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An autocorrelation and a discrete spectrum for dynamical systems on metric spaces

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Abstract. We study dynamical systems (X, G, m) with a compact metric space X, a locally compact, σ-compact, abelian group G and an invariant Borel probability measure m on X. We show that such a system has a discrete spectrum if and only if a certain space average over the metric is a Bohr almost periodic function. In this way, this average over the metric plays, for general dynamical systems, a similar role to that of the autocorrelation measure in the study of aperiodic order for special dynamical systems based on point sets.

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1. *Introduction*

There has been a substantial amount of interest in dynamical systems with discrete spectra in the past and they have attracted quite a lot of attention in recent years.

One reason for this is that such systems play an important role in the investigation of aperiodic order. Aperiodic order, also known as mathematical theory of quasicrystals, has emerged as a relevant topic of research over the last three decades (see, e.g., [**1**] for a recent monograph and [**13**] for a recent collection of surveys). A key feature of aperiodic order is the occurrence of (pure) point diffraction. Due to a collective effort over the years, pure point diffraction is now understood to be the discrete spectrum of suitable associated dynamical systems [**2**, **6**, **12**, **14**, **17**, **18**].

Another example of the recent interest in discrete spectra can be found in a series of works which analyze such spectra via weak notions of equicontinuity [**9**–**11**]. These works provide, in particular, a characterization of a discrete spectrum (see also [**8**, **25**] for related work) and a characterization of a discrete spectrum with continuous eigenfunctions and unique ergodicity.

The dynamical systems (X, G, m) underlying the investigation of aperiodic order (and defined in detail in §6) have a special structure. The compact space X, on which the locally compact, σ-compact, abelian group G acts, consists of point sets or, more generally, measures. Accordingly, these systems are known as *translation bounded measure dynamical systems* (TMDSs). The fact that the points of X are measures allows one to pair elements of X with elements from the vector space $C_c(G)$ of continuous compactly supported functions on the group, resulting in a map N from $C_c(G)$ to functions on X. Via this map, one can then define the *autocorrelation measure* γ associated to (X, G, m). The Fourier transform of the autocorrelation measure is known as a *diffraction measure*. The diffraction measure or, equivalently, the autocorrelation measure, encodes a remarkable amount of information on the original system. In fact, a main result of the theory (already mentioned above and discussed in §6 in more detail) can be stated as follows.

RESULT—TMDSS. [**2**, **12**, **14**, **17**, **18**] *The TMDS* (X, G, m) *has a discrete spectrum if and only if the measure* γ *is strongly almost periodic. In this case, the group generated by the frequencies of* γ *is the group of eigenvalues of* (X, G, m).

In a general dynamical system (X, G, m) (again with notation to be explained in detail later in §2), the points of X cannot be paired with elements of $C_c(G)$. Hence, such a system does not admit an autocorrelation. However, if d is a metric on X inducing the topology, then—as we will show below—the function

$$\underline{d} : G \longrightarrow [0, \infty), \; \underline{d}(t) = \int_X d(x, tx)\, dm(x)$$

can serve as a convenient analogue to the autocorrelation. Indeed, our main abstract result reads as follows.

MAIN RESULT. (Compare Theorem 5.1 below) *The dynamical system* (X, G, m) *has a discrete spectrum if and only if* $\underline{d}$ *is almost periodic in the sense of Bohr. In this case, the group of eigenvalues is generated by the frequencies of* $\underline{d}$.

For general dynamical systems over metric spaces, this result provides an analogue to the result above for TMDSs and this can be seen as the main achievement of the article. Moreover, as a consequence, we also obtain (in Corollary 5.3) a converse to a result of [**23**]. More specifically, we obtain a characterization of a discrete spectrum via denseness of suitable measure almost periods. In [**23**], it has already been shown that such a denseness is sufficient for a discrete spectrum. Our characterization shows that it is also necessary. This is particularly remarkable as it is mentioned in [**23**] that 'it is unlikely' that this necessity holds (see also the remark after Corollary 5.3, for further details). Finally—as is to be expected—our considerations allow us to also reprove the above result for TMDSs provided the group G is metrizable (see the discussion in §6).

Our approach to discrete spectra is somewhat complementary to the approach in the works mentioned above [**9–11**, **25**]. A central quantity in these works is the pseudometric $\overline{d}$ on X, which arises by averaging d over G, i.e.

$$\overline{d}(x, y) = \limsup_{n\to\infty} \frac{1}{m_G(F_n)} \int_{F_n} d(tx, ty)\, dm_G(t),$$

where (F_n) is a Følner sequence and m_G denotes Haar measure on G. The discreteness of the spectrum (and related phenomena) is then encoded in equicontinuity and covering

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properties with respect to the topology induced by $\overline{d}$. In comparison, our key quantity $\underline{d}$ can (essentially) be seen as a metric on G, which arises by averaging over X. Discreteness of the spectrum is then encoded in almost periodicity properties of $\underline{d}$. These two points of view are certainly related. For example, it is possible to use our result to reprove parts of the abstract considerations of [**11**] on spectral isomorphy. This and more will be considered elsewhere.

The necessary notation and set-up for our investigations is provided in §2. The main technical tools are gathered in §3. There we also introduce the domination relation $\prec$, which is a key ingredient in our analysis. With these tools, we can then provide various characterizations of almost periodicity of $\underline{d}$ in §4. The main result, given in §5, is then a rather direct consequence of these characterizations.

2. *Preliminaries*

In this section, we set up notation and recall a few standard facts on dynamical systems. The material is well known.

We consider a compact space X equipped with a continuous action

$$G \times X \longrightarrow X, (t, x) \mapsto tx$$

of a locally compact, σ-compact, abelian group G and a probability measure m, which is invariant under the action of G. We then call (X, G, m) a *dynamical system over the space* X. Throughout, we will assume that the topology of X is induced by a metric. This metric will usually be denoted by d.

We write the group operation on G additively and denote the neutral element of G by zero.

We do not assume that the topology of G is induced by a metric. Note that it still makes sense to speak about uniform continuity of functions on G. More precisely, a function $f : G \longrightarrow \mathbb{R}$ is called uniformly continuous if, for any $\varepsilon > 0$, there exists an open neighborhood U of $0 \in G$ with $|f(t) - f(s)| < \varepsilon$ whenever $s - t \in U$.

The action of G on X induces unitary operators $T_t : L^2(X, m) \longrightarrow L^2(X, m)$ with

$$T_t f = f(t\,\cdot)$$

for each $t \in G$. The complex Hilbert space $L^2(X, m)$ is equipped with the inner product

$$\langle f, g\rangle = \int_X \overline{f} g \, dm$$

and the associated norm

$$\|f\| := \|f\|_2 := \sqrt{\langle f, f\rangle}$$

for $f, g \in L^2(X, m)$. We will be particularly interested in continuous functions on X and will denote the vector space of all such functions by $C(X)$.

An $f \in L^2(X, m)$ with $f \neq 0$ is called an *eigenfunction to the eigenvalue* $\gamma \in \widehat{G}$ if $T_t f = \gamma(t) f$ holds for each $t \in G$. Here, $\widehat{G}$ is the *dual group of* G consisting of all continuous group homomorphisms $\gamma : G \longrightarrow \{z \in \mathbb{C} : |z| = 1\}$ equipped with multiplication of functions and complex conjugation as product and inverse, respectively. We denote the group generated by the set of eigenvalues as the *group of eigenvalues*. If the

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dynamical system is minimal (i.e. each orbit is dense) or ergodic (i.e. if any measurable invariant set has measure zero or one), then the set of eigenvalues forms a group already.

The dynamical system (X, G, m) is said to have a *discrete spectrum* if there exists an orthonormal basis of $L^2(X, m)$ consisting of eigenfunctions.

Whenever f is a bounded function from a set Y to the complex numbers, we define the *supremum norm* of f as $\|f\|_\infty := \sup\{|f(y)| : y \in Y\}$. We will be interested in the cases $Y = G$ and $Y = X$.

Whenever (X_1, G, m_1), (X_2, G, m_2) are dynamical systems with actions of the same group G, a map $\pi : X_1 \longrightarrow X_2$ is called *G-equivariant* if it respects the action of G in that $\pi(tx) = t\pi(x)$ holds for all $t \in G$ and $x \in X_1$.

3. *The functions $\underline{e}$ and e'*

Let (X, G, m) be a dynamical system over a compact space X. A *pseudometric* on a set Y is a function $e : Y \times Y \longrightarrow [0, \infty)$ satisfying $e(x, x) = 0$, $e(x, y) = e(y, x)$ and $e(x, y) \leq e(x, z) + e(z, y)$ for all $x, y, z \in Y$. We will be interested in pseudometrics on X and on G.

To a continuous pseudometric e on X, we associate the functions

$$\underline{e} : G \longrightarrow [0, \infty), \quad \underline{e}(t) := \int_X e(x, tx)\, dm(x)$$

and

$$e' : G \times G \longrightarrow [0, \infty), \quad e'(s, t) := \int_X e(sx, tx)\, dm(x).$$

A short computation (using the invariance of m) gives

$$e'(s, t) = \underline{e}(s - t) \quad \text{and} \quad \underline{e}(s) = e'(0, s).$$

For this reason, properties of $\underline{e}$ and of e' are strongly connected and it usually suffices to study one of these functions.

A few basic properties of e', $\underline{e}$ and functions that have a similar relationship are gathered next. For completeness, we include the simple proofs.

PROPOSITION 3.1. (Basic properties of e') *The function e' is a continuous, bounded and G-invariant pseudometric.*

Proof. Continuity of e' is clear from continuity of e and the group action and compactness of X. Boundedness of e' follows as the continuous e is bounded on the compact $X \times X$ and m is a probability measure. The G-invariance is clear from the invariance of m under the action of G. It remains to show that e' is a pseudometric. This follows easily as e is a pseudometric. □

LEMMA 3.2. (Functions inducing invariant metrics) *For $F : G \longrightarrow [0, \infty)$, the following assertions are equivalent.*

(i) *The function $F'(s, t) := F(s - t)$ is a pseudometric.*

(ii) *For all $s, t \in G$, the equality $\|F(s + \cdot) - F(t + \cdot)\|_\infty = F(s - t)$ holds.*

If one of these equivalent conditions holds, then F satisfies $F(0) = 0$, $F(-s) = F(s)$ and $|F(s) - F(t)| \leq F(s - t)$ for all $s, t \in G$ and, moreover, F is uniformly continuous if it is continuous (at $t = 0$).

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Proof. The implication (ii) $\Longrightarrow$ (i) is clear. As for the implication (i) $\Longrightarrow$ (ii), we note that

$$|F(s) - F(t)| = |F'(s, 0) - F'(t, 0)| \leq F'(s, t) = F(s - t)$$

for all $s, t \in G$ as F' is a pseudometric. From this inequality, we directly obtain the inequality $\|F(s + \cdot) - F(t + \cdot)\|_\infty \leq F(s - t)$ for all $s, t \in G$. The reverse inequality $\geq$ follows by inserting the value $-t$.

Now the last statement is clear. □

Definition 3.3. (Pseudonorm on G) We will refer to a function F satisfying the (equivalent) conditions given in the preceding lemma as a *pseudonorm* on G.

PROPOSITION 3.4. (Basic properties of $\underline{e}$) *Let e be a continuous pseudometric on X. The function $\underline{e}$ is a bounded continuous pseudornorm. In particular, $\underline{e}$ is uniformly continuous and satisfies $\underline{e}(0) = 0$, $\underline{e}(s) = \underline{e}(-s)$ as well as $|\underline{e}(s) - \underline{e}(t)| \leq \underline{e}(s - t)$ and*

$$\|\underline{e}(s + \cdot) - \underline{e}(t + \cdot)\|_\infty = \underline{e}(s - t)$$

for all $s, t \in G$.

Proof. As X is compact and e is continuous, the function $\underline{e}$ is bounded and continuous. As for the remaining statements, we note that e' is a continuous pseudometric due to Proposition 3.1. Hence, we can apply the previous lemma with $F = \underline{e}$ and $F' = e'$. □

Of course, the functions $\underline{e}$ and e' depend on e. So one may wonder how they change if e is replaced by another pseudometric. To investigate this, we introduce the following concept.

Definition 3.5. (The relation $\prec$) Let f and g be functions from a set Y to the complex numbers. Then, f is said to *dominate* g (written as $g \prec f$) if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|g(y)| \leq \varepsilon$ whenever $|f(y)| \leq \delta$ holds. If both f dominates g and g dominates f, we say that f and g are equivalent and write $f \sim g$.

The following statement is a rather direct consequence of uniform continuity of continuous functions on compact sets.

PROPOSITION 3.6. *Let e be a pseudometric on the compact metric space X. Let d be a metric on X inducing the topology. Then e is continuous if and only if it is dominated by d.*

By the previous proposition, two metrics d and e on the compact X giving the topology are equivalent. This will be used repeatedly (and tacitly) in what follows.

LEMMA 3.7. *Let (X, G, m) be a dynamical system. Let e_1 and e_2 be continuous pseudometrics on X with $e_1 \prec e_2$. Then*

$$\underline{e_1} \prec \underline{e_2} \quad \text{and} \quad e_1' \prec e_2'.$$

Proof. It suffices to show the statement for $\underline{e_1}$ and $\underline{e_2}$. We have to show that, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\underline{e_2}(t) < \delta$ (for $t \in G$) implies that $\underline{e_1}(t) < \varepsilon$. So let $\varepsilon > 0$ be given. Without loss of generality, we can assume that $e_1, e_2 \leq 1$.

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By $e_1 \prec e_2$, there exists $\varepsilon_1 > 0$ with

$$e_2(x, y) \le \varepsilon_1 \Longrightarrow e_1(x, y) \le \frac{\varepsilon}{2}.$$

Without loss of generality, we can assume that

$$\varepsilon_1 \le \frac{\varepsilon}{2}.$$

Set $\delta := \varepsilon_1^2$ and consider a $t \in G$ with $\underline{e_2}(t) \le \delta$. Setting

$$M := \{x : e_2(x, tx) \ge \varepsilon_1\},$$

we then find that

$$\varepsilon_1 m(M) \le \int_X e_2(x, tx)\, dm(x) = \underline{e_2}(t) < \varepsilon_1^2.$$

This gives

$$m(M) < \varepsilon_1.$$

By construction, we have $e_2(x, tx) < \varepsilon_1$ and hence $e_1(x, tx) \le \varepsilon/2$ for $x \in X \setminus M$. Given this, a short computation shows that

$$\begin{aligned}
\underline{e_1}(t) &= \int_X e_1(x, tx)\, dm(x) \\
&= \int_M e_1(x, tx)\, dm(x) + \int_{X\setminus M} e_1(x, tx)\, dm(x) \\
&\le \|e_1\|_\infty m(M) + \frac{\varepsilon}{2} m(X \setminus M) \\
&< \varepsilon_1 + \frac{\varepsilon}{2} \\
&\le \varepsilon.
\end{aligned}$$

This finishes the proof. □

For us, certain (pseudo)metrics will be of special interest: to each continuous $f : X \longrightarrow \mathbb{C}$, we associate the pseudometric

$$e_f : X \times X \longrightarrow [0, \infty),\ e_f(x, y) := |f(x) - f(y)|.$$

As f is continuous, so is e_f. In particular, e_f is dominated by the metric d. Now, for $n \in \mathbb{N}$, let the functions $f_n \in C(X)$ and $c_n \ge 0$ with $\sum c_n \|f_n\|_\infty < \infty$ be given. Then

$$e_{(f_n),(c_n)} := \sum_n c_n e_{f_n}$$

is a continuous pseudometric. To an $f \in C(X)$, we can also associate the function

$$F_f : G \longrightarrow [0, \infty), \qquad F_f(t) := \|f - T_t f\|_2.$$

To $f_n \in C(X)$ and $c_n \ge 0$ with $\sum c_n \|f\|_\infty < \infty$, we can, moreover, associate the function

$$F_{(f_n),(c_n)} := \sum_n c_n F_{f_n}.$$

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PROPOSITION 3.8.
(a) *Let f be a continuous function on X. Then $\underline{e}_f \prec \underline{d}$.*
(b) *Let f be a continuous function on X. Then $F_f \leq \sqrt{2\|f\|_\infty} \cdot \sqrt{\underline{e}_f}$ and $\underline{e}_f \leq F_f$ hold. In particular, $F_f \sim \underline{e}_f$ and $F_f \prec \underline{d}$.*
(c) *Let continuous functions f_n on X and $c_n > 0$, $n \in \mathbb{N}$, with $\sum_n c_n \|f_n\| < \infty$ be given such that the (f_n) separate the points of X. Then*

$$\underline{e}_{(f_n),(c_n)} \sim F_{(f_n),(c_n)} \sim \underline{d}.$$

Proof. (a) We have already discussed that e_f is continuous. Hence, it is dominated by d. Thus, the previous lemma gives $\underline{e}_f \prec \underline{d}$.

(b) To show the bound on F_f, we compute

$$\begin{aligned} F_f(t) &= \|f - T_t f\| \\ &= \bigg(\int |f(x) - f(tx)|^2 \, dm(x)\bigg)^{1/2} \\ &\leq \sqrt{2\|f\|_\infty}\bigg(\int |f(x) - f(tx)| \, dm(x)\bigg)^{1/2} \\ &= \sqrt{2\|f\|_\infty} \cdot \sqrt{\underline{e}_f(t)}. \end{aligned}$$

To show the bound on $\underline{e}_f$, we note that m is a probability measure, and hence the Cauchy–Schwarz inequality gives $\int_X |g| \, dm \leq \|g\|$ for all g in $L^2(X, m)$. Thus, we can estimate

$$\begin{aligned} \underline{e}_f(t) &= \int_X e_f(x, tx) \, dm(x) \\ &= \int |f(x) - f(tx)| \, dm(x) \\ &\leq \|f - T_t f\| \\ &= F_f(t) \end{aligned}$$

for each $t \in G$.

The preceding two bounds give $F_f \sim \underline{e}_f$. Invoking (a), we then also infer that $F_f \prec \underline{d}$.

(c) From (b) and the summability condition on the sequence (c_n), we directly infer that $\underline{e}_{(f_n),(c_n)} \leq F_{(f_n),(c_n)}$ as well as

$$F_{(f_n),(c_n)} \leq \sum_n c_n \sqrt{2\|f_n\|_\infty} \sqrt{\underline{e}_{f_n}} \leq \bigg(2 \sum_n c_n \|f_n\|_\infty\bigg)^{1/2} \sqrt{\underline{e}_{(f_n),(c_n)}}.$$

This gives

$$\underline{e}_{(f_n),(c_n)} \sim F_{(f_n),(c_n)}.$$

From (a) and the summability condition on the sequence (c_n), we also easily find that $\underline{e}_{(f_n),(c_n)} \prec \underline{d}$. It remains to show that $\underline{d} \prec \underline{e}_{(f_n),(c_n)}$. Now, by the assumptions, the function $e := e_{(f_n),(c_n)}$ is a continuous pseudometric, which separates the points

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of X. Hence, e is a continuous metric. As X is a compact Hausdorff space, this metric must then generate the topology as well by standard 'uniqueness' of a compact Hausdorff topology (see, e.g., [**19**]). Thus, d is continuous with respect to the metric e, and *vice versa*. Thus, d and e are equivalent. Therefore, the previous lemma gives $\underline{d} \sim \underline{e}$. This finishes the proof. $\square$

4. *Almost periodicity and* $\underline{d}$

We begin this section with a recollection on almost periodicity. We then discuss how almost periodicity is compatible with $\prec$. Building on this and the results of the previous section, we can then characterize almost periodicity of $\underline{d}$ in Lemma 4.4.

Recall that a subset $\mathcal{T}$ of G is called *relatively dense* if there exists a compact $K \subset G$ with Minkowski sum

$$\mathcal{T} + K := \{\tau + k : \tau \in \mathcal{T}, k \in K\}$$

equal to G. A uniformly continuous function $F : G \longrightarrow \mathbb{C}$ is called *almost periodic (in the sense of Bohr)* if, for any $\varepsilon > 0$, the set

$$\{t \in G : \|F(t + \cdot) - F\|_\infty \leq \varepsilon\}$$

is relatively dense. This is the case if and only if the *hull of* F, defined by

$$\mathbb{T}(F) := \overline{\{F(t + \cdot) : t \in G\}}^{\|\cdot\|_\infty},$$

is compact. In this case, the hull has a unique group structure making it into a topological group such that

$$j : G \longrightarrow \mathbb{T}(F), \quad t \mapsto F(t + \cdot)$$

is a continuous group homomorphism. Clearly, the map j has dense range. Hence, the group $\mathbb{T}(F)$ must be abelian. Moreover, the dual map

$$\widehat{\mathbb{T}(F)} \longrightarrow \widehat{G}, \quad \gamma \mapsto \gamma \circ j$$

is injective. Thus, any element $\gamma \in \widehat{\mathbb{T}(F)}$ can be considered as an element of $\widehat{G}$ and this is how we will think about elements of $\widehat{\mathbb{T}(F)}$.

As $\mathbb{T}(F)$ is a compact group, it carries a unique normalized invariant measure and the elements of $\widehat{\mathbb{T}(F)}$ form an orthonormal basis in the Hilbert space $L^2(\mathbb{T}(F))$ consisting of (classes of) square integrable functions on $\mathbb{T}(F)$ with respect to this measure. Consequently, any continuous function h on $\mathbb{T}(F)$ can then be expanded uniquely in a Fourier type series

$$h = \sum_{\gamma \in \widehat{\mathbb{T}(F)}} c_\gamma^h \gamma,$$

where the sum converges in the L^2-sense. Now consider a continuous function H on G that can be *lifted* to $\mathbb{T}(F)$, i.e. it has the form $H = h \circ j$ with a (necessarily unique) continuous h on $\mathbb{T}(F)$. Then the *frequencies* of H are the elements of

$$\{\gamma \circ j : c_\gamma^h \neq 0\} \subset \widehat{G}.$$

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Clearly, one such function is given by $H = \delta \circ j$ with the function $\delta : \mathbb{T}(F) \longrightarrow \mathbb{C}$, $\delta(E) = E(0)$. In this case, the group generated by the frequencies of H can easily be seen to be just $\widehat{\mathbb{T}(F)}$. For details on this reasoning as well as for further discussion of some basic properties of frequencies used in this article, we refer the reader to the Appendix A.

We will be interested in proving almost periodicity of functions such as $\underline{e}$ for a continuous pseudometric e on X, F_f for $f \in C(X)$ and $F_{(c_n),(f_n)}$ for $f_n \in C(X)$ and $c_n \geq 0$ with $\sum_n c_n \|f_n\| < \infty$. Clearly, all of these are continuous pseudonorms and we will use the following simple criterion to show their almost periodicity.

LEMMA 4.1. *A continuous pseudonorm $F : G \longrightarrow [0, \infty)$ is almost periodic if and only if, for every $\varepsilon > 0$, the set*

$$\{\tau \in G : |F(\tau)| \leq \varepsilon\} \tag{1}$$

is relatively dense.

Proof. This is an immediate consequence of the equality

$$\|F(s + \cdot) - F(t + \cdot)\|_\infty = F(s - t)$$

(applied with $t = 0$). □

Accordingly, proving relative denseness of sets as in (1) will be our main tool in dealing with almost periodicity. As relative denseness of sets as in (1) is clearly preserved under $\prec$, we easily obtain the following consequence.

PROPOSITION 4.2. (Preservation of almost periodicity under $\prec$) *Let $F_j : G \longrightarrow [0, \infty)$, $j \in \{1, 2\}$, be continuous pseudonorms with $F_1 \prec F_2$. Then F_1 is almost periodic if F_2 is almost periodic.*

Proof. By $F_1 \prec F_2$, relative denseness of $\{\tau \in G : |F_2(\tau)| \leq \varepsilon\}$ for each $\varepsilon > 0$ implies relative denseness of $\{\tau \in G : |F_1(\tau)| \leq \delta\}$ for each $\delta > 0$. Now the proof follows from Lemma 4.1. □

For our considerations, it will also be helpful that taking the hull is compatible with $\prec$. The corresponding statement is the content of the next lemma.

LEMMA 4.3. (Compatibility of the hull construction with $\prec$) *Let $F_j : G \longrightarrow [0, \infty)$, $j \in \{1, 2\}$, be continuous bounded pseudonorms with $F_1 \prec F_2$. Then there exists a unique uniformly continuous G-equivariant map $\pi : \mathbb{T}(F_2) \longrightarrow \mathbb{T}(F_1)$ with $\pi(F_2) = F_1$. If $\mathbb{T}(F_2)$ is compact, this map is onto.*

Proof. For $j = 1, 2$, set

$$O(F_j) := \{F_j(t + \cdot) : t \in G\} \subset \mathbb{T}(F_j)$$

and note that $O(F_j)$ is dense in $\mathbb{T}(F_j)$, $j = 1, 2$.

We now show existence and uniqueness of a map π as specified in the statement of the lemma. As $O(F_2)$ is dense in $\mathbb{T}(F_2)$, uniqueness is clear. As for existence, we use that F_1 and F_2 are pseudonorms and hence satisfy the equalities

$$(E_1) \quad \|F_1(t + \cdot) - F_1(s + \cdot)\|_\infty = F_1(t - s)$$

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and

$$(E_2) \quad \|F_2(t+\cdot) - F_2(s+\cdot)\|_\infty = F_2(t-s).$$

By $F_1 \prec F_2$, we immediately obtain that $F_2(r) = 0$ implies that $F_1(r) = 0$. Combining this with (E_1) and (E_2), we see that the map

$$\pi^* : O(F_2) \longrightarrow O(F_1),\ F_2(t+\cdot) \mapsto F_1(t+\cdot)$$

is well defined. Now, combining $F_1 \prec F_2$ once more with (E_1) and (E_2), we infer that π^* is uniformly continuous. Hence, it can be extended to a uniformly continuous map

$$\pi : \mathbb{T}(F_2) \longrightarrow \mathbb{T}(F_1).$$

By construction, $\pi(F_2) = \pi^*(F_2) = F_1$. The G-equivariance of π follows easily from the properties of π^*.

If $\mathbb{T}(F_2)$ is compact, then so is $\pi(\mathbb{T}(F_2))$ by continuity of π. Moreover, $\pi(\mathbb{T}(F_2))$ clearly contains $O(F_1)$, which is dense in $\mathbb{T}(F_1)$. This easily proves the last statement. □

Remark. If F_2 and F_1 are almost periodic, then $\mathbb{T}(F_2)$ and $\mathbb{T}(F_1)$ carry a group structure. In this case, it is not hard to see that the map π is, in fact, a group homomorphism. Indeed, as π is equivariant, it respects the group operations on the set $O(F_2)$ defined in the proof of the preceding lemma. By density of $O(F_2)$ in $\mathbb{T}(F_2)$ and continuity of π, this then carries over to the whole hull.

The following lemma gathers various equivalent versions of almost periodicity of $\underline{d}$. Our main result will be a rather direct consequence of this lemma.

LEMMA 4.4. *Let (X, G, m) be a dynamical system and let d be a metric on X generating the topology. Then the following assertions are equivalent.*

(i) *The function $\underline{d}$ is almost periodic.*
(ii) *For each $f \in C(X)$, the function $\underline{e}_f$ is almost periodic.*
(iii) *The set of $f \in C(X)$ for which $\underline{e}_f$ is almost periodic separates the points of X.*
(iv) *The function $\underline{e}_{(f_n),(c_n)}$ is almost periodic for one (each) set of functions f_n in $C(X)$ and $c_n > 0$, $n \in \mathbb{N}$, with $\sum_n c_n\|f_n\| < \infty$ such that the (f_n) separate the points of X.*
(v) *For any $f \in C(X)$, the function $G \longrightarrow \mathbb{C}$, $t \mapsto \langle f, T_t f\rangle$, is almost periodic.*
(vi) *For any $f \in L^2(X, m)$, the function $G \longrightarrow \mathbb{C}$, $t \mapsto \langle f, T_t f\rangle$, is almost periodic.*

If one of these equivalent conditions holds, the group generated by the frequencies of $\underline{d}$ is the same as the group generated by the frequencies of all functions of the form $G \longrightarrow \mathbb{C}$, $t \mapsto \langle f, T_t f\rangle$, for $f \in L^2(X, m)$.

Remark. It is not hard to see that, in all above statements, $\underline{e}_*$ could be replaced by F_* (with $* = f$ or $* = (f_n, c_n)$). Indeed, $\underline{e}_*$ and F_* are equivalent by Proposition 3.8, and hence, by Proposition 4.2, one of them is almost periodic if and only if the other is almost periodic.

Proof. We first discuss the equivalence statement. Note that the functions in statements (i) to (iv) are continuous pseudonorms. Thus, we can invoke Proposition 4.2 to study their almost periodicity. Given this, the equivalence between statements (i) to (iv) is a consequence of Proposition 3.8. Note that (iii) implies (iv) as each set of continuous

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functions that separates the points must contain a countable subset of functions that also separates the points of X (due to compactness of the metric space X).

The implication (vi) $\Longrightarrow$ (v) is clear and the implication (v) $\Longrightarrow$ (vi) follows easily by density of $C(X)$ in $L^2(X, m)$.

We now prove the equivalence between (ii) and (v). The crucial ingredient is the equality

$$F_f^2(t) = 2\|f\|^2 - \langle f, T_t f\rangle - \overline{\langle f, T_t f\rangle}, \tag{2}$$

which follows by a direct computation. Recall that almost periodicity of F_f is equivalent to almost periodicity of $\underline{e}_f$ (see the remark preceding the proof). Hence, the implication (v) $\Longrightarrow$ (ii) is immediate from (2) (as almost periodicity is preserved under taking square roots). Similarly, (ii) and (2) then yield almost periodicity of $\langle f, T_t f\rangle$ for any real-valued $f \in C(X)$. By a simple polarization argument, this then gives almost periodicity of $\langle f, T_t g\rangle$ for all real-valued continuous f, g. This, in turn, easily implies (v).

We now turn to the last statement. Set $S_f : G \longrightarrow \mathbb{C}$, $S_f(t) = \langle f, T_t f\rangle$ for $f \in L^2(X, m)$. Whenever $f \in C(X)$ is real valued, we have $F_f^2 = 2\|f\|^2 - 2S_f$ by (2). Hence, we can lift S_f to a continuous function on $\mathbb{T}(F_f)$. As $F_f \prec \underline{d}$ due to Proposition 3.8, we infer that $F_f' \prec d'$ from Lemma 3.7. By Lemma 4.3, this implies that there exists a continuous map $\pi : \mathbb{T}(\underline{d}) \longrightarrow \mathbb{T}(F_f)$ mapping $\underline{d}(t + \cdot)$ to $F_f(t + \cdot)$ for any $t \in G$. Hence, we can then lift S_f to a continuous function on $\mathbb{T}(\underline{d})$ as well. By considering real and imaginary parts, we can then lift S_f to a continuous function on $\mathbb{T}(\underline{d})$ for any $f \in C(X)$. Taking limits and using that $C(X)$ is dense in $L^2(X, m)$, we can then lift S_f for any $f \in L^2(X, m)$ to $\mathbb{T}(\underline{d})$. Hence, for any $f \in L^2(X, m)$, the set of frequencies of S_f is contained in the dual group of $\mathbb{T}(\underline{d})$, which, in turn, is the group generated by the frequencies of $\underline{d}$.

Conversely, consider $F := F_{(f_n),(c_n)}$ for a set of functions f_n in $C(X)$ and $c_n > 0$, $n \in \mathbb{N}$, with $\sum_n c_n\|f_n\| < \infty$ such that the (f_n) separate the points of X. Then $F \sim \underline{d}$ by Proposition 3.8. Hence, by Lemma 4.3, there exists a continuous surjective equivariant map $\mathbb{T}(F) \longrightarrow \mathbb{T}(\underline{d})$. Thus, $\underline{d}$ can be lifted to a continuous function on $\mathbb{T}(F)$. In particular, the frequencies of $\underline{d}$ are contained in the group generated by the frequencies of F. The frequencies of F are, in turn, contained in a union of the frequencies of the F_{f_n}, $n \in \mathbb{N}$. The frequencies of the F_{f_n}, however, are contained in the group generated by the frequencies of the S_{f_n} as F_{f_n} can clearly be lifted to $\mathbb{T}(S_{f_n})$ by (2). □

5. *A characterization of a discrete spectrum*

In this section, we state and prove our main result which gives a characterization of dynamical systems with discrete spectra.

THEOREM 5.1. (Characterizing a discrete spectrum) *Let (X, G, m) be a dynamical system. Let d be a metric on X inducing the topology. Then the following assertions are equivalent.*

(i) *The dynamical system (X, G, m) has a discrete spectrum.*

(ii) *The function $\underline{d}$ is almost periodic.*

If one of these equivalent conditions holds, then the group of eigenvalues of (X, G, m) equals the group generated by the frequencies of $\underline{d}$.

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Remark. If $\underline{d}$ is almost periodic, then the group generated by its frequencies is the dual group of $\mathbb{T}(\underline{d})$ (as follows from the general discussion above). Moreover, it is not hard to see that this group is also given as the Hausdorff completion of G with respect to d'.

Proof. We first deal with the equivalence statement. By Lemma 4.4, the function $\underline{d}$ is almost periodic if and only if $G \longrightarrow \mathbb{C},\ t \mapsto \langle f, T_t f\rangle = S_f(t)$ is almost periodic for any $f \in L^2(X, m)$. This, in turn, is a well-known characterization of a discrete spectrum. Indeed, for any $f \in L^2(X, m)$, there exists a unique measure μ_f on $\widehat{G}$ with

$$\langle f, T_t f\rangle = \int_{\widehat{G}} \gamma(t)\, d\mu_f(\gamma)$$

(see, e.g., [**19**]). Now, a discrete spectrum just means that all these measures are point measures and pure pointedness of μ_f is equivalent to almost periodicity of $t \mapsto \langle f, T_t f\rangle$ (see, e.g., [**18**] for a recent discussion).

We now turn to the second statement of the theorem. By the last statement of Lemma 4.4, the group generated by the frequencies of $\underline{d}$ is the group generated by the frequencies of the S_f, $f \in L^2(X, m)$. The frequencies of S_f, however, are just the atoms of the measures μ_f. Hence, they generate the group of eigenvalues. □

It is possible to rephrase the result in terms of almost periods. This will clarify the relationship between our result and earlier results. Whenever e is a continuous pseudometric on X, a $t \in G$ is called a *measure-theoretic ε-almost period of e* if

$$m(\{x \in X : e(x, tx) > \varepsilon\}) < \varepsilon.$$

LEMMA 5.2. (Almost periodicity and measure-theoretic ε-almost periods) *Let e be a continuous pseudometric on X. Then the following assertions are equivalent.*

(i) *For any $\varepsilon > 0$ the set of measure-theoretic ε-almost periods of e is relatively dense.*

(ii) *The function $\underline{e}$ is almost periodic.*

Proof. (i) $\Longrightarrow$ (ii). We have to show that the set $\{t \in G : \underline{e}(t) \le \varepsilon\}$ is relatively dense for any $\varepsilon > 0$ (compare Lemma 4.1). Let $\varepsilon_1 > 0$ with

$$\varepsilon_1 \|e\|_\infty + \varepsilon_1 < \varepsilon$$

be given. Choose a measure-theoretic ε_1-almost period t of e and set

$$M := \{x \in X : e(x, tx) > \varepsilon_1\}.$$

Then a direct computation gives

$$\underline{e}(t) = \int_M e(x, tx)\, dm + \int_{X\setminus M} e(x, tx)\, dm \le \varepsilon_1 \|e\|_\infty + m(X \setminus M)\varepsilon_1 < \varepsilon.$$

As the set of measure theoretic ε_1-almost periods is relatively dense by (i), the desired statement follows.

(ii) $\Longrightarrow$ (i). This follows by mimicking an argument given in the proof of Lemma 3.7. Let $\varepsilon > 0$ be given. By (ii), the set $\{t \in G : \underline{e}(t) < \varepsilon^2\}$ is relatively dense. For any t in this set, we obtain, with $N := \{x \in X : e(x, tx) > \varepsilon\}$,

$$\varepsilon m(N) \le \int_X e(x, tx)\, dm = \underline{e}(t) < \varepsilon^2$$

and, hence $m(N) < \varepsilon$. This finishes the proof. □

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From the previous lemma and the main result, Theorem 5.1, we directly obtain the following consequence.

COROLLARY 5.3. *Let (X, G, m) be a dynamical system and let d be a metric on X inducing the topology on X. Then the following assertions are equivalent.*

(i) *For any $\varepsilon > 0$, the set of measure-theoretic ε-almost periods of d is relatively dense.*

(ii) *The dynamical system has a discrete spectrum.*

Remark. The implication (i) $\Longrightarrow$ (ii) of the corollary is proved as [**23**, Theorem 3.2] (under an additional ergodicity assumption). A partial converse is also proved in Proposition 3.3 of that paper. This converse needs an additional requirement of continuity of eigenfunctions.

We can also derive the following consequence.

COROLLARY 5.4. *Let (X, G, m) be a dynamical system. Let $\mathcal{F}$ be a family of continuous functions on X, which separates the points of X. Then the following assertions are equivalent.*

(i) *For any $\varepsilon > 0$ and each $f \in \mathcal{F}$, the set of ε-almost periods of $\underline{e}_f$ is relatively dense.*

(ii) *The dynamical system has a discrete spectrum.*

Proof. (ii) $\Longrightarrow$ (i). This follows from combining the implication (i) $\Longrightarrow$ (ii) of Theorem 5.1, the implication (i) $\Longrightarrow$ (ii) of Lemma 4.4 and the first lemma of this section. (This reasoning actually works for any $f \in C(X)$ and not only for $f \in \mathcal{F}$.)

(i) $\Longrightarrow$ (ii). This follows from combining the first lemma of this section with the implication (iii) $\Longrightarrow$ (i) of Lemma 4.4 and Theorem 5.1 (ii) $\Longrightarrow$ (i). □

Remark. A possible choice of the family $\mathcal{F}$ is given as $d(x, \cdot)$, $x \in X$.

6. *Connection to the autocorrelation measure*

Our considerations are motivated by the study of diffraction theory for quasicrystals. Diffraction theory for quasicrystals and the relationship with dynamical systems has gained substantial attention in the last two decades. Indeed, from the very beginning, tiling and point set dynamical systems with discrete spectra have played a key role in the study of quasicrystals [**6**, **20–24**]. For recent discussions containing further references, we refer the interested reader to the surveys [**3**, **15**] and the corresponding parts of [**16**]. In this section, we briefly sketch the necessary background to put our main result into this context.

As discussed in [**2**], diffraction theory for quasicrystals can be conveniently set up in the framework of translation bounded measures on a locally compact, σ-compact abelian group G (see [**17**, **18**] for generalizations). Here, we follow [**2**] to which we refer for further details.

Let $C_c(G)$ be the space of continuous functions on G with compact support. A measure μ on G is called *translation bounded* if its total variation $|\mu|$ satisfies

$$\sup_{t\in G} |\mu|(t + U) < \infty$$

for one (all) relatively compact open U in G. The set of all translation bounded measures is denoted by $M^\infty(G)$. It is equipped with the vague topology. There is a natural action

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α of G on $M^{\infty}(G)$ by translations, where ,for $t \in G$ and $\mu \in M^{\infty}(G)$, the measure $\alpha_t(\mu)$ satisfies $\alpha_t(\mu)(\varphi) = \mu(\varphi(\cdot - t))$ for all $\varphi \in C_c(G)$. Whenever X is a compact subset of $M^{\infty}(G)$ which is invariant under the translation action and m is an invariant probability measure on X, we call (X, G, m) a *dynamical system of translation bounded measures* or just a TMDS for short. Such a system admits a canonical map

$$N : C_c(G) \longrightarrow C(X) \quad \text{with } N_{\varphi}(\mu) = \int \varphi(-s)\, d\mu(s).$$

Let us emphasize that the existence of such a map is a distinctive feature of TMDSs compared with general dynamical systems. Then there exists a unique translation bounded measure γ on (X, G, m) with

$$\gamma * \varphi * \widetilde{\varphi}(t) = \langle N_{\varphi}, T_t N_{\varphi} \rangle \tag{3}$$

for all $\varphi \in C_c(G)$ and all $t \in G$. Here, $*$ denotes the convolution and $\widetilde{\varphi}(t) = \overline{\varphi}(-t)$. The measure γ is called the *autocorrelation* of the TMDS. This measure allows for a Fourier transform $\widehat{\gamma}$ which is a (positive) measure on $\widehat{G}$. A main result of the theory is the following theorem.

THEOREM. *The TMDS (X, G, m) has a discrete spectrum if and only if the measure γ is strongly almost periodic. In this case, the group of eigenvalues of (X, G, m) is the group generated by $\{k \in \widehat{G} : \widehat{\gamma}(\{k\}) > 0\}$.*

Remarks.

(a) The measure γ is called strongly almost periodic of $\gamma * \varphi$ is almost periodic (in the sense of Bohr) for all $\varphi \in C_c(G)$.

(b) The above theorem is usually formulated with the assumption that the *diffraction measure* $\widehat{\gamma}$ is a pure point measure. However, as is well known (see, e.g., [**4**, Proposition 7 and Theorem 4] for a discussion in the context of aperiodic order), the measure γ is strongly almost periodic if and only if $\widehat{\gamma}$ is a pure point measure.

(c) The theorem has a long history. The connection between the autocorrelation and point spectrum goes back to work of Dworkin [**6**]. The first statement giving an equivalence (in the more restricted setting of uniquely ergodic dynamical systems of point sets satisfying the regularity requirement of finite local complexity) can be found in [**14**]. This was then generalized in [**12**] to rather general point processes and in [**2**] to the context discussed in this section. A unified treatment of [**2**] and [**12**] was given in [**18**]. Recently, an even more general result was given in [**17**].

Clearly, the preceding theorem is quite close to our main result. Pure pointedness of the spectrum is characterized by almost periodicity of a suitable function (in this case, the measure γ) and the group of eigenvalues is generated by the frequencies of the function (in this case, the atoms of the diffraction measure). In fact, it is easy to derive the previous result from our main result (provided the group is metrizable). We leave the details to the reader.

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A. *Appendix. Frequencies of almost periodic functions*

In this appendix, we gather some facts on frequencies of almost periodic functions used in the main body of the text. All of these facts are certainly well known but may not easily be found in the literature in the way we need it above. For background and further details on results used in this appendix, we refer the reader to [**5**, **7**, **19**].

The basic idea is that any almost periodic function on G can be 'expanded' in a sum of the form $\sum_{\gamma\in\widehat{G}} c_\gamma \gamma$ with $c_\gamma \in \mathbb{C}$. The (countably many) $\gamma \in \widehat{G}$ with $c_\gamma \neq 0$ are then the frequencies of the almost periodic function. It does require some care to make sense of the sum and we include a discussion next.

Equip $\widehat{G}$ with the discrete topology and denote the arising discrete group by $\widehat{G}_{\rm disc}$. The identity $\kappa : \widehat{G}_{\rm disc} \longrightarrow \widehat{G}$, $\gamma \mapsto \gamma$, is then clearly a continuous surjective group homomorphism. By Pontryagin duality, the dual group G_b of $\widehat{G}_{\rm disc}$ is then compact and there exists an injective group homomorphism $\iota : G \longrightarrow G_b$. The group G_b is known as *Bohr compactification* of G. Whenever F is an almost periodic function on G with associated map $j : G \longrightarrow \mathbb{T}(F)$, $j(t) = F(t + \cdot)$, the injective map

$$\widehat{\mathbb{T}(F)} \longrightarrow \widehat{G}, \gamma \mapsto \gamma \circ j$$

factors through $\widehat{G}_{\rm disc}$, i.e. it can be expressed as a $\kappa \circ \lambda$ with a continuous group homomorphism $\lambda : \mathbb{T}(F) \longrightarrow \widehat{G}_{\rm disc}$. Hence, Pontryagin duality gives that there exists a continuous surjective group homomorphism $p : G_b \longrightarrow \mathbb{T}(F)$ with $j = p \circ \iota$. This is a crucial feature of G_b. It implies, in particular, that any almost periodic function F on G can (uniquely) be written as $F_b \circ \iota$ with a continuous function F_b on G_b.

As G_b is a compact group, it carries a unique normalized Haar measure and the elements of $\widehat{G_b} = \widehat{G}_{\rm disc}$ form an orthonormal basis of the corresponding L^2-space over G_b. In particular, any continuous function H on G_b admits a Fourier type expansion

$$H = \sum_{\gamma\in\widehat{G}_{\rm disc}} c_\gamma^H \gamma,$$

where the $c_\gamma^H \in \mathbb{C}$ satisfy $\sum |c_\gamma^H|^2 < \infty$.

Now, whenever F is an almost periodic function on G, F_b admits such a Fourier type expansion and this provides a precise version of the desired expansion of F. More specifically, the $\gamma \in \widehat{G}$ (considered as elements of $\widehat{G}_{\rm disc}$) with $c_\gamma^{F_b} \neq 0$ are called the *frequencies* of F.

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As a direct consequence of the preceding considerations, we see that the frequencies of the almost periodic $F = \sum_{n\in\mathbb{N}} F_n$ are contained in the union of the frequencies of the F_n whenever the F_n are almost periodic on G with $\sum_{n\in\mathbb{N}} \|F_n\|_\infty < \infty$.

Of course, the frequencies of an almost periodic function can also be expressed in terms of its hull. More specifically, consider an almost periodic F. Then the elements of $\widehat{\mathbb{T}(F)}$ provide an orthonormal basis of the Hilbert space $L^2(\mathbb{T}(F))$ of (classes of) square integrable functions on $\mathbb{T}(F)$ with respect to the unique normalized Haar measure on $\mathbb{T}(F)$. In particular, any continuous $h : \mathbb{T}(F) \longrightarrow \mathbb{C}$ can be written as

$$h = \sum_{\gamma\in\widehat{\mathbb{T}(F)}} c_\gamma^h \gamma$$

and the $\gamma \in \widehat{\mathbb{T}(F)}$ with $c_\gamma^h \neq 0$ are called the frequencies of h. Now assume that the almost periodic function H can be lifted to $\mathbb{T}(F)$, i.e. it arises as $H = h \circ j$ with a continuous h on $\mathbb{T}(F)$. Then

$$H_b = h \circ p$$

(as $H_b \circ \iota = H = h \circ j = h \circ p \circ \iota$). This easily gives that the set of frequencies of H is just the set of frequencies of h, where $\widehat{\mathbb{T}(F)}$ is considered as a subgroup of $\widehat{G}$ via the map j. In particular, the set of frequencies of H is contained in $\widehat{\mathbb{T}(F)}$.

For us, the special case $H = \delta \circ j$ with $\delta : \mathbb{T}(F) \longrightarrow \mathbb{C}$, $\delta(E) = E(0)$ will be particularly relevant. In this case, the group generated by the frequencies of H is the whole group $\widehat{\mathbb{T}(F)}$. To see this, assume that $\gamma \in \widehat{\mathbb{T}(F)}$ is perpendicular to the group generated by the frequencies of H. As δ can clearly be expressed as a series invoking these frequencies, γ must then be perpendicular to δ. As we consider the group generated by the frequencies, γ must then also be perpendicular to the constant function with value 1 as well as to the complex conjugate $\overline{\delta}$ of δ. This can easily be seen to imply that γ is perpendicular to all functions from the algebra generated by the constant functions together with the functions $\delta(j(t) + \cdot)$ and $\overline{\delta(j(t) + \cdot)}$ for $t \in G$. As this algebra clearly separates the points of $\mathbb{T}(F)$, contains the constant functions and is invariant under complex conjugation, it is dense in the algebra of continuous functions on $\mathbb{T}(F)$ by the Stone–Weierstrass theorem. This gives that γ is perpendicular to all continuous functions on $\mathbb{T}(F)$, and hence it must vanish everywhere. This is a contradiction.

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