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POSITIVE MULTIPOINT PADÉ CONTINUED FRACTIONS

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1. Introduction

Multipoint Padé fractions were introduced in [2]. They are continued fractions defined in the following way:

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be given fixed points in the complex plane. For each $n \geqq 1$ let $\alpha_n = \alpha_m$ where $1 \leqq m \leqq p$ and $n \equiv m \pmod p$.

Let A_n, B_n, C_n be constants, $B_n \neq 0$, $C_n \neq 0$ for all $n \in \mathbb{N}$. We define

$$a_1(z) = \frac{C_1}{z - \alpha_1}, \; b_1(z) = \frac{A_1}{z - \alpha_1} + B_1 \tag{1.1a}$$

$$a_2(z) = \frac{C_2}{z - \alpha_2}, \; b_2(z) = A_2 \frac{z - \alpha_1}{z - \alpha_2} + \frac{B_2}{z - \alpha_2}, \tag{1.1b}$$

$$a_n(z) = C_n \frac{z - \alpha_{n-2}}{z - \alpha_n}, \; b_n(z) = A_n \frac{z - \alpha_{n-1}}{z - \alpha_n} + B_n \frac{z - \alpha_{n-2}}{z - \alpha_n} \quad \text{for } n = 3, 4, \ldots \tag{1.1c}$$

The continued fraction $K_{n=1}^{\infty} a_n(z)/b_n(z)$ is then called a multipoint Padé continued fraction, *or MP-fraction* (belonging to the set $\{\alpha_1, \ldots, \alpha_p\}$). The MP-fraction is called *positive* if the points $\{\alpha_1, \ldots, \alpha_p\}$ lie on the real axis and the coefficients A_n, B_n, C_n are real and satisfy the conditions

$$B_1 C_1 > 0 \tag{1.2a}$$

$$B_1 B_2 C_2 > 0 \tag{1.2b}$$

$$B_2 B_3 C_3 (\alpha_2 - \alpha_1) < 0 \tag{1.2c}$$

$$B_n B_{n+1} C_{n+1} (\alpha_n - \alpha_{n-1})(\alpha_{n-1} - \alpha_{n-2}) < 0 \quad \text{for } n = 3, 4, \ldots. \tag{1.2d}$$

Positive MP-fractions are related to positive linear functionals on the space $\mathscr{R}$ of

R-functions belonging to the points $\{a_1,\dots,a_p\}$. This space consists of all rational functions with no poles in the extended complex plane outside the set $\{a_1,\dots,a_p\}$. The R-functions play a central role in the treatment of moment problems connected with the set $\{a_1,\dots,a_p\}$, and in certain multipoint Padé approximation problems connected with series expansions about the points $\{a_1,\dots,a_p\}$. For more information on R-functions and their uses, see [7, **8**, **9**].

In [**2**] we showed that a positive linear functional Φ on $\mathcal{R}$ with corresponding regular orthogonal R-functions, gives rise to a positive MP-fraction. On the other hand, every positive MP-fraction whose denominators are regular and of exact degree originates in this way. (For the concepts of orthogonality of R-functions and of regularity, see [**2**, **7**, **8**]).

In this paper we treat positive MP-fractions without reference to the theory of functionals on $\mathcal{R}$. Our aim is to study mapping properties of the linear fractional transformations associated with a positive MP-fraction, and from these mapping properties to obtain results on the structure of the approximants (ordinary and generalized) of the continued fraction. We show that there exists a situation of nested discs defined by the linear fractional transformations, analogous to the situation for real J-fractions (see [**5**, **12**]), APT-fractions (see [**3**]) and contractive Laurent fractions (see [**10**]). These situations can also be treated on the basis of the theory of orthogonal functions connected with the linear functional Φ, see e.g., [**1**, **6**, **8**, **13**]. We use the obtained mapping properties to prove that, except for special values of the parameter τ, the generalized approximants $F_n(z,\tau)$ of the continued fraction have partial fraction decompositions of the form

$$F_n(z,\tau)=\sum_{v=1}^{n}\frac{\lambda_v}{z-t_v}, \tag{1.3}$$

where $t_v\in\mathbb{R}$, $t_v\notin\{a_1,\dots,a_p\}$, $\lambda_v>0$. The approach to this decomposition problem is similar to that found in e.g., [**3**, **4**, **10**, **11**].

For general standard information on continued fractions, we refer to [**4**].

2. Mapping properties

Let $K_{n=1}^{\infty}(a_n(z)/b_n(z))$ be an MP-fraction. We denote by $F_n(z)=(P_n(z)/Q_n(z))$ the nth approximant of the continued fraction, and set $P_0=0$, $Q_0=1$.

We define the linear fractional transformations associated with the continued fraction in the usual way:

$$s_1(w)=\frac{C_1(z-a_1)^{-1}}{A_1(z-a_1)^{-1}+B_1+w}, \tag{2.1a}$$

$$s_2(w)=\frac{C_2(z-a_2)^{-1}}{A_2(z-a_1)(z-a_2)^{-1}+B_2(z-a_2)^{-1}+w}, \tag{2.1b}$$

$$s_n(w)=\frac{C_n(z-a_{n-2})(z-a_n)^{-1}}{A_n(z-a_{n-1})(z-a_n)^{-1}+B_n(z-a_{n-2})(z-a_n)^{-1}+w} \quad \text{for } n=3,4,\ldots. \tag{2.1c}$$

We also define

$$S_n(w)=\frac{P_n(z)+wP_{n-1}(z)}{Q_n(z)+wQ_{n-1}(z)} \quad \text{for } n=1,2,\ldots. \tag{2.2}$$

Recall that $P_0=0$, $Q_0=1$, so that $S_1(w)=s_1(w)$. We thus have (cf. [4])

$$S_n(w)=s_1\circ s_2\circ\cdots\circ s_n(w). \tag{2.3}$$

We shall in the rest of this section assume that the MP-fraction is positive. We recall that this means that the points $a_1,\ldots,a_p$ and the coefficients A_n, B_n, C_n are real and that B_n, C_n satisfy (1.2).

We shall let Π_+ denote the closed upper half plane and Π_- the closed lower half plane. We always assume in this section that $\operatorname{Im} z>0$.

We write α_i for the angle $\alpha_i=\operatorname{Arg}(z-a_i)$, $i=1,\ldots,p$. For simplicity we may (without loss of generality) assume that the points $\{a_1,\ldots,a_p\}$ are arranged in increasing order: $a_1<a_2<\cdots<a_p$. It follows that $0<\alpha_i<\alpha_{i+1}<\pi$ for $i=1,2,\ldots,p-1$.

We define the half planes $\Omega_n=\Omega_n(z)$ by the conditions

$$\Omega_0=\Pi_- \tag{2.4a}$$

$$\begin{cases}\Omega_1=\{w\colon -\alpha_1\leqq \operatorname{Arg} w\leqq \pi-\alpha_1\} \text{ if } B_1>0\\ \Omega_1=\{w\colon -\alpha_1-\pi\leqq \operatorname{Arg} w\leqq -\alpha_1\} \text{ if } B_1<0\end{cases} \tag{2.4b}$$

$$\begin{cases}\Omega_n=\{w\colon \alpha_{n-1}-\alpha_n-\pi\leqq \operatorname{Arg} w\leqq \alpha_{n-1}-\alpha_n\} \text{ if } B_n(a_n-a_{n-1})>0\\ \Omega_n=\{w\colon \alpha_{n-1}-\alpha_n\leqq \operatorname{Arg} w\leqq \alpha_{n-1}-\alpha_n+\pi\} \text{ if } B_n(a_n-a_{n-1})<0\end{cases}$$
$$\text{for } n=1,2,\ldots. \tag{2.4c}$$

Theorem 2.1. *The inclusions $s_n(\Omega_n)\subset\Omega_{n-1}$ hold for $n=1,2,\ldots$.*

Proof. This result can be proved by using elementary mapping properties of linear fractional transformations, the various sign combinations of B_n, B_{n-1}, C_n being taken into account. We shall illustrate the method by going through the argument in one case.

Let $n\geqq 4 \pmod p$ and assume that $B_n>0$, $B_{n-1}<0$. Then by the positivity condition (1.2d) we have $C_n>0$. Let $w\in\Omega_n$. Then

$$\alpha_{n-1}-\alpha_n-\pi\leqq \operatorname{Arg} w\leqq \alpha_{n-1}-\alpha_n.$$

We have

$$\arg A_n(z-a_{n-1})(z-a_n)^{-1}=\alpha_{n-1}-\alpha_n-\pi$$

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or

$$\arg A_n(z-a_{n-1})(z-a_n)^{-1}=\alpha_{n-1}-\alpha_n$$

and so $A_n(z-a_{n-1})(z-a_n)^{-1}\in\Omega_n$. Since $0<\alpha_{n-2}<\alpha_{n-1}<\pi$ and $B_n>0$, we also have

$$\alpha_{n-1}-\alpha_n-\pi\leqq \operatorname{Arg} B_n(z-a_{n-2})(z-a_n)^{-1}\leqq\alpha_{n-1}-\alpha_n,$$

and so $B_n(z-a_{n-2})(z-a_n)^{-1}\in\Omega_n$. It follows that $D(z)=A_n(z-a_{n-1})(z-a_n)^{-1}+B_n(z-a_{n-2})(z-a_{n-1})^{-1}+w\in\Omega_n$. Consequently, since $C_n>0$, we have

$$\alpha_n-\alpha_{n-1}\leqq\operatorname{Arg}\left(C_n\cdot\frac{1}{D(z)}\right)\leqq\alpha_n-\alpha_{n-1}+\pi.$$

Since $\arg[(z-a_{n-2})(z-a_n)^{-1}]=\alpha_{n-2}-\alpha_n$, we conclude that

$$\alpha_{n-2}-\alpha_n+\alpha_n-\alpha_{n-1}\leqq\operatorname{Arg} s_n(w)\leqq\alpha_{n-2}-\alpha_n+\alpha_n-\alpha_{n-1}+\pi,$$

in other words

$$\alpha_{n-2}-\alpha_{n-1}\leqq\operatorname{Arg} s_n(w)\leqq\alpha_{n-2}-\alpha_{n-1}+\pi,$$

which means that $s_n(w)\in\Omega_{n-1}$, since $B_{n-1}<0$.

The other cases can be treated in the same way. □

We write $\Delta_n=\Delta_n(z)$ for the images of Ω_n by S_n, that is: $\Delta_n=S_n(\Omega_n)$, $n=1,2,\ldots$.

Theorem 2.2. *The following statements about* $\Delta_n=\Delta_n(z)$ *hold for every* z *with* $\operatorname{Im} z>0$:

(a) $\Delta_n\subset\Delta_{n-1}$ *for* $n=2,3,\ldots$.
(b) $\Delta_n\subset\Pi_-$ *for* $n=1,2,\ldots$.
(c) Δ_n *is a closed disc.*

Proof.

(a) By taking into account that $s_n(\Omega_n)\subset\Omega_{n-1}$ (Theorem 2.1) we obtain

$$\Delta_n=S_n(\Omega_n)=S_{n-1}(s_n(\Omega_n))\subset S_{n-1}(\Omega_{n-1})=\Delta_{n-1}.$$

(b) It follows from Theorem 2.1 that $\Delta_1=S_1(\Omega_1)=s_1(\Omega_1)\subset\Omega_0=\Pi_-$, and hence from (a) we get $\Delta_n\subset\Pi_-$ for all n.

(c) Since S_n is a linear fractional transformation, Δ_n is either a closed half plane, a closed disc or the exterior of an open disc. Since $B_1\neq 0$, $B_1\in\mathbb{R}$, the denominator of $s_1(w)$ is not zero for any w on the boundary $\partial\Omega_1$ of Ω_1. It follows that

$\infty \notin \partial\Delta_1 = s_1(\partial\Omega_1)$. Thus Δ_1 is not a half plane. Since $\Delta_1 \subset \Pi_-$, it follows that Δ_1 is a closed disc. We now conclude by (a) that all Δ_n are closed discs. □

It follows from Theorem 2.2 that the intersection $\Delta_\infty(z) = \bigcap_{n=1}^{\infty} \Delta_n(z)$ is either a single point or a closed disc. It was shown in [8] by methods using properties of orthogonal R-functions that $\Delta_\infty(z)$ is either a single point for every z with $\operatorname{Im} z > 0$ or a closed disc for every z with $\operatorname{Im} z > 0$. We may thus speak of a limit point—limit circle situation, independent of z. It is not our aim to undertake a treatment of this problem by continued fractions methods in this paper.

3. Partial fraction decomposition

Let $K_{n=1}^{\infty}(a_n(z)/b_n)(z))$ be an MP-fraction, and let $P_n(z)/Q_n(z)$ be the nth approximant. Then the denominators Q_n satisfy the following recurrence relations:

$$Q_1 = \left(\frac{A_1}{z-a_1} + B_1\right) Q_0 + \frac{C_1}{z-a_1} Q_{-1} \tag{3.1a}$$

$$Q_2 = \frac{A_2(z-a_1) + B_2}{(z-a_2)} Q_1 + \frac{C_2}{z-a_2} Q_0 \tag{3.1b}$$

$$Q_n = \frac{A_n(z-a_{n-1}) + B_n(z-a_{n-2})}{z-a_n} Q_{n-1} + \frac{C_n(z-a_{n-2})}{z-a_n} Q_{n-2} \quad \text{for } n = 3, 4, \ldots, \tag{3.1c}$$

with initial conditions $Q_0 = 1$, $Q_{-1} = 0$. The numerators P_n satisfy the same recurrence relations, with initial conditions $P_0 = 0$, $P_{-1} = 1$.

It is easily verified by induction that we may write

$$Q_n(z) = \beta_0^{(n)} + \frac{\beta_1^{(1)}}{z-a_1} + \cdots + \frac{\beta_p^{(n)}}{z-a_p} + \frac{\beta_{p+1}^{(n)}}{(z-a_1)^2} + \cdots + \frac{\beta_{n-1}^{(n)}}{(a-z_{n-1})^{q+1}} + \frac{\beta_n^{(n)}}{(z-a_n)^{q+1}} \tag{3.2}$$

$$P_n(z) = \frac{\alpha_1^{(n)}}{z-a_1} + \cdots + \frac{\alpha_p^{(n)}}{z-a_p} + \frac{\alpha_{p+1}^{(n)}}{(z-a_1)^2} + \cdots + \frac{\alpha_{n-1}^{(n)}}{(z-a_{n-1})^{q+1}} + \frac{\alpha_n^{(n)}}{(z-a_n)^{q+1}}, \tag{3.3}$$

where q is the integer part $[n/p]$ of n/p. Equivalently Q_n and P_n may be written in the following way:

$$Q_n(z) = \frac{V_n(z)}{N_n(z)}, \; P_n(z) = \frac{U_n(z)}{N_n(z)}, \tag{3.4}$$

where

$$N_n(z) = (z-a_1)^{q+1} \cdots (z-a_n)^{q+1}(z-a_{n+1})^q \cdots (z-a_p)^q, \tag{3.5}$$

and U_n and V_n are polynomials such that $\deg V_n \leq n$, $\deg U_n \leq n-1$. We call V_n and Q_n

degenerate if $\deg V_n \leqq n$. We shall say that Q_n is *of exact degree* if $\beta_n^{(n)} \neq 0$. This is equivalent to a_n not being a zero of $V_n(z)$. We say that Q_n is *regular* if $\beta_{n-1}^{(n)} \neq 0$. This is equivalent to a_{n-1} not being a zero of $V_n(z)$. It follows from the recurrence relations (3.1) that if all Q_n are of exact degree, then all Q_n are regular, since all $B_n \neq 0$.

Lemma 3.1. *Assume that all Q_n are of exact degree. Then the polynomials V_n and V_{n-1} do not both have a zero at a_i for any fixed $i=1,2,\ldots,p$. Similarly V_n and V_{n-1} are not both degenerate.*

Proof. The recurrence relations (3.1) can be rewritten in the following form:

$$V_1 = A_1 + B_1(z-a_1) \tag{3.6a}$$

$$V_2 = [A_2(z-a_1)+B_2]V_1 + C_2(z-a_1) \tag{3.6b}$$

$$V_n = [A_n(z-a_{n-1})+B_n(z-a_{n-2})]V_{n-1} + C_n(z-a_{n-1})(z-a_{n-2})V_{n-2} \quad \text{for } n=3,4,\ldots. \tag{3.6c}$$

Let $n \geqq 3$, and assume that V_n and V_{n-1} have a common factor $(z-a_i)$. This factor is not $(z-a_n)$ or $(z-a_{n-1})$, since Q_n and Q_{n-1} are of exact degree. It follows that $(z-a_i)$ is also a factor of V_{n+1}. By repeating this argument at most $p-2$ times, we conclude that $(z-a_i)$ is a factor of V_{pq+i} for some q, which contradicts the assumption that all Q_n are of exact degree. Also V_2 and V_1 have no common factor $(z-a_i)$, since $C_2 \neq 0$.

The proof of the second statement is similar: If $\deg V_n < n$, $\deg V_{n-1} < n-1$, then $\deg V_{n-2} < n-2$, and by repeating the argument we get $\deg V_1 < 1$, which is impossible since $B_1 \neq 0$. □

The generalized approximants of the MP-fraction are defined by

$$F_n(z,\tau) = \frac{P_n(z,\tau)}{Q_n(z,\tau)}, \quad \tau \in \mathbb{R}, \tag{3.7}$$

where for any $\tau \in C$,

$$P_n(z,\tau) = P_n(z) + \tau \frac{z-a_{n-1}}{z-a_n} P_{n-1}(z), \quad n=3,4,\ldots, \tag{3.8a}$$

$$Q_n(z,\tau) = Q_n(z) + \tau \frac{z-a_{n-1}}{z-a_n} Q_{n-1}(z), \quad n=3,4,\ldots. \tag{3.8b}$$

Lemma 3.2. *For an arbitrary $\tau \in C$, $P_n(z,\tau)$ and $Q_n(z,\tau)$ have no common zeros outside the set $\{a_1,\ldots,a_p\}$.*

Proof. Assume that $P_n(z,\tau)$ and $Q_n(z,\tau)$ have a common zero outside $\{a_1,\dots,a_p\}$, for some $\tau\in C$. Then the determinant

$$D_n(z)=\frac{z-a_{n-1}}{z-a_n}[P_n(z)Q_{n-1}(z)-P_{n-1}(z)Q_n(z)]$$

is zero for this value of z. The product formula for continued fractions in our case reads

$$P_n(z)Q_{n-1}(z)-P_{n-1}(z)Q_n(z)=(-1)^{n-1}\frac{C_1C_2\dots C_n}{(z-a_n)(z-a_{n-1})} \tag{3.9}$$

which gives a contradiction. □

Theorem 3.3. *Let a positive MP-fraction be given, and assume that the denominators are of exact degree. Then every generalized approximant $F_n(z,\tau)$, except for at most p values of τ, has a partial fraction decomposition of the following form:*

$$F_n(z,\tau)=\sum_{\nu=1}^{n}\frac{\lambda_\nu^{(n)}(\tau)}{z-t_\nu^{(n)}(\tau)}, \tag{3.10}$$

where $t_\nu^{(n)}(\tau)\in\mathbb{R}$, $t_\nu^{(n)}(\tau)\notin\{a_1,\dots,a_p\}$, $\lambda_\nu^{(n)}(\tau)>0$.

Proof. Comparing (3.7)–(3.8) with (2.2) and the definitions of $\Omega_n(z)$, $\Delta_n(z)$, we find (for a positive MP-fraction): For a fixed $z\in\Pi_+^0$ (the open upper half plane) and a fixed $\tau\in\mathbb{R}$,

$$F_n(z,\tau)=S_n\left(\tau\frac{z-a_{n-1}}{z-a_n}\right)\quad\text{and}\quad\tau\frac{z-a_{n-1}}{z-a_n}\in\partial\Omega_n(z),$$

so that $F_n(z,\tau)\in\partial\Delta_n(n)$. By Lemma 3.2, $P_n(z,\tau)$ and $Q_n(z,\tau)$ have no common zeros outside the set $\{a_1,\dots,a_p\}$ for any τ. Since $F_n(z,\tau)\in\Delta_n(z)$, we must then have $Q_n(z,\tau)\neq 0$, since otherwise $F_n(z,\tau)=\infty$, which contradicts Theorem 2.2c. It follows that for a fixed $\tau\in\mathbb{R}$, $Q_n(z,\tau)$ has no zeros in Π_+^0. Then also $Q_n(z,\tau)$ has no zeros in Π_-^0 (the open lower half plane), since non-real zeros appear in conjugate pairs. (All the coefficients A_n, B_n, C_n are real and hence all coefficients in $Q_n(z,\tau)$ are real.) Consequently all the zeros of $Q_n(z,\tau)$ are real.

The rational function $F_n(z,\tau)$ may be written as

$$F_n(z,\tau)=\frac{U_n(z)+\tau(z-a_{n-1})U_{n-1}(z)}{V_n(z)+\tau(z-a_{n-1})V_{n-1}(z)}. \tag{3.11}$$

(Recall that $N_n(z)=(z-a_n)N_{n-1}(z)$.) Except possibly for one value of τ, the denominator is a polynomial of degree n. (Recall that by Lemma 3.1, $\deg V_{n-1}=n-1$ if $\deg V_n<n$.) The numerator is a polynomial of degree at most $n-1$. Except possibly for $p-1$ values

of τ, none of the zeros of the denominator can be among the points $a_1, \ldots, a_p$. For since V_n and V_{n-1} have no common zeros among these points (by Lemma 3.1) and a_{n-1} is not a zero of V_n (since Q_n is regular), the value of τ must be $\tau_i = -V_n(a_i)[(a_i - a_{n-1})V_{n-1}(a_i)]^{-1}$ in order that a_i shall be a zero of the denominator. Except for these at most $p-1$ values of τ, the numerator and denominator have no common zeros, by Lemma 3.2.

Therefore $F_n(z,\tau)$ has, except for at most p values of τ, a partial fraction decomposition of the form

$$F_n(z,\tau) = \sum_{v=1}^{s} \left[\frac{c_{v,1}}{(z-t_v)} + \cdots + \frac{c_{v,m_v}}{(z-t_v)^{mv}} \right], \tag{3.12}$$

where $t_1, \ldots, t_s \in \mathbb{R}$, $t_j \notin \{a_1, \ldots, a_p\}$, $m_1 + m_2 + \cdots + m_s = n$. (Here t_v, $c_{v,j}$ are constants depending on τ and m.)

For points close to t_v the dominating term in the decomposition (3.12) of $F_n(z,\tau)$ is $c_{v,mv}/(z-t_v)^{mv}$ When $(z-t_v)$ varies over an angle π in Π_+, then $(1/(z-t_v)^{mv})$ varies over an angle $m_v \cdot \pi$. For this to be possible, the exponent m_v cannot be greater than 1, since we know that

$$F_n(z,\tau) \in \Delta_n(z) \subset \Pi_- \quad \text{for all } z \in \Pi_+^0$$

(Theorem 2.2(b)). Thus the zeros $t_1, \ldots, t_s$ are simple, and $s = n$. The decomposition (3.12) therefore must have the special form

$$F_n(z,\tau) = \sum_{v=1}^{n} \frac{c_v}{z - t_v}. \tag{3.13}$$

Again for z close to t_v the dominating term is $c_v/(z-t_v)$. For the inclusion $F_n(z,\tau) \in \Pi_-$ for all $z \in \Pi_+^0$ to hold, it is necessary that $c_v > 0$. Writing $\lambda_v^{(n)}(\tau)$ for c_v, $t_v^{(n)}(\tau)$ for t_v, we get (3.10).

□

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