RESONANT COUPLING BETWEEN SOLAR GRAVITY MODES*

(Invited Review)

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Abstract. It is shown that in consequence of the parametric resonance, g modes of low spherical harmonic degree l are strongly coupled to the modes of high degree. The coupling limits the growth of low l modes to very small amplitudes. For g_1, $l = 1$ mode, the final amplitude of the radial velocity is of the order of 10 cm s^{-1}. A mixing of solar core as a result of a finite-amplitude development of linear instability of this mode is thus highly unlikely.

1. Introduction

The observational evidence for gravity-mode excitation in the Sun is still considered as controversial. Nevertheless, there has been a significant amount of work devoted to the study of the theoretical properties of such modes. This interest was stimulated by Dilke and Gough's (1972) suggestion that the overstability of g modes may lead to mixing in the core, as well as by the potential importance of the detection of these modes for probing the solar interior (e.g. Hill and Caudell, 1979).

Following Dilke and Gough's work, numerical calculations of solar g modes have been made by several groups (Christensen-Dalsgaard *et al.*, 1974; Boury *et al.*, 1975; Shibahashi *et al.*, 1975; Saio, 1980). The common conclusion of those papers is that there is a significant driving effect in the core via ε-mechanism for the g_1 and g_2 modes corresponding to $l = 1$ harmonics. However, the question of whether there is an actual instability will be possible to answer only when a credible theory of the interaction between convection and oscillation is available.

The present paper deals with nonlinear effects in solar gravity modes. Such effects must be investigated if we want to determine whether the development of instability may lead to mixing in the Sun's interior or to predict the amplitudes of radial velocity variation caused by oscillation. The study of nonlinear effects may also help us to understand why, if the 160 min oscillation represents a gravity mode, there has only been one such mode detected so far.

It may be expected that the effect limiting the amplitude growth of an unstable g mode is the parametric resonance. This is the instability of an oscillating system with the angular frequency ω_1 relative to the growth of two modes with angular frequencies ω_2 and ω_3 such that $\omega_2 + \omega_3 \approx \omega_1$. These two newly-excited modes derive their energy from the originally-excited mode and the mutual interaction between all three leads, under certain conditions, to a limitation of the amplitudes.

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The importance of parametric (resonance) instability in the stellar pulsation theory was first pointed out by Vandakurov (1965). Now a more detailed theory of this phenomenon also covering the equilibrium solution for the interacting modes, is available (Dziembowski, 1982).

2. Onset of the Parametric Instability

The criterion of instability to the growth of modes 2 and 3 in the presence of an excited mode 1 may be written in the following form:

$$Q_1 > \frac{1}{\nu}\left\{\Delta\sigma^2\left[1-\left(\frac{\gamma_2-\gamma_3}{\gamma_2+\gamma_3}\right)^2\right]+4\gamma_2\gamma_3\right\}^{1/2}. \tag{1}$$

In this formula Q_1 is the r.m.s. amplitude of the relative distortion of stellar surface $\Delta R/R$, due to the excitation of mode 1. It is assumed that $Q_1 \ll 1$. $\Delta\sigma = \sigma_1 - \sigma_2 - \sigma_3$ is the frequency mismatch, where σ is the angular frequency in units $\sqrt{4\pi G\langle\rho\rangle}$ and $\langle\rho\rangle$ is the mean stellar density. $\gamma_{2,3}$ are the linear nonadiabatic damping rates in the same units. We assume that $\gamma_{2,3} > 0$. By ν we denoted a normalized coupling coefficient. Its explicit form (Dziembowski, 1982) containing integrals of products of the eigenfunctions for the involved modes is very complicated and will not be reproduced here.

At this point it is not important what was the cause of mode 1 excitation, but it is essential for the validity of the criterion that the present growth (or damping) rate is small in the sense that $|\gamma_1| \ll Q_1\nu$. The criterion also holds in the cases when some other modes are excited, provided that they are not coupled to any of the three modes considered.

Let us assume now that mode 1 is a gravity mode of low radial order k_1 and low degree l_1, and ask what are the characteristics of the pairs of modes that may be excited as a result of the parametric resonance at the lowest amplitudes of mode 1. As seen from Equation (1), the coupling coefficient must be nonzero and possibly large, while the frequency mismatch and the damping rates of modes 2 and 3 must possibly be small.

The nonzero values of ν occur only for certain combinations of the spherical harmonics, $Y_{l,j}$, of the involved modes. Choosing $l_3 \geq l_2$, the condition that $\nu \neq 0$ may be written as

$$j_1 = j_2 + j_3,$$

$$l_3 = \begin{cases} l_2 + 1 & \text{if } l_1 = 1, \\ l_2 \text{ or } l_2 + 2 & \text{if } l_1 = 2, \\ l_2 + 1 \text{ or } l_2 + 3 & \text{if } l_1 = 3, \end{cases} \tag{2}$$

and so on.

Moreover, it turns out that the coupling coefficient is large only, at least in the present case, if the radial order of modes 2 and 3 do not differ very much. As we shall see, the expected values of $l_{2,3}$ and $k_{2,3}$ are large. For such modes we have $k \sim l/\sigma$ which implies in conjunction with Equation (2) and the resonance condition ($\Delta\sigma \approx 0$) that

$$k_2 \approx k_3 , \qquad \sigma_2 \approx \sigma_3 \approx \sigma_1/2$$

for the relevant modes. Thus, for a given l_2 we have to consider a few values of l_3 and a few pairs with frequencies close to $\sigma_1/2$.

Consider now the r.h.s. of Equation (1) as a function of l_2. For large l_2, the value of v is virtually independent of l_2 (Dziembowski, 1982; see also Table I in Section 4).

The distance between the consecutive radial modes, $\delta\sigma$, is given by

$$\delta\sigma \approx \frac{\beta}{l}\left(\frac{\sigma}{\sigma_1/2}\right)^2 , \quad \text{where} \quad \beta = \text{const.} \tag{3}$$

Thus, the expected frequency mismatch decreases with l_2 like l_2^{-1}. On the other hand, for damping rates we have (Dziembowski, 1982)

$$\gamma \approx \alpha \left(\frac{\sigma_1/2}{\sigma}\right)^2 l^2 , \tag{4}$$

where α^{-1} is the thermal time-scale in the units $(4\pi G \langle\rho\rangle)^{-1}$ for the region where the g modes are trapped. For the Sun α is in the range 10^{-12}–10^{-11}. Clearly, there is a certain optimum range of the l_2-values for the instability to occur and as α is so small, in our case this range falls at rather high values (10^2–10^3). In such a situation, a very precise fitting of the resonance is possible.

It would be unreasonable to rely on the numerical determination of the eigenfrequencies with an accuracy better than, say, 10^{-4}. Therefore, adopting a probabilistic approach we look for the probability P that the instability occurs at a given amplitude Q_1. For this purpose we assume $\sigma_2 = \sigma_3 = \sigma_1/2$ and ignore the difference between γ_2 and γ_3. Using now Equation (4) in Equation (1) we get for the amplitude of mode 1 at the onset of the instability

$$Q_1 = \frac{1}{v}(\Delta\sigma^2 + 4\alpha^2 l_2^4)^{1/2} . \tag{5}$$

The probability, $P_{l,i}$, that the instability occurs for given $l_2 = l$ and for given pair of modes 2 and 3, i, is

$$P_{l,i} = \begin{cases} \dfrac{\sqrt{Q_1^2 v_i^2 - 4\alpha^2 l^4}}{\beta l^{-1}} & \text{for} \quad l < l_c \\ 0 & \text{for} \quad l > l_c , \end{cases} \tag{6}$$

where we used Equation (3) for $\delta\sigma$ and denoted

$$l_c = \sqrt{\frac{Q_1 v_i}{2\alpha}} . \tag{7}$$

Assuming the independence of $P_{l,i}$, which is actually not true but usually gives a good approximation, we get

$$P = 1 - \prod_l \left[\prod_i (1 - P_{l,i}) \right] \approx 1 - \exp\left(- \sum_l \sum_i P_{l,i} \right) .$$

Replacing the summation over l by the integral, we obtain

$$\sum_l P_{l,i} \approx \frac{Q_1 v_i}{\beta} \int_0^{l_c} l \sqrt{1 - \left(\frac{l}{l_c}\right)^4}\, \mathrm{d}l =$$

$$= \frac{Q_1^2 v_i^2}{2\alpha\beta} \int_0^1 x\sqrt{1 - x^4}\, \mathrm{d}x = \frac{\pi}{16} \frac{Q_1^2 v_i^2}{\alpha\beta} \tag{8}$$

and finally

$$P = 1 - \exp\left(- \frac{\pi}{16} \frac{Q_1^2}{\alpha\beta} \sum_i v_i^2 \right) . \tag{9}$$

3. Equilibrium Amplitudes of the Interacting Modes

The initial growth of the amplitudes of modes 2 and 3 is exponential. This, however, is quickly terminated, once the amplitudes are high enough.

In the case of $\gamma_1 < 0$ (linear instability) and $\gamma_{2,3} > 0$, the equilibrium solution exists. However, it is stable only in a certain range of parameters $\Delta\sigma/\gamma_1$, $\gamma_{2,3}/\gamma_1$ and only then does it describe the final stage of the parametric resonance phenomenon.

In the case of our interest, namely $|\gamma_1| \ll \gamma_2 \approx \gamma_3$, the stability criteria (Wersinger *et al.*, 1980; Dziembowski, 1982) are reduced to $\Delta\sigma > 2\gamma_2$, which, using Equations (4), (5) and (7), may be written as

$$l \leq l_c(\tfrac{1}{2})^{1/4} .$$

Performing similar integration as in Equation (8) but in the limits $[0, l_c(\frac{1}{2})^{1/4}]$ it is easy to calculate that the conditional probability that the instability leads to the stable equilibrium is 0.82.

The equilibrium amplitude of mode 1 is

$$Q_1 = \frac{2}{v} \sqrt{\gamma_2\gamma_3\left[1 + \left(\frac{\Delta\sigma}{\gamma_2 + \gamma_3 + \gamma_1}\right)^2\right]} \approx \frac{1}{v}\sqrt{\Delta\sigma^2 + 4\gamma_2^2}; \tag{10}$$

thus it is approximately the same as at the onset of the instability. For the remaining

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two amplitudes we have

$$Q_k = Q_1 \sqrt{-\frac{I_1 \gamma_1 \sigma_1}{I_k \gamma_k \sigma_k}}\,, \qquad k = 1, 2 \tag{11}$$

with $I_k = \int \rho |\mathbf{h}_k|^2 \, d^3x$, where $\mathbf{h}_k$ is the displacement eigenvector.

Since the expected values of l_2 and l_3 are large, these modes are effectively trapped in the radiative cores and, consequently, their surface amplitudes are exceedingly small. More relevant measure of the amplitude for such modes is the maximum ratio of the amplitude of the horizontal displacement $|\mathbf{h}_H|$ to the radial half-wavelength λ denoted by ζ. Using Equation (11) we have

$$\zeta = \max(|\mathbf{h}_H|/\lambda) = \mu Q_1 \sqrt{-\gamma_1}\,, \tag{12}$$

where μ may be considered as constant, because for $\sigma \approx \sigma_1/2$ we have $\lambda \sim l_2^{-1}$ and $\gamma_2 \sim l_2^2$. This constant must be calculated numerically. As long as $\zeta \ll 1$, the motion may be regarded as purely oscillatory and the mixing cannot be expected.

4. A Numerical Example

We shall provide here an estimate of the maximum amplitude for the g_1, $l = 1$ mode for the solar model calculated by Dziembowski and Pamjatnykh (1978). The model is characterized by the following parameters:

envelope composition $X = 0.775$, $Z = 0.01$,

$X_c = 0.41$, $T_c = 145 \times 10^6$, $\rho_c = 133$,

bottom of the convective envelope at $r = 0.77\, R_\odot$.

The g_1, $l = 1$ mode has nondimensional frequency $\sigma = 1.345$, that corresponds to a period of 71.6 min. The mode exhibits the net driving in the radiative core and if the convection effects are ignored, its excitation rate is $-\gamma_1 = 3.4 \times 10^{-12}$.

The damping rates for high-order modes with frequencies close to $\sigma_1/2$ are accurately described by Equation (4) with $\alpha = 6.9 \times 10^{-12}$, while the distance between the consecutive radial modes by Equation (3) with $\beta = 0.18$.

The normalized coupling coefficients, ν_i, were calculated according to the formulae given by Dziembowski (1982). The values of ν_i for modes 2 and 3 having frequencies close to $\sigma_1/2$ at some chosen l_2 are given in Table I. The were evaluated for $j_1 = 0$ and $j_3 = 0$. The latter choice corresponds to the maximum of ν. In the case $j_1 = \pm 1$ the maximum occurs for $j_2 = \mp l_2$ and is by a factor $\sqrt{2}$ higher. In the same table we given values of the frequency mismatch, $\Delta\sigma$, as obtained from the numerical calculations. It is important to observe the constancy of ν with varying l_2 and a rapid decline, on the average, of ν with increasing $|k_2 - k_3|$.

To evaluate the probability of the onset of parametric instability according to

TABLE I

The coupling coefficient (ν) and frequency mismatch ($\Delta\sigma$) for some pairs of modes coupled to the g_1, $l = 1$ mode

k_2	k_3	ν	$\Delta\sigma$
$l_2 = 100$	$l_3 = 101$		
300	300	12.58	1.73 E−3
300	299	14.07	−8.21 E−3
299	300	5.63	−7.14 E−3
301	299	8.30	1.71 E−3
299	301	0.98	1.73 E−3
301	298	2.66	−1.12 E−4
298	301	0.25	−8.05 E−4
302	298	0.53	1.67 E−3
298	302	0.21	1.71 E−3
302	297	0.26	−1.60 E−4
297	302	0.35	−1.09 E−4
303	297	0.18	1.61 E−3
297	303	0.21	1.68 E−3
303	296	0.21	−2.27 E−4
296	303	0.29	−1.56 E−4
304	296	0.10	1.53 E−3
296	304	0.14	1.62 E−3
304	295	0.15	−3.12 E−4
295	304	0.21	−2.22 E−4
305	295	0.06	1.45 E−3
395	305	0.05	1.54 E−3
305	294	0.11	−4.16 E−4
294	305	0.12	−3.07 E−4
$l_2 = 200$	$l_3 = 201$		
596	596	12.41	7.82 E−4
596	595	14.00	−1.28 E−4
595	596	5.53	−1.25 E−4
597	595	8.22	8.03 E−4
595	597	0.84	7.82 E−4
597	594	2.63	−1.35 E−4
$l_2 = 500$	$l_3 = 501$		
1484	1484	12.48	1.97 E−4
1484	1483	14.04	−1.67 E−4
1483	1484	5.64	−1.67 E−4
1485	1483	8.26	1.96 E−4
1483	1485	0.90	1.97 E−4
1485	1482	2.66	−1.69 E−4

Equation (9), the values of ν for the first six pairs were used along with the values of α and β quoted above. The results are shown in Table II. It is seen that the instability is likely to occur already at the amplitude Q_1 of the order of 10^{-7}. The values of $l_{\max}$ given in this table are the values of l at the maximum of the integrand in Equation (8)

TABLE II

Probability of the parametric instability (P), l_2 at the maximum of probability (l_{max}), and the corresponding frequency mismatch $\Delta\sigma$ as a function of amplitude of the g_1, $l = 1$ mode (Q_1)

Q_1	P	l_{max}	$\Delta\sigma$
2.5 E−8	0.044	121	2.9 E−7
5.0 E−8	0.165	172	5.7 E−7
1.0 E−7	0.514	243	1.1 E−6
2.0 E−7	0.944	343	2.3 E−6
3.0 E−7	0.998	420	3.4 E−6

for the largest v_i. It is easy to see that $l_{max} = (\frac{1}{3})^{1/4} l_c$. The corresponding values of $\Delta\sigma$ were calculated with the use of Equation (5).

As we already noted in the previous section, the equilibrium amplitudes Q_1 are practically the same as at the onset of instability. We therefore conclude that the maximum amplitude for the g_1, $l =.1$ mode is not likely to exceed 2×10^{-7}. At this level, the nonresonant nonlinear effects are completely negligible. For instance, as the maximum value of the relative pressure perturbation, $\Delta p/p$, is of the same order as Q_1, the nonlinear effects in the excitation rate γ_1 may be safely ignored.

The motion is purely oscillatory, as the parameter ζ for all three modes involved is much smaller than unity. For modes 2 and 3 it was found that

$$\zeta = 9 \times 10^7 Q_1 \sqrt{-\gamma_1} = 1.7 \times 10^2 Q_1 ,$$

thus it is of the order of 10^{-5}.

It is easy to convert the amplitude Q_1 to the amplitude of the radial velocity for the whole-disc measurements ΔV_{rad}. In such measurements, only the modes with the equatorial symmetry, i.e., with $j_1 = \pm 1$, are visible. We have

$$\Delta V_{rad} = \sqrt{2} \times \sqrt{3/2} \times 0.55 \times 7 \times 10^{10} \times 1.5 \times 10^{-3} Q_1 \text{ cm s}^{-1} =$$
$$= 1.0 \times 10^6 Q_1 \text{ cm s}^{-1} .$$

In this formula, factor $\sqrt{2}$ follows from the difference in v between $j = 0$ and $j = 1$ cases, factor $\sqrt{3/2}$ converts the r.m.s. amplitude to the actual amplitude, 0.55 is an averaging factor for radial velocity, 7×10^{10} cm $\approx R_\odot$, 1.5×10^{-3} s^{-1} $= \omega_1$. Thus, the expected radial velocity amplitude is of the order of 10 cm s^{-1}.

5. Conclusions

We have seen that the resonant coupling of g_1, $l = 1$ to higher-order g modes limits its amplitude to an unexpectedly low level. It is believed that with the accuracy to $\pm$ one order of magnitude our estimated value of Q_1 is valid for all low-order and low-degree g modes in all conventional solar modes.

The reason why the nonlinear effects are important at so low amplitudes is that there is a large number of highly adiabatic modes that may strongly interact with the unstable modes. This large number of possibilities permits us to tune the resonance very precisely.

In view of the presented results, the hypothesis that the finite amplitude development of the linear instability of the $l = 1$ g modes leads to mixing in the solar core seems highly implausible.

At its present stage, the theory of mode coupling does not cause any difficulty to the interpretation of the 160 min oscillation in terms of the excitation of a low-degree g mode. The observed amplitudes of 0.56 m s^{-1} (Severny *et al.*, 1979) and 0.22 m s^{-1} (Scherrer *et al.*, 1979) are certainly within the limits of uncertainty of our estimate. The apparent absence of other modes corresponding to different j at the same l and k may be consequence of the fact that their frequencies fall closer to the perfect resonance condition $\Delta\sigma = 0$, with some pairs of high-order g modes. The probability of such an event remains to be calculated.

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