\xi)E_s(\eta))\|_{L^1} &\le \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\partial^{2\alpha}(E_t(\xi)E_s(\eta))\|_{L^1} \\
&\le \sum_{|\beta|+|\gamma|=k} \frac{k!}{\beta!\gamma!} \|\partial^\beta_\xi E_t(\xi)\|_{L^1} \|\partial^\gamma_\eta E_s(\eta)\|_{L^1} \\
&\le \sum_{|\beta|+|\gamma|=k} \frac{k!(2\beta)!^{1/2}(2\gamma)!^{1/2} M^{2k}}{\beta!\gamma!} t^{-|\beta|} s^{-|\gamma|} \\
&\le \sum_{|\beta|+|\gamma|=k} k!(2M)^{2k} t^{-|\beta|} s^{-|\gamma|} \\
&\le k!(2\sqrt{n}M)^{2k}(1/t + 1/s)^k.
\end{aligned}$$

Now, by the structure (4.4) of bounded hyperfunctions together with the growth condition of $a_k, k = 0, 1, 2, \ldots,$

$$\begin{aligned}
&|[\tau * (E_t(\xi)E_s(\eta))](x,y)| \\
&\quad\le \sum_{k=0}^\infty \|a_k(\Delta^k g) * (E_t(\xi)E_s(\eta))\|_{L^\infty} + \|h * (E_t(\xi)E_s(\eta))\|_{L^\infty} \\
&\quad\le \|g\|_{L^\infty} \sum_{k=0}^\infty \|a_k \Delta^k(E_t(\xi)E_s(\eta))\|_{L^1} + \|h\|_{L^\infty} \|E_t(\xi)E_s(\eta)\|_{L^1} \\
&\quad\le C_1 \sum_{k=0}^\infty \frac{1}{k!}(4nM^2L)^k(1/t + 1/s)^k + \|h\|_{L^\infty} \\
&\quad\le C_2 \sum_{k=0}^\infty \frac{1}{k!}\epsilon^k(1/t + 1/s)^k + \|h\|_{L^\infty} \le C_\epsilon e^{\epsilon(1/t+1/s)},
\end{aligned}$$

where L is chosen so that $4nM^2L < \epsilon$ and the constant C_ϵ depends only on τ and ϵ. Thus, we have the inequality

$$|\tilde{u}(x+y,t+s) - \tilde{v}(x,t)\tilde{w}(y,s) - \tilde{k}(y,s)| \le \Phi(t,s)$$

for all $x, y \in \mathbb{R}^n, t, s > 0$, where $\Phi(t,s) = C(1/t + 1/s)^N$ [respectively $C_\epsilon e^{\epsilon(1/t+1/s)}$]. Assume that v is not constant. Then $\tilde{v}$ is not constant. Thus, we can apply Theorem 3.1, replacing f, g, h, k by $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{k}$, respectively, in (3.1). If $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{k}$ satisfy (3.4), then, by Lemma 4.2, we have $u, v, w, k \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [respectively $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$]. This gives case (i).

Assume that $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{k}$ satisfy (3.5). Since $\tilde{w}$ is a smooth solution of the heat equation, we have $\tilde{w}(x,t) = e^{bt+a\cdot x}$, where $a = (a_1, \ldots, a_n) \in \mathbb{C}^n, b = a_1^2 + \cdots + a_n^2$. Thus,

$$\begin{gathered} \tilde{w}(x,t) = \beta e^{bt+a\cdot x}, \quad \tilde{v}(x,t) = \alpha\beta e^{bt+a\cdot x} + \gamma, \\ |\tilde{k}(x,t) + \alpha\gamma e^{bt+a\cdot x} - \mu| \le 2\Phi(t,t), \\ |\tilde{u}(x,t) - \alpha\beta^2 e^{bt+a\cdot x} - \mu| \le 3\Phi(t,t) \end{gathered} \tag{4.12}$$

for all $x \in \mathbb{R}^n, 0 < t < \delta$. Letting $t \to 0^+$ in (4.12),

$$w = \beta e^{a\cdot x}, \quad v = \alpha\beta e^{a\cdot x} + \gamma.$$

Since $\tilde{k}(x,t) + \alpha\gamma e^{bt+a\cdot x} - \mu$ is the Gauss transform of $k + \alpha\gamma e^{a\cdot x} - \mu$, by applying Lemma 4.2,

$$k_0 := k + \alpha\gamma e^{a\cdot x} \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n) \quad [\text{respectively } \mathcal{A}'_{L^\infty}(\mathbb{R}^n)].$$

Similarly,

$$u_0 := u - \alpha\beta^2 e^{a\cdot x} \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n) \quad [\text{respectively } \mathcal{A}'_{L^\infty}(\mathbb{R}^n)].$$

Thus, we have (4.9). Finally, we assume that $v :\equiv \gamma$ is constant. Then $\tilde{v} \equiv \gamma$. By (3.29) and (3.30) in Remark 3.2,

$$|\tilde{k}(x,t) + \gamma\tilde{w}(x,t) - \mu| \le 2\Phi(t,t), \tag{4.13}$$

$$|\tilde{u}(x,t) - \mu| \le 3\Phi(t,t) \tag{4.14}$$

for all $x \in \mathbb{R}^n, 0 < t < \delta$. By Lemma 4.2, it follows from (4.13) and (4.14) that

$$k + \gamma w := k_0, \quad u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n) \quad [\text{respectively } \mathcal{A}'_{L^\infty}(\mathbb{R}^n)].$$

Thus, we have (4.10). This completes the proof. □

Let f be a Lebesgue measurable function on $\mathbb{R}^n$. If, for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that the inequality

$$|f(x)| \le C_\epsilon e^{\epsilon|x|^2}$$

holds for all $x \in \mathbb{R}^n$, then the function f is said to be an *infra-exponential function of order two*. It is easy to see that every infra-exponential function f of order two defines an element of $\mathcal{G}'(\mathbb{R}^n)$ via the correspondence

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)\,dx, \quad \varphi \in \mathcal{G}.$$

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Lemma 4.6 [16, page 122]. *Let $\tilde{f}(x,t)$ be a solution of the heat equation satisfying*

$$|\tilde{f}(x,t)| \le M, \quad x \in \mathbb{R}^n, t \in (0,\delta)$$

for some $M > 0$ and $\delta > 0$. Then $\tilde{f}$ can be written as

$$\tilde{f}(x,t) = (f * E_t)(x) = \int f(y)E_t(x-y)\,dy$$

for some bounded measurable function f defined in $\mathbb{R}^n$. In particular, $\tilde{f}(x,t) \to f(x)$ almost everywhere in $x \in \mathbb{R}^n$ as $t \to 0^+$.

As a consequence of Theorem 4.5 together with Lemma 4.6, we have the following L^∞-version of the Hyers–Ulam stability theorem.

Corollary 4.7. *Nonconstant infra-exponential functions $f, g, h, k : \mathbb{R}^n \to \mathbb{C}$ of order two satisfy*

$$\|f(x+y) - g(x)h(y) - k(y)\|_{L^\infty(\mathbb{R}^{2n})} \le M \tag{4.15}$$

for some $M > 0$ if and only if either

$$f, g, h, k \in L^\infty(\mathbb{R}^n)$$

or else

$$h(x) = \beta e^{a\cdot x}, \quad g(x) = \alpha\beta e^{a\cdot x} + \gamma,$$
$$k(x) = \beta\gamma e^{a\cdot x} + r_1(x), \quad f(x) = \alpha\beta^2 e^{a\cdot x} + r_2(x),$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $a \in \mathbb{C}^n$ and $r_1, r_2 \in L^\infty(\mathbb{R}^n)$.

Proof. As in the proof of Theorem 4.5, convolving $E_t(x)E_s(y)$ in (4.15) yields

$$|\tilde{f}(x+y, t+s) - \tilde{g}(x,t)\tilde{h}(y,s) - \tilde{k}(y,s)| \le M$$

for all $x, y \in \mathbb{R}^n, t, s > 0$, where $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{k}$ denote the Gauss transforms of f, g, h, k, respectively. Thus, the proof of Theorem 4.5 is reduced to the case when $\Phi(s,t) \equiv M$. Now, using Lemma 4.6, we get the result. □

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References

[1] J. Chung, 'Hyers–Ulam stability on a generalized quadratic functional equation in distributions and hyperfunctions', *J. Math. Phys.* **50** (2009), 113519.

[2] J. Chung, 'A heat kernel approach to the stability of exponential equations in Schwartz distributions and hyperfunctions', *J. Math. Phys.* **51** (2010), 053523.

[3] J. Chung, H. Hunt, A. Perkins and P. K. Sahoo, 'Stability of a simple Levi-Civitá functional on non-unital commutative semigroup', *Proc. Indian Acad. Sci.* **124**(4) (2014), 365–381.

[4] S.-Y. Chung, 'A heat equation approach to distributions with Lp growth', *Commun. Korean Math. Soc.* **9**(4) (1994), 897–903.
[5] S. Y. Chung, D. Kim and E. G. Lee, 'Periodic hyperfunctions and Fourier series', *Proc. Amer. Math. Soc.* **128** (2000), 2421–2430.
[6] B. Ebanks, 'General solution of a simple Levi-Civitá functional equation on non-abelian groups', *Aequationes Math.* **85**(3) (2013), 359–378.
[7] I. M. Gelfand and G. E. Shilov, *Generalized Functions II* (Academic Press, New York, 1968).
[8] I. M. Gelfand and G. E. Shilov, *Generalized Functions IV* (Academic Press, New York, 1968).
[9] L. Hörmander, *The Analysis of Linear Partial Differential Operators I* (Springer, Berlin, 1983).
[10] D. H. Hyers, 'On the stability of the linear functional equation', *Proc. Natl. Acad. Sci. USA* **27** (1941), 222–224.
[11] T. Levi-Civitá, 'Sulle funzioni che ammetono una formula d'addizione del tipo $f(x+y) = \sum_{i=1}^{n} X_i(x)Y_i(y)$', *R.C. Accad. Lincei* **22** (1913), 181–183.
[12] T. Matsuzawa, 'A calculus approach to hyperfunctions III', *Nagoya Math. J.* **118** (1990), 133–153.
[13] P. K. Sahoo and Pl. Kannappan, *Introduction to Functional Equations* (CRC Press–Taylor and Francis, Boca Raton, FL, 2011).
[14] L. Schwartz, *Théorie des Distributions* (Hermann, Paris, 1966).
[15] E. V. Shulman, 'Group representations and stability of functional equations', *J. Lond. Math. Soc.* (2) **54** (1996), 111–120.
[16] D. V. Widder, *The Heat Equation* (Academic Press, New York, 1975).

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