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$\mathbb{Z}_2$-SYMMETRIC CRITICAL POINT THEOREMS FOR NON-DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper, some min–max theorems for even and C^1 functionals established by Ghoussoub are extended to the case of functionals that are the sum of a locally Lipschitz continuous, even term and a convex, proper, lower semi-continuous, even function. A class of non-smooth functionals admitting an unbounded sequence of critical values is also pointed out.

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1. Introduction. Already in the early days of the min–max methods for real-valued C^1 functionals f defined on an infinite-dimensional Banach space $(X, \|\cdot\|)$, it has been pointed out that the connection between symmetry and multiplicity of the critical points of f, which are not minima or maxima, is one of the most fascinating phenomena occurring in this framework. Starting with the results of Ljusternik and Schnirelmann, several techniques have widely been developed and successfully applied to many problems that stem from the calculus of variations. In this direction, it is by now well known that one of the most meaningful results is the symmetric version of the Mountain Pass Lemma (briefly, SMPL) due to Ambrosetti and Rabinowitz. For more details on this subject, we refer the reader to the excellent monographs [**1**, **14**, **15**].

However, many variational problems, which arise in the modelling of important mechanical and engineering questions, naturally lead to consider functionals lacking the smoothness properties usually required for the application of classical results [**13**, **14**, **16**]. As an example, we only mention both variational inequalities and elliptic equations with discontinuous non-linearities. Concerning the first case, the indicator function of some convex closed subset of X must appear in the expression of f; for the second case, f turns out to be locally Lipschitz continuous at most.

Recently, Motreanu and Panagiotopoulos established in [**12**] (see also [**7**] and [**11**]) a non-smooth version of SMPL for functionals $f : X \to \mathbb{R} \cup \{+\infty\}$ satisfying the following structural hypothesis:

(H_f) $f(x) := \Phi(x) + \psi(x)$ *for all* $x \in X$, *where* $\Phi : X \to \mathbb{R}$ *is even and locally Lipschitz continuous while* $\psi : X \to \mathbb{R} \cup \{+\infty\}$ *is even, convex, proper and lower semi-continuous.*

Critical points of f are defined as solutions to the following problem:

$$\textit{Find } x \in X \textit{ such that} \qquad \Phi^0(x; z - x) + \psi(z) - \psi(x) \geq 0 \quad \forall z \in X, \tag{$*$}$$

with $\Phi^0(x; z - x)$ being the generalized directional derivative [4, p. 25] of Φ in x along the direction $z - x$. Here, the standard Palais–Smale condition becomes the following:

$(\mathrm{PS})_f$ *Every sequence* $\{x_n\} \subseteq X$ *such that* $\{f(x_n)\}$ *is bounded and*

$$\Phi^0(x_n; z - x_n) + \psi(z) - \psi(x_n) \geq -\epsilon_n \|z - x_n\| \quad \forall n \in \mathbb{N},\ z \in X,$$

where $\epsilon_n \to 0^+$, *possesses a convergent subsequence.*

When Φ is C^1, problem $(*)$ reduces to a variational inequality, and the relevant critical point theory as well as significant applications are developed in [**16**]. If $\psi \equiv 0$, then $(*)$ coincides with the problem treated by Chang [**3**]. Finally, when both Φ turn out C^1 and $\psi \equiv 0$, problem $(*)$ simplifies to the Euler equation $\Phi'(x) = 0$, and the theory is classical. Let us explicitly observe that the above-mentioned results are obtained when the relevant mountain-pass-type inequalities are strict.

The aim of this paper is to establish a $\mathbb{Z}_2$-symmetric version of the main results contained in [**8**, Ch. 7] for functionals f satisfying the condition

(H'_f) $f : X \to \mathbb{R} \cup \{+\infty\}$ *satisfies* (H_f) *and* ψ *is continuous on any non-empty compact set* $A \subseteq X$ *such that* $\sup_{x \in A} \psi(x) < +\infty$

under a relaxed boundary condition that covers the limit case, i.e., a non-strict inequality is allowed.

Although less general than (H_f), this condition still works in all of the most important concrete situations. We emphasize that a non-symmetric version of (H'_f) has been introduced in [**9**] for extending some well-known min–max theorems to the non-differentiable setting. See also [**2**] and [**10**] for other applications of these abstract results in the framework of variational–hemi-variational inequalities.

Our goal is achieved by adapting the technical approach developed by Ghoussoub in [**8**] for C^1 functionals. It is worthwhile to stress that, in our framework, we cannot use the property that the sets $\{x \in X : f(x) \geq c\}$, $c \in \mathbb{R}$, are closed, which seems to be crucial in the arguments adopted therein. Likewise, it is not possible to follow the approach developed in [**7**] because the topological reasoning in that work involving the Hausdorff metric does not apply here. Such technical difficulties have been overcome mainly by means of a $\mathbb{Z}_2$-symmetric version of [**9**, Theorem 3.1] given as Theorem 3.1 below for functionals satisfying (H'_f), which furnishes a general method to construct a Palais–Smale sequence around a min–max sequence of suitable compact and symmetric subsets of X.

Basic definitions and preliminary results are contained in Section 2, while the main abstract results are established in Section 3. Some applications leading to infinitely many critical points for general non-smooth functionals are pointed out in Section 4.

Specifically, Theorem 4.1 provides a non-smooth $\mathbb{Z}_2$-symmetric version of SMPL and Theorem 4.3 ensures the existence of an unbounded sequence of critical values in the general non-smooth framework permitting to treat new classes of variational–hemivariational inequalities with symmetries.

2. Basic definitions and preliminary results. Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X, we write $\mathrm{int}(V)$ for the interior of V, $\overline{V}$ for the closure of V and ∂V for the boundary of V. When V is non-empty, $x \in X$ and $\delta > 0$, we define $B(x, \delta) := \{z \in X : \|z - x\| < \delta\}$ as well as $N_\delta(V) := \{z \in X : d(z, V) \leq \delta\}$. If Y is a subspace of X, we define $B_\delta(Y) = B(0, \delta) \cap Y$ and $S_\delta(Y) = \{z \in Y : \|z\| = \delta\}$. Given $x, z \in X$, the notation $[x, z]$ indicates the line segment joining x to z, namely

$$[x, z] := \{(1 - t)x + tz : t \in [0, 1]\}.$$

Moreover, $]x, z] := [x, z] \setminus \{x\}$. We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and X. A function $\Phi : X \to \mathbb{R}$ is called locally Lipschitz continuous when, to every $x \in X$, there correspond a neighbourhood V_x of x and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \, \forall \, z, w \in V_x.$$

If $x, z \in X$, we write $\Phi^0(x; z)$ for the generalized directional derivative of Φ at the point x along the direction z, i.e.,

$$\Phi^0(x; z) := \limsup_{w \to x, \, t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}.$$

It is known [**4**, Proposition 2.1.1] that Φ^0 is upper semi-continuous on $X \times X$. The generalized gradient of the function Φ at $x \in X$, denoted by $\partial \Phi(x)$, is the set

$$\partial \Phi(x) := \left\{x^* \in X^* : \langle x^*, z \rangle \leq \Phi^0(x; z) \, \forall \, z \in X\right\}.$$

In [**4**, Proposition 2.1.2], it is ensured that $\partial \Phi(x)$ turns out to be non-empty, convex, in addition to weak* compact.

Let f be an even function on X taking values in $\mathbb{R} \cup \{+\infty\}$ satisfying hypothesis (H_f). Set $D_\psi := \{x \in X : \psi(x) < +\infty\}$. Since ψ turns out to be continuous on $\mathrm{int}(D_\psi)$ (see, e.g., [**5**, Exercise 1, p. 296]), the same holds regarding f. To simplify the notation, always denote by $\partial \psi(x)$ the sub-differential of ψ at x in the sense of convex analysis, while

$$D_{\partial \psi} := \{x \in X : \partial \psi(x) \neq \emptyset\}.$$

Further, [**5**, Theorem 23.5] gives $\mathrm{int}(D_\psi) = \mathrm{int}(D_{\partial \psi})$. Moreover, by [**5**, Theorems 23.5 and 23.3], $\partial \psi(x)$ is always convex and weak* closed. We say that $x \in D_\psi$ is a critical point of f when

$$\Phi^0(x; z - x) + \psi(z) - \psi(x) \geq 0 \quad \forall \, z \in X.$$

If $\psi \equiv 0$, this clearly means $0 \in \partial \Phi(x)$; thus, x is a critical point of Φ according to [**3**, Definition 2.1]. The notation $K(f)$ indicates the set of all critical points of f. Given a real number c, we denote

$$K_c(f) := K(f) \cap f^{-1}(c) \quad \text{and} \quad f_c := \{x \in X : f(x) \leq c\}.$$

If $K_c(f) \neq \emptyset$, then $c \in \mathbb{R}$ is said to be a critical value of f. In this general framework, the classical Palais–Smale condition takes the form $(\mathrm{PS})_f$ in the preceding section (see [**12**, Definition 3.2]).

We say that a set $A \subseteq X$ is symmetric if $A = -A$. Let us recall that the $\mathbb{Z}_2$-index of Krasnoselski $\gamma : \Sigma \to \mathbb{N} \cup \{+\infty\}$ is defined on $\Sigma = \{A \subseteq X :$ A is closed and symmetric$\}$ as follows:

$$\gamma(A) := \inf\{k \in \mathbb{N} : \text{ there exists } \eta : A \to \mathbb{R}^k \setminus \{0\} \text{ odd and continuous}\}.$$

If no such a finite k exists, we set $\gamma(A) = +\infty$. In particular, if $0 \in A$, then $\gamma(A) = +\infty$. We also set $\gamma(\emptyset) = 0$. Moreover, γ satisfies the following properties:

(I_1) $\gamma(A) = 0$ if and only if $A = \emptyset$.
(I_2) $\gamma(A_1) \leq \gamma(A_2)$ if there exists $\eta \in C^0(A_1, A_2)$ odd.
(I_3) If $K \subset X$ is a compact and symmetric set, there exists $\delta > 0$ such that $\gamma(N_\delta(K)) = \gamma(K)$.
(I_4) $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$ for every $A_1, A_2 \in \Sigma$.
(I_5) If $K \subset X$ is a compact and symmetric set with $0 \notin K$, then K contains at least n pairs of points provided $\gamma(K) \geq n$.
(I_6) If $K \subset X$ is a compact and symmetric set with $0 \notin K$, then $\gamma(K) < +\infty$.

The following version [**6**, pp. 444, 456] of the famous variational principle of Ekeland will be repeatedly employed.

THEOREM 2.1. *Let (Z, d) be a complete metric space and let Π be a proper, lower semi-continuous, bounded below function from Z into $\mathbb{R} \cup \{+\infty\}$. Then to every $\epsilon, \delta > 0$ and every $\bar{z} \in Z$ satisfying $\Pi(\bar{z}) \leq \inf_{z \in Z} \Pi(z) + \epsilon$, there corresponds a point $z_0 \in Z$ such that*

$$\Pi(z_0) \leq \Pi(\bar{z}), \quad d(z_0, \bar{z}) \leq \frac{1}{\delta}, \quad \Pi(z) - \Pi(z_0) \geq -\epsilon\delta d(z, z_0) \quad \forall z \in Z.$$

Let us state a $\mathbb{Z}_2$-symmetric version of [**9**, Theorem 2.2].

THEOREM 2.2. *Let (H_f) be satisfied, $\epsilon > 0$ and B, C be two non-empty closed, symmetric sets in X. Suppose C is compact, $B \cap C = \emptyset$, $C \subseteq D_\psi$, and moreover, the following also holds:*

(a_1) *To each $x \in C$, there corresponds a point $\xi_x \in X$ such that*

$$\Phi^0(x; \xi_x - x) + \psi(\xi_x) - \psi(x) < -\epsilon\|\xi_x - x\|.$$

Then for every $k > 1$, there exists $t_0 \in (0, 1]$, $\alpha \in C^0([0, 1] \times X, X)$ such that $\alpha(t, \cdot)$ is odd for every $t \in [0, 1]$ and $\varphi \in C^0(X, \mathbb{R}_+)$ even, with the following properties:

(i_1) *$\alpha(t, D_\psi) \subseteq D_\psi$ $\forall t \in [0, t_0)$ and $\alpha(t, x) = x$ $\forall (t, x) \in [0, t_0) \times B$.*
(i_2) *$\|\alpha(t, x) - x\| \leq kt$ $\forall (t, x) \in [0, t_0) \times X$.*
(i_3) *$f(\alpha(t, x)) - f(x) \leq -\epsilon\varphi(x)t$ $\forall (t, x) \in [0, t_0) \times D_\psi$.*
(i_4) *$\varphi(x) = 1$ $\forall x \in C$.*

Proof. First of all, we can observe that, since Φ and ψ are even, one has that $0 \in \partial\Phi(0)$ and $\psi(0) = \min_{x \in X} \psi(x)$. Hence, $0 \in K(f)$. Moreover, (a_1) implies that $C \cap K(f) = \emptyset$. Thus, in particular, $0 \notin C$. Again by the evenness of Φ, it is easy to verify that

$$\Phi^0(-x; -v) = \Phi^0(x; v) \ \forall x, v \in X.$$

Hence, it is not restrictive to assume that $\xi_{-x} = -\xi_x$ for every $x \in C$. In fact, if $x \in C$ and $\xi_x \in X$ satisfies (a_1), one has that $-x \in C$ and

$$\begin{aligned} \Phi^0(-x; -\xi_x + x) + \psi(-\xi_x) - \psi(-x) &= \Phi^0(x; \xi_x - x) + \psi(\xi_x) - \psi(x), \\ &< -\epsilon \|\xi_x - x\|, \\ &= -\epsilon \| - \xi_x + x\|, \end{aligned}$$

that is, $\xi_{-x} = -\xi_x$ satisfies (a_1).

For every $x \in C$, by (a_1) and the upper semi-continuity of the function $(z, w) \mapsto \Phi^0(z; \xi_x - w) + \psi(\xi_x) - \psi(w) + \epsilon\|\xi_x - w\|$ on $X \times X$, we can find a positive number $\delta_x < \|\xi_x - x\|$ such that

$$\Phi^0(z; \xi_x - w) + \psi(\xi_x) - \psi(w) < -\epsilon\|\xi_x - w\| \quad \forall z, w \in B(x, \delta_x). \tag{1}$$

Moreover, δ_x can be chosen to be small enough such that $B(x, \delta_x) \cap B = \emptyset$, Φ is Lipschitz continuous in $B(x, \delta_x)$, $\delta_{-x} = \delta_x$ and bearing in mind that $0 \notin C$,

$$B(x, \delta_x) \cap B(-x, \delta_x) = \emptyset. \tag{2}$$

Define

$$V_x = B\left(x, \frac{\delta_x}{4}\right) \cup B\left(-x, \frac{\delta_x}{4}\right), \quad U_x = B\left(x, \frac{\delta_x}{2}\right) \cup B\left(-x, \frac{\delta_x}{2}\right).$$

The family $\mathcal{B} = \{V_x : x \in C\}$ represents an open covering of C. Since this set is compact, $\mathcal{B}$ possesses a finite sub-covering $\{V_{x_j} : j = 1, 2, \dots, m\}$ to which we can associate a continuous partition of unity $\{\hat{\chi}_j : j = 1, 2, \dots, m\}$, with $\sum_{j=1}^m \hat{\chi}_j(x) = 1$ for every $x \in C$ and $\sum_{j=1}^m \hat{\chi}_j(x) \leq 1$ for all $x \in X$. To simplify the notation, write

$$\xi_j = \xi_{x_j},\ \xi_{-j} = -\xi_j,\ \delta_j = \delta_{x_j},\ V_j = V_{x_j},\ U_j = U_{x_j}.$$

Let us assume that

$$\chi_j(x) = \frac{1}{2}\left(\hat{\chi}_j(x) + \hat{\chi}_j(-x)\right), \quad j = 1, 2, \dots, m \ \ \forall x \in X.$$

Obviously, $\{\chi_j : j = 1, 2, \dots, m\}$ is still a continuous partition of unity associated to $\{V_{x_j} : j = 1, 2, \dots, m\}$ such that $\sum_{j=1}^m \chi_j(x) = 1$ for every $x \in C$ and every χ_j is even. Let $\hat{l} : X \to [0, 1]$ be a continuous function such that

$$\hat{l}(x) = 1 \quad \forall x \in C, \quad \hat{l}(x) = 0 \quad \forall x \in X \setminus \cup_{j=1}^m V_j.$$

The function $l : X \to [0, 1]$ is defined by assuming that

$$l(x) = \frac{1}{2}\left(\hat{l}(x) + \hat{l}(-x)\right) \ \forall x \in X$$

is even, continuous and such that

$$l(x) = 1 \quad \forall x \in C, \quad l(x) = 0 \quad \forall x \in X \setminus \cup_{j=1}^m V_j.$$

Moreover, since $\delta_j < \|\xi_j - x_j\|$, one has that

$$\xi_j \notin \bar{B}(x_j, \delta_j),\ \xi_{-j} \notin \bar{B}(-x_j, \delta_j) \quad \forall j = 1, 2, \ldots, m. \tag{3}$$

At this point, if we assume that $\delta_0 = \frac{1}{2}\min_{1\le j\le m}\delta_j$ and fix $k > 1$, we can choose a positive number t_0 with $t_0 \le 1$ such that

$$t_0 < \min\left\{\frac{\delta_0}{1+k^2},\ \min_{1\le j\le m} d\left(\xi_j, B\left(x_j, \frac{\delta_j}{2}\right)\right)\right\}. \tag{4}$$

Starting with $\alpha_0(t,x) = x$, for every $(t,x) \in [0,1]\times X$, we define by induction on j, $1 \le j \le m$, the functions $\alpha_j : [0,1]\times X \to X$ as follows:

$$\alpha_j(t,x) = \begin{cases} \alpha_{j-1}(t,x) + tl(x)\chi_j(x)\dfrac{\xi_j - \alpha_{j-1}(t,x)}{\|\xi_j - \alpha_{j-1}(t,x)\|}, & \text{if } \alpha_{j-1}(t,x) \in B\left(x_j, \frac{\delta_j}{2}\right), \\ \alpha_{j-1}(t,x) + tl(x)\chi_j(x)\dfrac{\xi_{-j} - \alpha_{j-1}(t,x)}{\|\xi_{-j} - \alpha_{j-1}(t,x)\|}, & \text{if } \alpha_{j-1}(t,x) \in B\left(-x_j, \frac{\delta_j}{2}\right), \\ \alpha_{j-1}(t,x), & \text{otherwise.} \end{cases}$$

Observe that in view of (2) and (3), α_j is well defined. Moreover, a simple computation shows that for each $t \in [0,1]$, the function $\alpha_j(t,\cdot)$ is odd. We claim that for any $t \in [0,t_0)$, one has

$$\alpha_j(t, D_\psi) \subseteq D_\psi, \tag{5}$$

$$\|\alpha_j(t,x) - \alpha_{j-1}(t,x)\| \le kl(x)\chi_j(x)t \quad \forall x \in X, \tag{6}$$

$$f(\alpha_j(t,x)) - f(\alpha_{j-1}(t,x)) \le -\epsilon l(x)\chi_j(x)t \quad \forall x \in D_\psi. \tag{7}$$

Indeed, fix $j = 1$. If $x = \alpha_0(t,x) \notin U_1$, then (5) and (6) are obvious. Since $x \notin V_1$, we get $\chi_1(x) = 0$, which yields (7). Suppose now $x = \alpha_0(t,x) \in B(x_1, \delta_1/2)$. In this case, (5) immediately follows from $\xi_1 \in D_\psi$ and the convexity of D_ψ while (6) is trivial. In order to verify (7), by Lebourg's theorem ([**3**, Theorem 2.3.7]), there exists a suitable $z \in]\alpha_0(t,x), \alpha_1(t,x)[$ and $z^* \in \partial\Phi(z)$ such that

$$\begin{aligned} f(\alpha_1(t,x)) - f(\alpha_0(t,x)) &= \langle z^*, \alpha_1(t,x) - \alpha_0(t,x)\rangle + \psi(\alpha_1(t,x)) - \psi(\alpha_0(t,x)) \\ &\le \tau\left[\Phi^0(z; \xi_1 - \alpha_0(t,x)) + \psi(\xi_1) - \psi(\alpha_0(t,x))\right], \end{aligned} \tag{8}$$

where

$$\tau = \frac{tl(x)\chi_1(x)}{\|\xi_1 - \alpha_0(t,x)\|} < 1.$$

On account of (4) and (6), one has

$$\begin{aligned} \|z - x_1\| &\le \|z - \alpha_0(t,x)\| + \|\alpha_0(t,x) - x_1\| \\ &< \|\alpha_1(t,x) - \alpha_0(t,x)\| + \frac{\delta_1}{2} \le kl(x)\chi_1(x)t + \frac{\delta_1}{2} < \delta_1. \end{aligned}$$

Therefore, by (1), one has

$$\Phi^0(z; \xi_1 - \alpha_0(t, x)) + \psi(\xi_1) - \psi(\alpha_0(t, x)) < -\epsilon \|\xi_1 - \alpha_0(t, x)\|. \tag{9}$$

Combining (8) and (9) yields (7). Suppose now $x \in B(-x_1, \delta_1/2)$. In this case, (5) immediately follows from $\xi_{-1} = -\xi_1$, the symmetry and the convexity of D_ψ. While (6) is trivial, the proof of (7) can be obtained as in the previous case with ξ_{-1} and $-x_1$ in place of ξ_1 and x_1, respectively. Assume now that (5)–(7) hold up to $j-1$; we will prove them for $\alpha_j(t, x)$. The same reasoning made before gives (5) and (6). Let us next observe that

$$\chi_j(x) = 0 \ \text{ as soon as } \ \alpha_{j-1}(t, x) \notin U_j. \tag{10}$$

Indeed, if $\chi_j(x) \neq 0$ for some point x satisfying $\alpha_{j-1}(t, x) \notin U_j$, then $x \in V_j$. So, if $x \in B(x_j, \delta_j/4)$, by (6), one has

$$\begin{aligned}\|\alpha_{j-1}(t, x) - x_j\| &\le \|\alpha_{j-1}(t, x) - x\| + \|x - x_j\| < \sum_{h=1}^{j-1} \|\alpha_h(t, x) - \alpha_{h-1}(t, x)\| + \frac{\delta_j}{4} \\ &\le kl(x)t \sum_{h=1}^{j-1} \chi_h(x) + \frac{\delta_j}{4} < \frac{\delta_j}{2},\end{aligned}$$

that is, $\alpha_{j-1}(t, x) \in B(x_j, \delta_j/2)$ against the condition $\alpha_{j-1}(t, x) \notin U_j$. Analogous is the case $x \in B(-x_j, \delta_j/4)$. Therefore, if $\alpha_{j-1}(t, x) \notin U_j$, inequality (7) immediately follows from (10), while if $\alpha_{j-1}(t, x) \in U_j$, it can be established arguing exactly as in the case $j = 1$, with

$$\tau = \frac{tl(x)\chi_j(x)}{\|\xi_j - \alpha_{j-1}(t, x)\|} < \frac{d(\xi_j, B(x_j, \delta_j/2))}{\|\xi_j - \alpha_{j-1}(t, x)\|} \le 1$$

or

$$\tau = \frac{tl(x)\chi_j(x)}{\|\xi_{-j} - \alpha_{j-1}(t, x)\|} < \frac{d(\xi_{-j}, B(-x_j, \delta_j/2))}{\|\xi_{-j} - \alpha_{j-1}(t, x)\|} \le 1$$

if $\alpha_{j-1}(t, x) \in B(x_j, \delta_j/2)$ or $\alpha_{j-1}(t, x) \in B(-x_j, \delta_j/2)$, respectively. Suitable arguments relying on (10) imply that each α_j is continuous. Define $\alpha(t, x) = \alpha_m(t, x)$ and $\varphi(x) = l(x) \sum_{j=1}^m \chi_j(x)$, $(t, x) \in [0, 1] \times X$. Since $x \in B$ implies $l(x) = 0$, by (6), we get

$$\alpha(t, x) = \alpha_{m-1}(t, x) = \cdots = \alpha_0(t, x) = x,$$

and (i_1) is proved. Assertions (i_2) and (i_3) are consequences of the properties of l and χ_j $(j = 1, 2, \ldots, m)$ and (6), (7), respectively. Finally, since $l_{|C} \equiv 1$, one has $\varphi(x) = l(x) \sum_{j=1}^m \chi_j(x) = 1$ for all $x \in C$, which shows (i_4). □

3. Main results. The main purpose of this section is to prove Theorems 3.1 and 3.3, which give symmetric versions of [**9**, Theorem 3.1] and [**8**, Theorem 7.13], respectively, when the structural assumption (H'_f) is satisfied.

Let $B \subseteq X$ be closed and symmetric, and let $\mathcal{F}$ be a class of compact and symmetric sets in X. We say that $\mathcal{F}$ is a *symmetric homotopy-stable family with extended boundary*

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B when for every $A \in \mathcal{F}$ and every $\eta \in C^0([0,1] \times X, X)$ such that $\eta(t, \cdot)$ is odd for every $t \in [0,1]$ and $\eta(t,x) = x$ for every $(t,x) \in (\{0\} \times X) \cup ([0,1] \times B)$, one has $\eta(\{1\} \times A) \in \mathcal{F}$.

Moreover, if $B \subseteq A$ for every $A \in \mathcal{F}$, we say that $\mathcal{F}$ is a *symmetric homotopy-stable family with boundary* B.

The following assumptions will be posited in the sequel:

(a_2) *Let* $\{\mathcal{F}^\alpha\}$ *be a family of symmetric homotopy-stable classes with extended boundaries* $\{B^\alpha\}$ *and assume that* $\widetilde{\mathcal{F}} = \bigcup_\alpha \mathcal{F}^\alpha$. *The function* f *satisfies the condition* (H'_f) *and*

$$c := \inf_{A \in \widetilde{\mathcal{F}}} \sup_{x \in A} f(x) < +\infty.$$

(a_3) *There exists a closed symmetric subset* F *of* X *such that for each* α, *one has*

$$(F \cap A) \setminus B^\alpha \neq \emptyset \qquad \forall A \in \widetilde{\mathcal{F}}$$

and

$$\sup_{x \in B^\alpha} f(x) \leq \inf_{x \in F} f(x).$$

We explicitly observe that, by (a_2) and (a_3), one has that

$$\inf_{x \in F} f(x) \leq c. \tag{11}$$

THEOREM 3.1. *Assume that the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *verifies assumption* (H'_f). *Let* (a_2) *and* (a_3) *be satisfied. Then, to every sequence* $\{A_n\}$ *in* $\widetilde{\mathcal{F}}$ *such that* $\lim_{n \to +\infty} \sup_{x \in A_n} f(x) = c$, *there corresponds a sequence* $\{x_n\}$ *in* X *satisfying the following properties:*

(i_5) $\lim_{n \to +\infty} f(x_n) = c$.

(i_6) $\Phi^0(x_n; v - x_n) + \Psi(v) - \Psi(x_n) \geq -\epsilon_n \|v - x_n\|$ $\forall n \in \mathbb{N}$, $\forall v \in X$, *where* $\epsilon_n \to 0^+$.

(i_7) $\lim_{n \to +\infty} d(x_n, F) = 0$ *provided* $\inf_{x \in F} f(x) = c$.

(i_8) $\lim_{n \to +\infty} d(x_n, A_n) = 0$.

Proof. First we consider the case

$$\inf_{x \in F} f(x) = c. \tag{12}$$

Fix $\epsilon > 0$ and let $A_\epsilon \in \widetilde{\mathcal{F}}$ be such that

$$c \leq \sup_{x \in A_\epsilon} f(x) < c + \frac{\epsilon^2}{8}. \tag{13}$$

We claim the existence of $x_\epsilon \in X$, such that

$$c - \frac{\epsilon^2}{8} \leq f(x_\epsilon) \leq c + \frac{5\epsilon^2}{4}, \tag{14}$$

$$\Phi^\circ(x_\epsilon; v - x_\epsilon) + \Psi(v) - \Psi(x_\epsilon) \geq -5\epsilon \|v - x_\epsilon\| \;\; \forall v \in X, \tag{15}$$

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$$d(x_\epsilon, F) \le \frac{3\epsilon}{2}, \tag{16}$$

$$d(x_\epsilon, A_\epsilon) \le \frac{\epsilon}{2}, \tag{17}$$

which provides a sequence $\{x_n\}$ in X satisfying properties (i_5)–(i_8).

Since $A_\epsilon \in \widetilde{\mathcal{F}}$, there exists α such that $A_\epsilon \in \mathcal{F}^\alpha$. To simplify the notation, let

$$A'_\epsilon = \{1\} \times A_\epsilon, \quad F_\epsilon = N_\epsilon(F), \quad G_\epsilon = (\{0\} \times X) \cup ([0,1] \times ((A_\epsilon \setminus F_\epsilon) \cup B^\alpha)),$$

and let $\mathcal{L}$ be the space of all $\eta \in C^0([0,1] \times X, X)$ such that $\eta(t, \cdot)$ is odd for every $t \in [0, 1]$,

$$\eta(t,x) = x \;\forall (t,x) \in G_\epsilon, \qquad \sup_{(t,x)\in[0,1]\times X} \|\eta(t,x) - x\| < +\infty.$$

An easy computation shows that $\mathcal{L}$ equipped with the metric ρ of uniform convergence is complete. Moreover, since $(\{0\} \times X) \cup ([0,1] \times B^\alpha) \subseteq G_\epsilon$, it leads to

$$\eta\left(A'_\epsilon\right) \in \mathcal{F}^\alpha \subseteq \widetilde{\mathcal{F}} \quad \forall \eta \in \mathcal{L}. \tag{18}$$

Define now for every $x \in X$,

$$f_1(x) = \max\left\{0, \epsilon^2 - \epsilon d(x,F)\right\}, \quad f_2(x) = \min\left\{\frac{\epsilon^2}{8}, \epsilon d\left(x, (A_\epsilon \setminus F_\epsilon) \cup B^\alpha\right)\right\},$$

$$g(x) = f(x) + f_1(x) + f_2(x).$$

Moreover, let $I : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$I(\eta) = \sup_{z\in\eta(A'_\epsilon)} g(z) \quad \forall \eta \in \mathcal{L},$$

which is clearly lower semi-continuous. Combining (18) with assumption (a_3), one has

$$(\eta(A'_\epsilon) \cap F) \setminus B^\alpha \neq \emptyset.$$

Hence, for each $\eta \in \mathcal{L}$, we get

$$I(\eta) \ge \sup_{z\in\eta(A'_\epsilon)\cap F} g(z) \ge \sup_{z\in\eta(A'_\epsilon)\cap F} (f(z) + f_1(z))$$

$$= \sup_{z\in\eta(A'_\epsilon)\cap F} f(z) + \epsilon^2 \ge c + \epsilon^2,$$

that is,

$$\inf_{\eta\in\mathcal{L}} I(\eta) \ge c + \epsilon^2. \tag{19}$$

Therefore, bearing in mind the previous inequality and (13), assuming that $\overline{\eta}(t,x) = x$ for each $(t,x) \in [0,1] \times X$, one has

$$I(\overline{\eta}) < \inf_{\eta\in\mathcal{L}} I(\eta) + \epsilon^2/4. \tag{20}$$

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Then, by Theorem 2.1, there exists $\eta_0 \in \mathcal{L}$ such that

$$I(\eta_0) \leq I(\overline{\eta}), \tag{21}$$

$$\|\overline{\eta} - \eta_0\| \leq \frac{\epsilon}{2}, \tag{22}$$

$$I(\eta) \geq I(\eta_0) - \frac{\epsilon}{2}\|\eta - \eta_0\| \quad \forall \eta \in \mathcal{L}. \tag{23}$$

From (21), one has

$$\sup_{z \in \eta_0(A'_\epsilon)} \Psi(z) \leq I(\overline{\eta}) - \min_{z \in \eta_0(A'_\epsilon)} \Phi(z) < +\infty.$$

Thus, by (H'_f), Ψ becomes continuous on $\eta_0(A'_\epsilon)$ as well as the function $x \to g(\eta_0(1, x))$ $\forall x \in A_\epsilon$. So, the set

$$C = \left\{ z \in \eta_0(A'_\epsilon) : g(z) = \max_{w \in \eta_0(A'_\epsilon)} g(w) \right\}$$

is non-empty, symmetric and compact. Now, we write $B' = (A_\epsilon \backslash F_\epsilon) \cup B^\alpha$ and let us prove that

$$B' \cap C = \emptyset.$$

To this end, we first show that there exists $z_0 \in (\eta_0(A'_\epsilon) \cap F) \setminus B^\alpha$ such that

$$f(z_0) = \max_{x \in \eta_0(A'_\epsilon) \cap F} f(x). \tag{24}$$

Let $\widehat{z} \in \eta_0(A'_\epsilon) \cap F$ such that $f(\widehat{z}) = \max_{\eta_0(A'_\epsilon) \cap F} f$. If $\widehat{z} \notin B^\alpha$, then we can choose $z_0 = \widehat{z}$. Otherwise, by (a_3), there exists $z_0 \in (\eta_0(A'_\epsilon) \cap F) \setminus B^\alpha$ such that

$$\max_{z \in \eta_0(A'_\epsilon) \cap F} f(z) = f(\widehat{z}) \leq \sup_{x \in B^\alpha} f(x) \leq \inf_{x \in F} f(x) \leq f(z_0) \leq \max_{z \in \eta_0(A'_\epsilon) \cap F} f(z)$$

and (24) holds. Moreover, we have

$$\max_{z \in \eta_0(A'_\epsilon)} g(z) \geq f(z_0) + f_1(z_0) + f_2(z_0) \geq c + \epsilon^2 + f_2(z_0), \tag{25}$$

with $z_0 \notin B^\alpha \cup (\overline{A_\epsilon \setminus F_\epsilon})$ by the fact that $(\eta_0(A'_\epsilon) \cap F) \cap (\overline{A_\epsilon \setminus F_\epsilon}) = \emptyset$. Hence, one has $f_2(z_0) > 0$. On the other hand, $f_{1|X \backslash F_\epsilon} \equiv f_{2|A_\epsilon \backslash F_\epsilon} \equiv 0$ implies

$$\sup_{z \in A_\epsilon \backslash F_\epsilon} g(z) \leq \sup_{z \in A_\epsilon} f(z) < c + \epsilon^2/8, \tag{26}$$

and taking into account that $\sup_{B^\alpha} f \leq c$, we get

$$\sup_{z \in B^\alpha \cap F_\epsilon} g(z) = \sup_{z \in B^\alpha \cap F_\epsilon} (f + f_1)(z) \leq \sup_{z \in B^\alpha} f(z) + \epsilon^2 \leq c + \epsilon^2. \tag{27}$$

From (25)–(27), one has

$$\sup_{z\in B'} g(z) = \max\left\{\sup_{z\in A_\epsilon\setminus F_\epsilon} g(z),\ \sup_{z\in B^\alpha\cap F_\epsilon} g(z),\ \sup_{z\in B^\alpha\setminus F_\epsilon} g(z)\right\} \leq \max_{z\in\eta_0(A'_\epsilon)} g(z) - f_2(z_0),$$

which clearly ensures that $B'\cap C = \emptyset$.

Now we prove that there exists $x_\epsilon \in C$ satisfying (15). Suppose not, for $k \in]1, 2[$, we can apply Theorem 2.2 to B' and C with 5ϵ in place of ϵ. Therefore, there exist $t_0 \in]0, 1]$, $\beta \in C^0([0, 1]\times X, X)$ with $\beta(t, \cdot)$ odd for every $t \in [0, 1]$ and $\varphi \in C^0\left(X, \mathbb{R}^+\right)$ even satisfying (i$_1$)–(i$_4$). Pick $\lambda \in [0, t_1]$ where $t_1 \in]0, t_0]$ and define

$$\eta_\lambda(t, x) = \beta(\lambda t, \eta_0(t, x)) \quad \forall (t, x) \in [0, 1]\times X,$$

then we have $\eta_\lambda \in \mathcal{L}$ and $\|\eta_0 - \eta_\lambda\| \leq \lambda k$ for each λ. Hence, by (23), one has

$$I(\eta_\lambda) \geq I(\eta_0) - \frac{\lambda k\epsilon}{2}. \tag{28}$$

While (i$_3$) furnishes

$$\sup_{z\in\eta_\lambda\left(A'_\epsilon\right)} f(z) \leq \sup_{z\in\eta_0\left(A'_\epsilon\right)} f(z) - 5\epsilon\lambda \min_{z\in\eta_0\left(A'_\epsilon\right)} \varphi(z) \leq I(\eta_0) - 5\epsilon\lambda \min_{z\in\eta_0\left(A'_\epsilon\right)} \varphi(z) < +\infty.$$

Hence, by (H$'_f$), the function $x \to \Psi(\eta_\lambda(1, x))$ is continuous in A_ϵ as well as $x \to g(\eta_\lambda(1, x))$. So there exists $x_\lambda \in A_\epsilon$ such that

$$g(\eta_\lambda(1, x_\lambda)) = I(\eta_\lambda).$$

Then, for each $x \in A_\epsilon$, one has

$$g\left(\eta_\lambda\left(1, x_\lambda\right)\right) \geq g\left(\eta_0\left(1, x\right)\right) - \frac{\lambda\epsilon k}{2}. \tag{29}$$

At this point, we can show that

$$\begin{aligned} -\frac{\lambda\epsilon k}{2} \leq g\left(\eta_\lambda\left(1, x_\lambda\right)\right) - g\left(\eta_0\left(1, x\right)\right) &\leq -5\lambda\epsilon\varphi\left(\eta_0\left(1, x_\lambda\right)\right) + 2\epsilon\|\eta_\lambda - \eta_0\| \\ &\leq -5\lambda\epsilon\varphi\left(\eta_0\left(1, x_\lambda\right)\right) + 2\epsilon\lambda k. \end{aligned}$$

From this we get

$$\varphi\left(\eta_0\left(1, x_\lambda\right)\right) \leq \frac{k}{2} \ \ \forall\lambda \in [0, t_1]. \tag{30}$$

If $\widehat{x} \in A_\epsilon$ is a cluster point of $\{x_\lambda : \lambda \in [0, t_1]\}$ as $\lambda \downarrow 0$, bearing in mind that the function $(\lambda, x) \to g\left(\beta\left(\lambda, \eta_0\left(1, x\right)\right)\right)$ is continuous in $[0, t_1]\times A_\epsilon$, by (29), for every $x \in A_\epsilon$, one has

$$g\left(\eta_0(1, \widehat{x})\right) \geq g\left(\eta_0(1, x)\right),$$

which ensures $\eta_0\left(1, \widehat{x}\right) \in C$ and $\varphi\left(\eta_0\left(1, \widehat{x}\right)\right) = 1$ from (i$_4$). On the other hand, (30) implies that $\varphi\left(\eta_0\left(1, \widehat{x}\right)\right) < 1$. This contradiction shows that there exists $x_\epsilon \in C$ satisfying (15). Let $x_\epsilon = \eta_0\left(1, \widetilde{x}\right)$, with $\widetilde{x} \in A_\epsilon$. Arguing as in [**8**, p. 97], it is seen that the point x_ϵ satisfies (14), (16) and (17).

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Let us now consider the case

$$\inf_{x\in F} f(x) < c. \tag{31}$$

Fix $\epsilon > 0$ and choose $A_\epsilon \in \widetilde{\mathcal{F}}$ satisfying

$$c \le \sup_{x\in A_\epsilon} f(x) < c + \frac{\epsilon^2}{4}. \tag{32}$$

We claim the existence of $x_\epsilon \in X$ such that

$$c \le f(x_\epsilon) \le c + \frac{\epsilon^2}{4}, \tag{33}$$

$$\Phi^0(x_n; v - x_\epsilon) + \Psi(v) - \Psi(x_\epsilon) \ge -\epsilon\|v - x_\epsilon\| \;\; \forall v \in X, \tag{34}$$

$$d(x_\epsilon, A_\epsilon) \le \frac{\epsilon}{2}, \tag{35}$$

which provides a sequence $\{x_n\}$ in X satisfying properties (i_5), (i_6) and (i_8). To obtain this, observing that $A_\epsilon \in \mathcal{F}^\alpha$ for some α, we can make use of the reasoning in the previous case of (12) replacing the function g with f, the space $\mathcal{L}$ with the space $\widehat{\mathcal{L}}$ of all $\eta \in C^0([0,1]\times X, X)$ such that $\eta(t,\cdot)$ is odd for every $t \in [0,1]$,

$$\eta(t,x) = x \; \forall (t,x) \in (\{0\}\times X) \cup ([0,1]\times B^\alpha), \qquad \sup_{[0,1]\times X} \|\eta(t,x) - x\| < +\infty,$$

and the set B' with B^α. □

As a consequence of Theorem 3.1, we have the following critical point result.

THEOREM 3.2. *Let f be a function satisfying* (H_f') *and* $(\mathrm{PS})_f$ *in addition to* (a_2) *and* (a_3). *Then, for every sequence $\{A_n\} \subseteq \widetilde{\mathcal{F}}$ such that $\lim_{n\to+\infty} \sup_{x\in A_n} f(x) = c$, if we set $A_\infty = \{x \in X : \liminf_{n\to+\infty} d(x, A_n) = 0\}$, one has $K_c(f) \cap A_\infty \neq \emptyset$. If, moreover, $\inf_{x\in F} f(x) = c$, then $K_c(f) \cap F \cap A_\infty \neq \emptyset$.*

The following lemma will be useful for proving Theorem 3.3.

LEMMA 3.1. *Let f be a function satisfying* (a_2) *and* $(\mathrm{PS})_f$ *condition. In addition, we assume the following:*

(a_3') *There exists a symmetric, closed set $F \subseteq X$ such that for every α*

$$F \cap B^\alpha = \emptyset \quad \text{and} \quad F \cap A \neq \emptyset \quad \textit{for every } A \in \widetilde{\mathcal{F}}$$

$$\sup_{x\in B^\alpha} f(x) \le \inf_{x\in F} f(x) = c.$$

Then one has $\gamma(K_c(f) \cap F \cap A_\infty) \ge \inf\{\gamma(A\cap F) : A \in \widetilde{\mathcal{F}}\}$.

Proof. Let $n = \inf\{\gamma(A\cap F) : A \in \widetilde{\mathcal{F}}\}$. Theorem 3.2 ensures that $K_c(f) \cap F \cap A_\infty \neq \emptyset$, and by the index properties, there exists an open set U such that $K_c(f) \cap$

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$F \cap A_\infty \subset U$ and $\gamma(\overline{U}) = \gamma(K_c(f) \cap F \cap A_\infty)$ besides, for each $A \in \widetilde{\mathcal{F}}$,

$$\begin{aligned} n \leq \gamma(A \cap F) &\leq \gamma(A \cap F \cap (X \setminus U)) + \gamma(\overline{U}) \\ &= \gamma(A \cap F \cap (X \setminus U)) + \gamma(K_c(f) \cap F \cap A_\infty). \end{aligned}$$

Hence, if it were true that $\gamma(K_c(f) \cap F \cap A_\infty) \leq n - 1$, then it would follow

$$\gamma(A \cap F \cap (X \setminus U)) \geq 1 \quad \forall A \in \widetilde{\mathcal{F}}$$

and

$$\inf_{x \in F \cap (X \setminus U)} f(x) = c.$$

In other words, Theorem 3.2 applies to the symmetric, closed set $F \cap (X \setminus U)$ and we obtain

$$K_c(f) \cap A_\infty \cap F \cap (X \setminus U) \neq \emptyset,$$

which is absurd. □

THEOREM 3.3. *Let f be a function satisfying* (H'_f) *and* $(\mathrm{PS})_f$. *Let $\widetilde{\mathcal{F}}_j = \cup_\alpha \mathcal{F}_j^\alpha$ where, for every $j = 1, 2, \ldots, N$ and for every α, $\mathcal{F}_j^\alpha$ is a symmetric homotopy-stable family with boundary B_j^α such that $\mathcal{F}_{j+1}^\alpha \subseteq \mathcal{F}_j^\alpha$. Assume that*

(E) *For every $1 \leq j \leq j + p \leq N$, any $A \in \mathcal{F}_{j+p}^\alpha$ and any U open and symmetric such that $U \cap B_j^\alpha = \emptyset$ and $\gamma(\overline{U}) \leq p$, we have $A \setminus U \in \mathcal{F}_j^\alpha$.*

In addition, assume the following:

(a'_2) *For every $j = 1, 2, \ldots, N$, there exists $\widetilde{A}_j \in \widetilde{\mathcal{F}}_j$ such that*

$$\sup_{x \in \widetilde{A}_j} f(x) < +\infty.$$

(a''_3) *There exists a symmetric and closed set $F \subset X$ such that for every $j = 1, 2, \ldots, N$, and for every α*

$$F \cap B_j^\alpha = \emptyset, \qquad F \cap A \neq \emptyset \quad \textit{for every } A \in \widetilde{\mathcal{F}}_j,$$

$$\sup_{x \in B_j^\alpha} f(x) \leq \inf_{x \in F} f(x).$$

For every $j = 1, 2, \ldots, N$, set $c_j := \inf_{A \in \widetilde{\mathcal{F}}_j} \sup_{x \in A} f(x)$, $d = \inf_{x \in F} f(x)$,

$$J = \{k : c_k = d\} \quad \textit{and} \quad M = \begin{cases} \max J, & \textit{if } J \neq \emptyset \\ 0, & \textit{if } J = \emptyset. \end{cases}$$

Then

(a) *for every sequence $\{A_n\} \in \widetilde{\mathcal{F}}_M$ such that $\lim_{n \to +\infty} \sup_{x \in A_n} f(x) = c_M$, one has*

$$\gamma(K_{c_M}(f) \cap F \cap A_\infty) \geq M;$$

(b) *for every $M < j \leq j + p \leq N$ such that $c_j = c_{j+p} = \bar{c}$ and for every sequence $\{A_n\} \in \widetilde{\mathcal{F}}_{c_{j+p}}$ such that $\lim_{n \to +\infty} \sup_{x \in A_n} f(x) = \bar{c}$, one has $\gamma(K_{\bar{c}}(f) \cap A_\infty) \geq p + 1$.*

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Moreover, if $0 \in f_d \setminus F$, *then*

(c) f *has at least* N *distinct pairs of critical points;*

(d) $\lim_{j\to+\infty} c_j = +\infty$ *provided* $N = +\infty$.

Proof. First of all, we observe that assumption (a_2') ensures that every c_j is finite.

(a) For $M = 0$, $M = 1$, there is nothing to prove (see Theorem 3.2). Whereas for $M \geq 2$, bearing in mind Lemma 3.1, it is enough to show that for every $A \in \widetilde{\mathcal{F}}_M$, $\gamma(A \cap F) \geq M$. Arguing by contradiction, there exists $A \in \widetilde{\mathcal{F}}_M$ such that $\gamma(A \cap F) \leq M - 1$ and $A \in \mathcal{F}_M^\alpha$ for some α. By (a_3''), we have $F \cap B_1^\alpha = \emptyset$ and $F \cap A \neq \emptyset$; hence, there exists an open and symmetric set U such that $\gamma(\overline{U}) = \gamma(A \cap F)$, $U \cap B_1^\alpha = \emptyset$ and $A \cap F \subset U$. Therefore, condition (E) ensures that $A \setminus U \in \mathcal{F}_1^\alpha$. Thus, $(A \setminus U) \cap F \neq \emptyset$, which is absurd.

(b) Let $M < j \leq j + p \leq N$ with $c_j = c_{j+p} = \bar{c}$ and let $\{A_n\} \subseteq \widetilde{\mathcal{F}}_{c_{j+p}}$ be such that $\lim_{n\to+\infty} \sup_{x\in A_n} f(x) = \bar{c}$. By contradiction, assume that $\gamma(K_{\bar{c}}(f) \cap A_\infty) \leq p$. Since $d < \bar{c}$, one has

$$f_d \cap (K_{\bar{c}}(f) \cap A_\infty) = \emptyset.$$

Hence, there exists $\rho > 0$ such that

$$\gamma(N_{2\rho}(K_{\bar{c}}(f) \cap A_\infty)) = \gamma(K_{\bar{c}}(f) \cap A_\infty)$$

and for every α

$$N_{2\rho}(K_{\bar{c}}(f) \cap A_\infty) \cap B_j^\alpha = \emptyset.$$

Further, for every $k \in \mathbb{N}$, there exists $A_{n_k} \in \widetilde{\mathcal{F}}_{c_{j+p}}$ such that

$$\bar{c} \leq \sup_{x\in A_{n_k}} f(x) < \bar{c} + \frac{1}{k}, \tag{36}$$

and we assume that

$$\widehat{A}_k = A_{n_k} \setminus \operatorname{int}(N_{2\rho}(K_{\bar{c}}(f) \cap A_\infty)).$$

Condition (E) implies that $\{\widehat{A}_k\} \subseteq \widetilde{\mathcal{F}}_{c_j}$, and by (36), it follows that

$$\bar{c} = c_j \leq \sup_{x\in\widehat{A}_k} f(x) \leq \sup_{x\in A_{n_k}} f(x) < \bar{c} + \frac{1}{k}.$$

Hence,

$$\lim_{k\to+\infty} \sup_{x\in\widehat{A}_k} f(x) = \bar{c}.$$

Thus, applying Theorem 3.1 to $\widetilde{\mathcal{F}}_{c_j}$ and $\{\widehat{A}_k\}$, and taking into account $(PS)_f$ condition, we get a sequence $\{x_k\}$ satisfying (i_5), (i_6) and (i_8) such that

$$x_k \to x^* \in K_{\bar{c}} \cap \widehat{A}_\infty \subseteq K_{\bar{c}} \cap A_\infty, \tag{37}$$

where $\widehat{A}_\infty = \{x \in X : \liminf_{k\to+\infty} d(x, \widetilde{A}_k) = 0\}$. Now we claim that for every $q \in \mathbb{N}$,

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there exists $k_q > q$ such that

$$d(x_{k_q}, K_{\bar{c}}(f) \cap A_\infty) > \rho. \tag{38}$$

Suppose (38) is false. Then, there exists $\bar{q} \in \mathbb{N}$ such that for every $k > \bar{q}$, one has

$$d(x_k, K_{\bar{c}}(f) \cap A_\infty) \leq \rho.$$

Hence, if $k \in \mathbb{N}$ is large enough, $d(x_k, x^*) < \rho$ and, by (i_8), there exists $y_k \in \widehat{A}_k$ such that $d(x_k, y_k) < \rho$. Finally, we can write

$$2\rho \leq d(y_k, K_{\bar{c}} \cap A_\infty) \leq d(y_k, x_k) + d(x_k, K_{\bar{c}} \cap A_\infty) < \rho + d(x_k, x^*) < 2\rho,$$

which is absurd and (37) is proved. The contradiction between (37) and (38) justifies (b).

(c) The assertion follows from (a), (b) and property (I_5).

(d) First of all, we can observe that $M < +\infty$. In fact, since by hypothesis $0 \notin K_d(f) \cap F$, combining together (a) and the index property (I_6), one has

$$M \leq \gamma(K_{c_M}(f) \cap F) = \gamma(K_d(f) \cap F) < +\infty.$$

In a similar way, it is possible to verify that the sequence $\{c_j\}$ is not stationary. Clearly, the sequence $\{c_j\}$ is monotone. Let us prove that $c_j \to +\infty$. Arguing by contradiction, let us verify that $K_c(f)$ is non-empty, where $\lim_{j\to+\infty} c_j = c < +\infty$. Since $K_{c_j}(f) \neq \emptyset$, there exists a sequence $\{x_j\}$, with $f(x_j) = c_j$ satisfying

$$\Phi^0(x_j; z - x_j) + \psi(z) - \psi(x_j) \geq 0 \quad \forall j \in \mathbb{N},\ z \in X. \tag{39}$$

Moreover, by $(PS)_f$ condition, there exists $x^* \in K(f)$ such that $x_j \to x^*$, where a subsequence is considered if it is necessary. On the other hand, (39) implies that

$$\psi(x_j) \leq \Phi^0(x_j; x^* - x_j) + \psi(x^*) \quad \forall j \in \mathbb{N}.$$

Hence,

$$\psi(x^*) \leq \liminf_{k\to+\infty} \psi(x_{n_k}) \leq \limsup_{k\to+\infty} \psi(x_{n_k}) \leq \psi(x^*),$$

and we conclude that $x^* \in K_c(f)$ since

$$f(x^*) = \lim_{j\to+\infty} f(x_j) = c.$$

Clearly, $K_c(f)$ is a symmetric, compact set, and since $f(0) \leq d < c$, one has that $0 \notin K_c(f)$ and $f_d \cap K_c(f) = \emptyset$. Thus, properties ($I_6$) and ($I_3$) ensure that $\gamma(K_c(f)) = q < +\infty$ and that there exists a symmetric, open set U containing $K_c(f)$, which is disjoint from f_d such that $\gamma(\bar{U}) = q$.

Fix $\epsilon > 0$, we claim that there exist $j_\epsilon \in \mathbb{N}$, $A_\epsilon \in \widetilde{\mathcal{F}}_{j_\epsilon}$ such that

$$c_{j_\epsilon} \leq \sup_{x\in A_\epsilon} f(x) < c_{j_\epsilon} + \frac{\epsilon^2}{4}. \tag{40}$$

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In fact, let $j_\epsilon \in \mathbb{N}$ be such that $c_{j_\epsilon+q} < c_{j_\epsilon} + \frac{\epsilon^2}{8}$ and choose $\widehat{A}_\epsilon \in \widetilde{\mathcal{F}}_{j_\epsilon+q}$ with

$$c_{j_\epsilon} \leq c_{j_\epsilon+q} \leq \sup_{x\in\widehat{A}_\epsilon} f(x) < c_{j_\epsilon+q} + \epsilon^2/8 < c_{j_\epsilon} + \frac{\epsilon^2}{4}.$$

At this point, there are no difficulties in verifying that the set $A_\epsilon = \widehat{A}_\epsilon \backslash U \in \widetilde{\mathcal{F}}_{j_\epsilon}$ satisfies (40). Consequently, observing that (40) reduces to (32) with $\widetilde{\mathcal{F}}_{j_\epsilon}$ in place of $\widetilde{\mathcal{F}}$, reasoning as in the proof of Theorem 3.1, we find a point $x_\epsilon \in X$ satisfying (33)–(35). The previous arguments, in addition to $(\mathrm{PS})_f$, provide the sequences $\{x_n\}$ in X, convergent to a point $x^* \in K_c(f)$, and $\{A_n\}$ in $\bigcup_j \widetilde{\mathcal{F}}_j$ with $A_n \subseteq X\backslash U$ for every $n \in \mathbb{N}$ and $d(x_n, A_n) \to 0$. Hence,

$$d(x^*, X\backslash U) \leq d(x^*, x_n) + d(x_n, A_n)$$

for every $n \in \mathbb{N}$. That is, $x^* \notin U$ against the fact that $K_c(f) \subseteq U$. □

In the sequel, following [**8**], we will use the F-intersection index Int_B^F, where F and B are two closed, symmetric, disjoint sets in X, while Int_B^F is defined on $\Sigma_B = \{A \subseteq X :$ A is closed, symmetric and $B \subseteq A\}$ in the following way:

$$\mathrm{Int}_B^F(A) = \inf\{\gamma(\eta(A) \cap F) : \ \eta \in \mathcal{L}_B\}$$

and $\mathcal{L}_B = \{\eta \in C^0(X, X) : \ \eta$ is odd and $\eta_{|B} \equiv \mathrm{id}_{|B}\}$. Recall that $\mathrm{Int}_B^F : \Sigma_B \to \mathbb{N} \cup \{+\infty\}$ satisfies the following properties:

(T_1) $\mathrm{Int}_B^F(A_1) \leq \mathrm{Int}_B^F(A_2)$ if $\eta(A_1) \subseteq A_2$ for some $\eta \in \mathcal{L}_B$.
(T_2) $\mathrm{Int}_B^F(A_1 \cup A_2) \leq \mathrm{Int}_B^F(A_1) + \mathrm{Int}_B^F(A_2)$.
(T_3) If $\mathrm{Int}_B^F(A) \geq 1$, then $A \cap F \neq \emptyset$.

As a consequence of Theorem 3.3, we can obtain the following non-smooth version of [**8**, Corollary 7.17].

THEOREM 3.4. *Let f be a function satisfying (H_f') and $(\mathrm{PS})_f$. Suppose that $\{B_j\}_{j=1,\dots,N}$ is an increasing family of symmetric, closed sets, F is a closed, symmetric set such that $F \cap B_N = \emptyset$ and*

$$\sup_{x\in B_N} f(x) \leq \inf_{x\in F} f(x). \tag{41}$$

Assume that there exists a symmetric, compact set A_0 such that

$$B_N \subseteq A_0,\ \mathrm{Int}_{B_N}^F(A_0) \geq N,\ \sup_{x\in A_0} f(x) < +\infty. \tag{42}$$

Then, there exist critical values c_j $(1 \leq j \leq N)$ for f such that if $d = \inf_{x\in F} f(x)$ then

(a) *$\gamma(K_{c_M}(f) \cap F) \geq M$;*
(b) *for every $M < j \leq j + p \leq N$ with $c_j = c_{j+p} = \bar{c}$, we have $\gamma(K_{\bar{c}}(f)) \geq p + 1$ (where M is as in Theorem 3.3).*

In particular, if $0 \in f_d \setminus F$, then

(c) *f has at least N distinct pairs of critical points;*
(d) *f has an unbounded sequence of critical values provided the hypothesis hold for arbitrarily large N.*

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Proof. For every $1 \leq j \leq N$, let us consider the family

$$\mathcal{F}_j = \{A \subseteq X : \ A \text{ is symmetric and compact, } B_j \subseteq A \text{ and } \mathrm{Int}^F_{B_j}(A) \geq j\}.$$

By the monotonicity of $\{B_j\}$, it is simple to verify that

$$\mathcal{F}_N \subseteq \mathcal{F}_{N-1} \subseteq \cdots \subseteq \mathcal{F}_1.$$

Assumption (42) ensures that $A_0 \in \mathcal{F}_N$. Let us verify that every $\mathcal{F}_j$ is a symmetric homotopy-stable family with boundary B_j, satisfying the excision property (E). Fix j, $A \in \mathcal{F}_j$ and $\eta \in C^0([0, 1] \times X, X)$ such that $\eta(t, \cdot)$ is odd for every $t \in [0, 1]$ and $\eta(t, x) = x$ for every $(t, x) \in (\{0\} \times X) \cup ([0, 1] \times B_j)$. Of course, $\hat{A} = \eta(\{1\} \times A)$ is a symmetric and compact set such that $B_j \subseteq \hat{A}$. Moreover,

$$\mathrm{Int}^F_{B_j}(\hat{A}) = \inf\{\gamma(\nu(\eta(\{1\} \times A)) \cap F) : \ \nu \in \mathcal{L}_{B_j}\} \geq \mathrm{Int}^F_{B_j}(A) \geq j,$$

that is, $\mathcal{F}_j$ is a symmetric homotopy-stable family with boundary B_j.

Let $A \in \mathcal{F}_{j+p}$ and U be a symmetric open set such that $\gamma(\bar{U}) \leq p$ and $\bar{U} \cap B_j = \emptyset$. Then, $A \setminus U \in \Sigma_{B_j}$, and bearing in mind ($T_1$) and ($T_2$), one has

$$j + p \leq \mathrm{Int}^F_{B_{j+p}}(A) \leq \mathrm{Int}^F_{B_j}(A) \leq \mathrm{Int}^F_{B_j}((A \setminus U) \cup \bar{U}) \leq \mathrm{Int}^F_{B_j}(A \setminus U) + \gamma(\bar{U})$$

$$\leq \mathrm{Int}^F_{B_j}(A \setminus U) + p,$$

that is, $\mathrm{Int}^F_{B_j}(A \setminus U) \geq j$ and $A \setminus U \in \mathcal{F}_j$. By the definition of $\mathcal{F}_j$ and property (T_3), one has that $F \cap A \neq \emptyset$ for every $A \in \mathcal{F}_j$ and each $j = 1, \ldots, N$. Finally, from (42), (a'_2) follows immediately. Hence, all the assumptions of Theorem 3.3 are satisfied and the conclusion is achieved. □

4. Some applications. The following two results are meaningful consequences of Theorems 3.4 and 3.3, respectively.

THEOREM 4.1. *Let f be a function satisfying* (H'_f) *and* $(PS)_f$ *on* $X = Y \oplus Z$ *with* $\dim(Y) = k < +\infty$. *Assume that* $f(0) = 0$ *in addition to the following:*

(i) *There exist* $\rho > 0$ *and* $\beta \geq 0$ *such that* $\inf_{x \in S_\rho(Z)} f(x) \geq \beta$.

(ii) *There exist* $R > \rho$ *and a subspace* E *of* X *containing* Y *such that* $\dim(E) = n > k$ *and* $\sup_{x \in S_R(E)} f(x) \leq 0$.

Then, there exist critical values c_j ($1 \leq j \leq n - k$) *for* f *such that*

(a) $0 \leq c_1 \leq \cdots \leq c_{n-k}$;

(b) *f has at least $n - k$ distinct pairs of non-trivial symmetric critical points.*

Moreover, if $c_j = \beta$ for some $1 \leq j \leq n - k$, we have

$$\gamma(K_\beta(f) \cap S_\rho(Z)) \geq j.$$

Proof. Assumption (ii) ensures the existence of $e_1, \ldots, e_N \in E$ such that $E = Y \oplus \mathrm{span}\{e_1, \ldots, e_N\}$, where $N = n - k$. For every $j = 1, \ldots, N$, let us assume that

$$E_j = Y \oplus \mathrm{span}\{e_1, \ldots, e_j\}, \quad B_j = S_R(E_j).$$

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Moreover, let

$$F = S_\rho(Z), \quad A_0 = \overline{B_R(E)}.$$

Clearly, $\{B_j\}_{j=1,\dots,N}$ is an increasing family of symmetric, closed sets as well as F is symmetric, closed, $F \cap B_N = \emptyset$ and $\sup_{x\in S_R(E)} f(x) \le \inf_{x\in S_\rho(Z)} f(x)$. A_0 is a symmetric, compact set such that $B_N \subseteq A_0$, and by [**8**, Lemma 7.21], $\mathrm{Int}^F_{B_N}(A_0) \ge N$. Finally, by assumption (ii) and the convexity of ψ, one has $\sup_{x\in A_0} f(x) < +\infty$. So the conclusion follows at once from Theorem 3.4 observing that $0 \in f_\beta \setminus F$. □

THEOREM 4.2. *Let f be a function satisfying* (H'_f) *and* $(\mathrm{PS})_f$ *on $X = Y \oplus Z$ with* $\dim(Y) = k < +\infty$. *Assume that $f(0) = 0$,* (i) *in Theorem 4.1 holds:*

(ii′) *There exists an increasing sequence $\{E_n\}$ of finite-dimensional subspaces of X, containing Y such that $\lim_{n\to+\infty} \dim E_n = +\infty$ and for each $n \in \mathbb{N}$, $\sup_{x\in S_{R_n}(E_n)} f(x) \le 0$ for some $R_n > \rho$.*

Then, f has an unbounded sequence of critical values.

Proof. The conclusion follows from (d) of Theorem 3.3 and [**8**, Lemma 7.21] if, for every $j > k$, we consider

$$\widetilde{\mathcal{F}}_j = \bigcup_n \{\mathcal{F}^n_j : \ \dim(E_n) \ge j\},$$

where

$$\mathcal{F}^n_j = \{A \subseteq X : \ A \text{ is symmetric and compact, } B^n_j \subseteq A \text{ and } \mathrm{Int}^F_{B^n_j}(A) \ge j - k\},$$

$$F = S_{R_n}(Z), \quad B^n_j = S_{R_n}(E_n),$$

as well as

$$\widetilde{A}_j = \overline{B_{R_{n_j}}(E_{n_j})},$$

with $n_j = \min\{n \in \mathbb{N} : \ \dim(E_n) \ge j\}$. □

THEOREM 4.3. *Let f be a function satisfying* (H'_f) *on a reflexive Banach space $X = Y \oplus Z$ with* $\dim(Y) = k < +\infty$. *Assume that* (i) *of Theorem 4.1 holds, $f(0) = 0$, $D(\Psi)$ is a cone, in addition to the following:*

(H_1) *There exist constants $c_0 > 0$, $c_1 > 0$, $\alpha > 1$, $\mu > 0$ such that*

$$\mu\Phi(x) - \Phi^0(x; x) + (\mu + 1)\Psi(x) - \Psi(2x) \ge c_0\|x\|^\alpha - c_1 \ \textit{for all } x \in D(\Psi).$$

(H_2) *Every sequence $\{x_n\}$ in X such that $x_n \rightharpoonup x$ and $\liminf_{n\to+\infty} \Phi^0(x_n; x - x_n) \ge 0$ possesses a strongly convergent subsequence.*

(H_3) *There exists an increasing sequence $\{E_n\}$ of finite-dimensional subspaces of X containing Y with $\lim_{n\to+\infty} \dim E_n = +\infty$ and there exists a constant $a > 0$ such that for each n*

$$\mu\Phi(x) - \Phi^0(x; x) \ge 0 \ \textit{for all } x \in E_n \textit{ with } \|x\| > a$$

and

$$\limsup_{t\to+\infty} \frac{\Psi(tx)}{t^\mu} < -\max_{S_a(E_n)} \Phi \text{ uniformly with respect to } x \in S_a(E_n).$$

Then f has an unbounded sequence of critical values.

Proof. Let us verify that f satisfies the $(\mathrm{PS})_f$-condition. Let $\{x_n\}$ be a sequence in X such that $\{f(x_n)\}$ is bounded and holds

$$\Phi^0(x_n; v - x_n) + \Psi(v) - \Psi(x_n) \geq -\epsilon_n \|v - x_n\| \text{ for all } v \in X, \tag{43}$$

where $\epsilon_n \downarrow 0$. Therefore, for some $M > 0$, we have

$$\Phi(x_n) + \Psi(x_n) \leq M \text{ for all } n.$$

Combining this and (43) written with $v = 2x_n$ yields

$$\mu M + \|x_n\| \geq \mu\Phi(x_n) - \Phi^0(x_n; x_n) + (\mu+1)\Psi(x_n) - \Psi(2x_n) \geq c_0\|x_n\|^\alpha - c_1.$$

Since $\alpha > 1$, we infer that $\{x_n\}$ is bounded in X. Let $x \in X$ be such that $x_n \rightharpoonup x$, where a subsequence is considered if necessary. Exploiting (43) with $v = x$, one has

$$\liminf_{n\to+\infty} \Phi^0(x_n; x - x_n) \geq \liminf_{n\to+\infty} \Psi(x_n) - \Psi(x) \geq 0.$$

Thus, the $(\mathrm{PS})_f$-condition follows from (H_2).

For every $t > 1$ and $x \in X$, using Lebourg's mean value theorem, we find $z^* \in \partial_s(s^{-\mu}\Phi(sx))(\tau)$ with some $\tau \in]1, t[$ such that

$$t^{-\mu}\Phi(tx) - \Phi(x) = z^*(t-1) \leq \tau^{-(\mu+1)}\left(\Phi^0(\tau x; \tau x) - \mu\Phi(\tau x)\right)(t-1).$$

Now fix a positive integer n. Making use of the above inequality and the first part of assumption (H_3), we find for every $x \in S_a(E_n)$ that

$$f(tx) \leq \frac{t^\mu(t-1)}{\tau^{\mu+1}}\left(\Phi^0(\tau x; \tau x) - \mu\Phi(\tau x)\right) + t^\mu\Phi(x) + \Psi(tx)$$

$$\leq t^\mu\left(\frac{\Psi(tx)}{t^\mu} + \max_{S_a(E_n)} \Phi\right).$$

Then the second part of (H_3) provides $b_n \in \mathbb{R}$ and $t_n > \frac{\varrho}{a}$ such that

$$\frac{\Psi(t_n x)}{t_n^\mu} \leq b_n < -\max_{S_a(E_n)} \Phi \text{ for all } x \in S_a(E_n).$$

We see that condition (ii′) holds with $R_n = t_n a$ for every n. So, owing to Theorem 4.2, our conclusion is achieved. □

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