



Equivariant homology and K -theory of affine Grassmannians and Toda lattices

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Dedicated to Vladimir Drinfeld on the occasion of his 50th birthday

ABSTRACT

For an almost simple complex algebraic group G with affine Grassmannian $\mathrm{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$, we consider the equivariant homology $H^{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$ and K -theory $K^{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$. They both have a commutative ring structure with respect to convolution. We identify the spectrum of homology ring with the universal group-algebra centralizer of the Langlands dual group \check{G} , and we relate the spectrum of K -homology ring to the universal group-group centralizer of G and of \check{G} . If we add the loop-rotation equivariance, we obtain a noncommutative deformation of the (K -)homology ring, and thus a Poisson structure on its spectrum. We relate this structure to the standard one on the universal centralizer. The commutative subring of $G(\mathbb{C}[[t]])$ -equivariant homology of the point gives rise to a polarization which is related to Kostant's Toda lattice integrable system. We also compute the equivariant K -ring of the affine Grassmannian Steinberg variety. The equivariant K -homology of Gr_G is equipped with a canonical basis formed by the classes of simple equivariant perverse coherent sheaves. Their convolution is again perverse and is related to the Feigin–Loktev fusion product of $G(\mathbb{C}[[t]])$ -modules.

1. Introduction

Let G be an almost simple complex algebraic group, and let Gr_G be its affine Grassmannian. Recall that if we set $\mathbf{O} = \mathbb{C}[[t]]$ and $\mathbf{F} = \mathbb{C}((t))$, then $\mathrm{Gr}_G = G(\mathbf{F})/G(\mathbf{O})$.

It is well known that the subgroup ΩK of polynomial loops into a maximal compact subgroup $K \subset G$ projects isomorphically to Gr_G ; thus Gr_G acquires the structure of a topological group. An algebro-geometric counterpart of this structure is provided by the *convolution diagram* $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \rightarrow \mathrm{Gr}_G$.

This allows one to define the *convolution* of two $G(\mathbf{O})$ -equivariant geometric objects (such as sheaves, or constructible functions) on Gr_G . A famous example of such a structure is the category of $G(\mathbf{O})$ -equivariant perverse sheaves on Gr ('Satake category' in the terminology of Beilinson and Drinfeld [BD00]); this is a semisimple abelian category, and convolution provides it with a symmetric monoidal structure. By results of [Gin95], [MV00] and [BD00], this category is identified with the category of (algebraic) representations of the Langlands dual group.

The starting point for the present work was the observation that a similar definition works in another setting, yielding a monoidal structure on the category of $G(\mathbf{O})$ -equivariant *perverse coherent*

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sheaves on Gr (the ‘coherent Satake category’). The latter is a nonsemisimple artinian abelian category, the heart of the middle perversity t -structure on the derived category of $G(\mathbf{O})$ -equivariant coherent sheaves on Gr_G ; existence of this t -structure is due to the fact that dimensions of all $G(\mathbf{O})$ -orbits inside a given component of Gr_G are of the same parity (cf. [Bez00]). The resulting monoidal category turns out to be nonsymmetric, though its Grothendieck ring $K^{G(\mathbf{O})}(\text{Gr}_G)$ is commutative. One of the results of this paper is a computation of this ring. Along with $K^{G(\mathbf{O})}(\text{Gr}_G)$, we compute its ‘graded version’, the ring $H^{G(\mathbf{O})}(\text{Gr})$ of equivariant homology of Gr , where the algebra structure is again provided by convolution.¹ (The ring $H^{G(\mathbf{O})}(\text{Gr}_G)$ was essentially computed by Peterson [Pet97] cf. also [Kos96].)

To describe the answer, let \check{G} be the Langlands dual group to G , and let $\check{\mathfrak{g}}$ be its Lie algebra. Consider the *universal centralizers* $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$ and $\mathfrak{Z}_{\check{G}}^{\check{G}}$: if we denote by $C_{\check{G}, \check{\mathfrak{g}}} \subset \check{G} \times \check{\mathfrak{g}}$ (respectively $C_{\check{G}, \check{G}} \subset \check{G} \times \check{G}$) the locally closed subvariety formed by all the pairs (g, x) such that $\text{Ad}_g(x) = x$ and x is regular (respectively all the pairs (g_1, g_2) such that $\text{Ad}_{g_1}g_2 = g_2$ and g_2 is regular), then $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$ (respectively $\mathfrak{Z}_{\check{G}}^{\check{G}}$) is the categorical quotient $C_{\check{G}, \check{\mathfrak{g}}}/\check{G}$ (respectively $C_{\check{G}, \check{G}}/\check{G}$) with respect to the diagonal adjoint action of \check{G} .

We identify $\text{Spec}(H^{G(\mathbf{O})}(\text{Gr}_G))$ with $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$. Also, we identify $\text{Spec}(K^{G(\mathbf{O})}(\text{Gr}_G))$ with a *variant* of $\mathfrak{Z}_{\check{G}}^{\check{G}}$ (the isomorphism $\text{Spec}(K^{G(\mathbf{O})}(\text{Gr}_G)) \simeq \mathfrak{Z}_{\check{G}}^{\check{G}}$ holds true if and only if G is of type E_8).

Notice that $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$ inherits a canonical symplectic structure as a Hamiltonian reduction of the cotangent bundle $\mathbb{T}^*\check{G}$. Also, $\mathfrak{Z}_{\check{G}}^{\check{G}}$ inherits a canonical Poisson structure as a q -Hamiltonian reduction of the q -Hamiltonian \check{G} -space *internal fusion double* $\mathbf{D}(\check{G})$ (see [AMM98]); this Poisson structure is in fact symplectic if and only if \check{G} is simply connected (that is, G is adjoint).

The corresponding Poisson structures on $K^{G(\mathbf{O})}(\text{Gr}_G)$ and $H^{G(\mathbf{O})}(\text{Gr}_G)$ come from a deformation of these commutative algebras to noncommutative algebras $H^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ (respectively $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$); here \mathbb{G}_m acts on Gr_G by loop rotation. We conjecture that the noncommutative algebra $H^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ can also be obtained from the ring of differential operators on \check{G} by quantum Hamiltonian reduction.

The space $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$ contains an open piece $\mathfrak{Z}(\check{G})$ which for \check{G} adjoint (that is, for G simply connected) is a complexification of the Kostant phase space of the classical Toda lattice [Kos79, Theorem 2.6]. We remark in passing that the Toda lattice also appears in the (apparently related) computations by Givental, Kim and others of quantum cohomology of flag varieties (see e.g. [Kim99]).

Our computation should be compared with (and is to a large extent inspired by) [Gin95] where equivariant cohomology $H_{G(\mathbf{O})}(\text{Gr}_G)$ was computed² in terms of the \check{G} . (The precise relation between the two computations is spelled out in Remark 2.13.)

The second main object considered in this paper is another derived category of coherent sheaves with a convolution monoidal structure, namely the derived category $D^b \text{Coh}_{\Lambda_G}^{G(\mathbf{O})}(\mathbb{T}^* \text{Gr})$ of $G(\mathbf{O})$ -equivariant coherent sheaves on the cotangent bundle of Gr_G supported on the union Λ_G of conormal bundles to the $G(\mathbf{O})$ -orbits (the definition of involved objects requires extra work since Gr_G is infinite dimensional). (In this case we do not find a t -structure compatible with convolution, so all we get is a monoidal triangulated category.) Notice that the singular support of a $G(\mathbf{O})$ -equivariant D -module on Gr_G is an object of $\text{Coh}_{\Lambda_G}^{G(\mathbf{O})}(\mathbb{T}^* \text{Gr})$; thus this category can be considered a ‘classical limit’ of the (derived) Satake category. We compute the Grothendieck ring of $D^b \text{Coh}_{\Lambda_G}^{G(\mathbf{O})}(\mathbb{T}^* \text{Gr})$, identifying its spectrum with $(T \times \check{T})/W$, where $T \subset G$ and $\check{T} \subset \check{G}$

¹The two rings are related via the Chern character homomorphism from $K^{G(\mathbf{O})}(\text{Gr})$ to the completion of $H^{G(\mathbf{O})}(\text{Gr})$.

²Another description for $H_{G(\mathbf{O})}(\text{Gr}_G)$ is provided by a general result of [KK86]; in fact, its extension from [KK90] also gives an answer for $K^{G(\mathbf{O})}(\text{Gr}_G)$, and a similar technique can be applied to compute $H^{G(\mathbf{O})}(\text{Gr}_G)$. However, this form of the answer does not make the relation to the (dual) group geometry explicit.

are Cartan subgroups. This is a singular variety birationally equivalent to $\text{Spec}(K^{G(\mathbf{O})}(\text{Gr}_G))$. Unlike the latter, the former remains unchanged if we replace G by \tilde{G} . This motivates a conjecture that the corresponding triangulated monoidal categories for G and \tilde{G} are equivalent. The conjecture is compatible with a ‘classical limit’ of the geometric Langlands conjecture of Beilinson and Drinfeld (see § 7.9 for a more precise statement of the conjecture).

Finally, we remark that the convolution of $G(\mathbf{O})$ -equivariant perverse coherent sheaves is closely related to the *fusion product* of $G(\mathbf{O})$ -modules introduced by Feigin³ [FL99] (see § 8). In fact, our desire to understand the category $\mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$, and the work of Feigin and Loktev [FL99], was one of the motivations for the present work.

2. Notation and statements of the results

2.1 Kostant slices. The group G is an almost simple algebraic group with the Lie algebra \mathfrak{g} . We choose a principal \mathfrak{sl}_2 triple (e, h, f) in \mathfrak{g} . Let $\phi : \mathfrak{sl}_2 \rightarrow \mathfrak{g}$ (respectively $\Phi : SL_2 \rightarrow G$) be the corresponding homomorphism. We denote by e_G (respectively f_G) the image $\Phi(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix})$ (respectively $\Phi(\begin{smallmatrix} 1 & 0 \\ & 1 \end{smallmatrix})$). We denote by $\mathfrak{z}(e)$ the centralizer of e in \mathfrak{g} , and by $Z(e)$ (respectively $Z^0(e)$) the centralizer of e (equivalently, of e_G) in G (respectively its neutral connected component). We denote by $\Sigma_{\mathfrak{g}} \subset \mathfrak{g}$ (respectively $\Sigma_G \subset G$) the *Kostant slice* $\mathfrak{z}(e) + f$ (respectively $Z^0(e) \cdot f_G$). It is known that $\Sigma_{\mathfrak{g}} \subset \mathfrak{g}^{\text{reg}}$ (respectively $\Sigma_G \subset G^{\text{reg}}$), and the projection to the categorical quotient $\Sigma_{\mathfrak{g}} \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\text{Ad}_G = \mathfrak{t}/W$ induces an isomorphism $\Sigma_{\mathfrak{g}} \simeq \mathfrak{t}/W$. Similarly, if G is simply connected, the projection to the categorical quotient $\Sigma_G \hookrightarrow G \rightarrow G/\text{Ad}_G = T/W$ induces an isomorphism $\Sigma_G \simeq T/W$.

2.2 The universal centralizers. We consider the locally closed subvariety $C_{\mathfrak{g},\mathfrak{g}} \subset \mathfrak{g} \times \mathfrak{g}$ (respectively $C_{\mathfrak{g},G} \subset \mathfrak{g} \times G$, $C_{G,\mathfrak{g}} \subset G \times \mathfrak{g}$, $C_{G,G} \subset G \times G$) formed by all the pairs (x_1, x_2) such that $[x_1, x_2] = 0$ and x_2 is regular (respectively all the pairs (x, g) such that $\text{Ad}_g(x) = x$ and g is regular; all the pairs (g, x) such that $\text{Ad}_g(x) = x$ and x is regular; all the pairs (g_1, g_2) such that $\text{Ad}_{g_1}(g_2) = g_2$ and g_2 is regular). The categorical quotients with respect to the diagonal adjoint action of G are denoted respectively $C_{\mathfrak{g},\mathfrak{g}}//G = \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$, $C_{\mathfrak{g},G}//G = \mathfrak{Z}_G^{\mathfrak{g}}$, $C_{G,\mathfrak{g}}//G = \mathfrak{Z}_{\mathfrak{g}}^G$, and $C_{G,G}//G = \mathfrak{Z}_G^G$. The projections to the second (regular) factor are denoted by $\varpi : \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}} \rightarrow \mathfrak{g}^{\text{reg}}//G = \mathfrak{t}/W$, $\varpi : \mathfrak{Z}_G^{\mathfrak{g}} \rightarrow G^{\text{reg}}//G = T/W$, $\varpi : \mathfrak{Z}_{\mathfrak{g}}^G \rightarrow \mathfrak{g}^{\text{reg}}//G = \mathfrak{t}/W$, and $\varpi : \mathfrak{Z}_G^G \rightarrow G^{\text{reg}}//G = T/W$. In all the four cases ϖ is flat.

We consider the restrictions of our centralizer varieties to the Kostant slices: $C_{\mathfrak{g},\mathfrak{g}}^{\Sigma} = C_{\mathfrak{g},\mathfrak{g}} \cap (\mathfrak{g} \times \Sigma_{\mathfrak{g}})$, $C_{\mathfrak{g},G}^{\Sigma} = C_{\mathfrak{g},G} \cap (\mathfrak{g} \times \Sigma_G)$, $C_{G,\mathfrak{g}}^{\Sigma} = C_{G,\mathfrak{g}} \cap (G \times \Sigma_{\mathfrak{g}})$, and $C_{G,G}^{\Sigma} = C_{G,G} \cap (G \times \Sigma_G)$.

Then the locally closed embedding $C_{\mathfrak{g},\mathfrak{g}}^{\Sigma} \hookrightarrow C_{\mathfrak{g},\mathfrak{g}} \twoheadrightarrow \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$ induces an isomorphism $C_{\mathfrak{g},\mathfrak{g}}^{\Sigma} \simeq \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$. Similarly, we have isomorphisms $C_{G,\mathfrak{g}}^{\Sigma} \simeq \mathfrak{Z}_{\mathfrak{g}}^G$ and (for simply connected G) $C_{\mathfrak{g},G}^{\Sigma} \simeq \mathfrak{Z}_G^{\mathfrak{g}}$, and $C_{G,G}^{\Sigma} \simeq \mathfrak{Z}_G^G$.

Thus both $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}} \rightarrow \mathfrak{t}/W$ and $\mathfrak{Z}_G^{\mathfrak{g}} \rightarrow T/W$ (for simply connected G) are the sheaves of abelian Lie algebras, while both $\mathfrak{Z}_{\mathfrak{g}}^G \rightarrow \mathfrak{t}/W$ and $\mathfrak{Z}_G^G \rightarrow T/W$ (for simply connected G) are the sheaves of abelian Lie groups.

2.3 Isogenies. The center $Z(G)$ acts naturally on $\mathfrak{Z}_G^{\mathfrak{g}}$ (respectively $\mathfrak{Z}_{\mathfrak{g}}^G$) by $z(x, g) = (x, zg)$ (respectively $z(g, x) = (zg, x)$). The center $Z(G)$ acts on \mathfrak{Z}_G^G on both sides: $z_1(g_1, g_2)z_2 = (z_1g_1, z_2g_2)$. Let \tilde{G} denote the universal cover of G . Then the fundamental group $\pi_1(G)$ is embedded into $Z(\tilde{G})$, and we have $\mathfrak{Z}_G^{\mathfrak{g}} = \pi_1(G) \backslash \mathfrak{Z}_{\tilde{G}}^{\mathfrak{g}}$, $\mathfrak{Z}_{\mathfrak{g}}^G = \pi_1(G) \backslash \mathfrak{Z}_{\tilde{G}}^G$, and $\mathfrak{Z}_G^G = \pi_1(G) \backslash \mathfrak{Z}_{\tilde{G}}^G / \pi_1(G)$.

³The relation between convolution and fusion was known to B. Feigin in 1997.

2.4 Symplectic structures. We fix an invariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$; hence $\mathfrak{t} \simeq \mathfrak{t}^*$. Then $\mathfrak{g} \times \mathfrak{g}$ gets identified with $\mathfrak{g} \times \mathfrak{g}^* = \mathbb{T}^*\mathfrak{g}$ (the cotangent bundle), and $G \times \mathfrak{g}$ gets identified with $G \times \mathfrak{g}^* = \mathbb{T}^*G$. After this, $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$ (respectively $\mathfrak{Z}_{\mathfrak{g}}^G$) can be viewed as a Hamiltonian reduction of $\mathbb{T}^*\mathfrak{g}$ (respectively \mathbb{T}^*G) with respect to the adjoint action of G ; thus it inherits a canonical symplectic structure.

Identifying $\mathfrak{g} \times G$ with $\mathfrak{g}^* \times G = \mathbb{T}^*G$, we can view $\mathfrak{Z}_G^{\mathfrak{g}}$ as a Hamiltonian reduction of \mathbb{T}^*G as well; thus it inherits a canonical Poisson structure. Note that $\mathfrak{Z}_G^{\mathfrak{g}}$ is smooth and symplectic if and only if G is simply connected. We have symplectic isomorphisms $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}} \simeq \mathbb{T}^*(\mathfrak{t}/W)$, and (when G is simply connected) $\mathfrak{Z}_G^{\mathfrak{g}} \simeq \mathbb{T}^*(T/W)$.

Note that $\mathfrak{Z}_G^{\mathfrak{g}}$ and $\mathfrak{Z}_{\mathfrak{g}}^G$ share a common open piece $Z(G)$ formed by the classes of pairs (g, x) where both g and x are regular. The canonical symplectic structures agree on $\mathfrak{Z}_G^{\mathfrak{g}} \supset Z(G) \subset \mathfrak{Z}_{\mathfrak{g}}^G$. Note also that for adjoint G the space $Z(G)$ contains (a complexification of) the Kostant phase space $\mathfrak{Z}(G)$ of the classical Toda lattice [Kos79], and the embedding $\mathfrak{Z}(G) \hookrightarrow \mathfrak{Z}_{\mathfrak{g}}^G$ is given by Theorem 2.6 of [Kos79].

Alekseev, Malkin and Meinrenken introduced in [AMM98, Example 6.1] the q -Hamiltonian G -space *internal fusion double* $\mathbf{D}(G)$. Its q -Hamiltonian reduction is \mathfrak{Z}_G^G , so it inherits a canonical Poisson structure. For a simply connected G , the space \mathfrak{Z}_G^G is smooth and symplectic.

2.5 Affine blow-ups. The set of roots of G (respectively \check{G}) is denoted by R (respectively \check{R}). We will view $\alpha \in R$ (respectively $\check{\alpha} \in \check{R}$) as a homomorphism $\mathfrak{t} \rightarrow \mathbb{C}$ (respectively $\check{\mathfrak{t}} \rightarrow \mathbb{C}$) or as a homomorphism $T \rightarrow \mathbb{C}^*$ (respectively $\check{T} \rightarrow \mathbb{C}^*$) depending on context. Also, for a root $\alpha \in R$ we denote by ${}^1\alpha$ (respectively ${}^2\alpha$) the linear function on $\mathfrak{t} \times \mathfrak{t}$ obtained as a composition of α with the projection on the first (respectively second) factor.

We consider the following affine blow-up of $\mathfrak{t} \times \mathfrak{t}$ at the diagonal walls: $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} = \text{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, {}^1\alpha/{}^2\alpha, \alpha \in R])$. We also set $\mathfrak{B}_G^G = \text{Spec}(\mathbb{C}[\mathfrak{t} \times T, {}^1\alpha/({}^2\alpha - 1), \alpha \in R])$; $\mathfrak{B}_G^G = \text{Spec}(\mathbb{C}[T \times T, ({}^1\alpha - 1)/({}^2\alpha - 1), \alpha \in R])$; $\mathfrak{B}_G^{\check{G}} = \text{Spec}(\mathbb{C}[\check{\mathfrak{t}} \times T, ({}^1\check{\alpha} - 1)/({}^2\alpha - 1), \alpha \in R])$; and let $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} = \mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}}/W$, $\mathfrak{B}_G^G = \mathfrak{B}_G^G/W$, $\mathfrak{B}_G^{\check{G}} = \mathfrak{B}_G^{\check{G}}/W$ (thus $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} = \text{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, {}^1\alpha/{}^2\alpha, \alpha \in R]^W)$, etc.). We denote by ϖ the projection of \mathfrak{B} to the second factor; thus we have $\varpi : \mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} \rightarrow \mathfrak{t}/W$, $\mathfrak{B}_G^G \rightarrow T/W$, $\mathfrak{B}_G^{\check{G}} \rightarrow \mathfrak{t}/W$, $\mathfrak{B}_G^G \rightarrow T/W$ and $\mathfrak{B}_G^{\check{G}} \rightarrow T/W$.

2.6 Poisson structures. We have the canonical trivializations of the tangent bundles $\mathbb{T}(\mathfrak{t} \times \mathfrak{t}) = (\mathfrak{t} \times \mathfrak{t}) \times (\mathfrak{t} \times \mathfrak{t})$, $\mathbb{T}(\mathfrak{t} \times T) = (\mathfrak{t} \times T) \times (\mathfrak{t} \times \mathfrak{t})$, $\mathbb{T}(T \times \mathfrak{t}) = (T \times \mathfrak{t}) \times (\mathfrak{t} \times \mathfrak{t})$, $\mathbb{T}(T \times T) = (T \times T) \times (\mathfrak{t} \times \mathfrak{t})$, and $\mathbb{T}(T \times \check{T}) = (T \times \check{T}) \times (\mathfrak{t} \times \check{\mathfrak{t}})$. Making use of the identification $\check{\mathfrak{t}} = \mathfrak{t}^* \simeq \mathfrak{t}$, we obtain the W -invariant symplectic structures on the above varieties. Thus the above affine blow-ups carry the rational Poisson structures (regular off the discriminants $\mathbf{D} \subset \mathfrak{B}$).

PROPOSITION 2.7. *The Poisson structure on $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} - \mathbf{D}$ (respectively $\mathfrak{B}_G^G - \mathbf{D}$, $\mathfrak{B}_{\mathfrak{g}}^G - \mathbf{D}$, $\mathfrak{B}_G^G - \mathbf{D}$, $\mathfrak{B}_G^{\check{G}} - \mathbf{D}$) extends to the global Poisson structure; it is a symplectic structure if the corresponding variety is smooth.*

PROPOSITION 2.8. *We are in the setup of § 2.5.*

- (a) *The projection ϖ is flat if G is simply connected.*
- (b) *There are natural identifications $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} \simeq \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$, $\mathfrak{B}_G^G \simeq \mathfrak{Z}_G^G$, $\mathfrak{B}_{\mathfrak{g}}^G \simeq \mathfrak{Z}_{\mathfrak{g}}^G$, and $\mathfrak{B}_G^G \simeq \mathfrak{Z}_G^G$ commuting with ϖ .*
- (c) *If G is simply laced and adjoint, we have an identification $\mathfrak{B}_G^G \simeq Z(\check{G}) \setminus \mathfrak{Z}_G^{\check{G}}$ commuting with ϖ .*
- (d) *If G is simply laced and simply connected, we have an identification $\mathfrak{B}_G^G \simeq \mathfrak{Z}_G^G/Z(G)$ commuting with ϖ .*
- (e) *The above identifications respect the Poisson structures.*

2.9 Flat group sheaves. We consider the functor $\mathfrak{F}_{\mathfrak{g}}^{\mathfrak{g}}$ on the category $\text{Flat}_{\mathfrak{t}/W}$ of schemes flat over \mathfrak{t}/W to the category of sets, sending a test scheme S to the set of W -invariant morphisms $(\text{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, \mathfrak{t}))^W$. Similarly, we consider the functor $\mathfrak{F}_G^{\mathfrak{g}}$ on the category $\text{Flat}_{T/W}$ sending a test scheme S to the set of W -invariant morphisms $(\text{Mor}(S \times_{T/W} T, \mathfrak{t}))^W$. Also, we consider the functor \mathfrak{F}_G^G on the category $\text{Flat}_{\mathfrak{t}/W}$ sending a test scheme S to the set of W -invariant morphisms $(\text{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, T))_0^W \subset (\text{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, T))^W$ subject to the condition (cf. [DG02, 4.2])

$$\alpha(f(\alpha^{-1}(0))) = 1 \quad \forall \alpha \in R. \tag{1}$$

(Note that the W -invariance condition automatically implies $\alpha(f(\alpha^{-1}(0))) = \pm 1 \quad \forall \alpha \in R$.)

Furthermore, we consider the functor \mathfrak{F}_G^G on the category $\text{Flat}_{T/W}$ sending a test scheme S to the set of W -invariant morphisms $(\text{Mor}(S \times_{T/W} T, T))_0^W \subset (\text{Mor}(S \times_{T/W} T, T))^W$ subject to the condition

$$\alpha(f(\alpha^{-1}(1))) = 1 \quad \forall \alpha \in R. \tag{2}$$

(Note that the W -invariance condition automatically implies $\alpha(f(\alpha^{-1}(1))) = \pm 1 \quad \forall \alpha \in R$.)

Finally, we consider the functor $\mathfrak{F}_G^{\check{G}}$ on the category $\text{Flat}_{T/W}$ sending a test scheme S to the set of W -invariant morphisms $(\text{Mor}(S \times_{T/W} T, \check{T}))_0^W \subset (\text{Mor}(S \times_{T/W} T, \check{T}))^W$ subject to the condition

$$\check{\alpha}(f(\alpha^{-1}(1))) = 1 \quad \forall \alpha \in R. \tag{3}$$

(Note that the W -invariance condition automatically implies $\check{\alpha}(f(\alpha^{-1}(1))) = \pm 1 \quad \forall \alpha \in R$.)

The following proposition is a generalization of [DG02, 11.6].

PROPOSITION 2.10. *Assume that G is simply connected. The functor $\mathfrak{F}_{\mathfrak{g}}^{\mathfrak{g}}$ (respectively $\mathfrak{F}_G^{\mathfrak{g}}$, $\mathfrak{F}_{\mathfrak{g}}^G$, \mathfrak{F}_G^G , $\mathfrak{F}_G^{\check{G}}$) is representable by the scheme $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}}$ (respectively $\mathfrak{B}_G^{\mathfrak{g}}$, $\mathfrak{B}_{\mathfrak{g}}^G$, \mathfrak{B}_G^G , $\mathfrak{B}_G^{\check{G}}$).*

2.11 Equivariant Borel–Moore homology. For the definition of convolution in equivariant Borel–Moore homology, we refer the reader to [CG97, 2.7, 8.3] or [Lus95, Chapter 2].

We have $H_{G(\mathbf{O})}^{G(\mathbf{O})}(pt) = H_{G(\mathbf{O})}^{\bullet}(pt) = \mathbb{C}[\mathfrak{t}/W]$, and $H_{G(\mathbf{O}) \times \mathbb{G}_m}^{G(\mathbf{O}) \times \mathbb{G}_m}(pt) = H_{G(\mathbf{O}) \times \mathbb{G}_m}^{\bullet}(pt) = \mathbb{C}[\mathfrak{t}/W][\hbar]$, where \hbar is the generator of $H_{\mathbb{G}_m}^2(pt)$. We will consider the $\mathbb{C}[\mathfrak{t}/W]$ -algebra (respectively $\mathbb{C}[\mathfrak{t}/W][\hbar]$ -algebra) (with respect to convolution) $H_{G(\mathbf{O})}^{G(\mathbf{O})}(\text{Gr}_G)$ (respectively $H_{G(\mathbf{O}) \times \mathbb{G}_m}^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$). Note that setting $\hbar = 0$ in $H_{G(\mathbf{O}) \times \mathbb{G}_m}^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ we obtain $H_{G(\mathbf{O})}^{G(\mathbf{O})}(\text{Gr}_G)$; indeed for any group H , a space X with an $H \times \mathbb{G}_m$ action, and an $H \times \mathbb{G}_m$ -equivariant complex \mathcal{F} on X , we have a long exact sequence $\dots \rightarrow H_{H \times \mathbb{G}_m}^{i-2}(X, \mathcal{F}) \xrightarrow{\hbar} H_{H \times \mathbb{G}_m}^i(X, \mathcal{F}) \rightarrow H_H^i(X, \mathcal{F}) \rightarrow H_{H \times \mathbb{G}_m}^{i-1}(X, \mathcal{F}) \rightarrow \dots$ coming from the principal \mathbb{G}_m -bundle $E(H \times \mathbb{G}_m) \times_H X \rightarrow E(H \times \mathbb{G}_m) \times_{H \times \mathbb{G}_m} X$; if the space of $H \times \mathbb{G}_m$ -equivariant cohomology is \hbar -torsion free, then we get $H_H^i(X, \mathcal{F}) = H^i(X, \mathcal{F})|_{\hbar=0}$.

THEOREM 2.12.

- (a) *The algebra $H_{G(\mathbf{O})}^{G(\mathbf{O})}(\text{Gr}_G)$ is commutative.*
- (b) *Its spectrum together with the projection onto $\mathfrak{t}/W = \check{\mathfrak{t}}/W$ is naturally isomorphic to $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}} \xrightarrow{\cong} \check{\mathfrak{t}}/W$.*
- (c) *The Poisson structure on $H_{G(\mathbf{O})}^{G(\mathbf{O})}(\text{Gr}_G)$ arising from the \hbar -deformation $H_{G(\mathbf{O}) \times \mathbb{G}_m}^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ corresponds under the above identification to the Poisson structure of § 2.4 on $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}}$.*

Remark 2.13. The equivariant cohomology ring $H_{G(\mathbf{O})}^{\bullet}(\text{Gr}_G, \mathbb{C}) = H_{G(\mathbf{O})}^{\bullet}(\text{Gr}_G)$ was computed by Ginzburg [Gin95]. More precisely, the projection to the second (regular) factor $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}} \rightarrow \check{\mathfrak{g}}^{\text{reg}}//\check{G} = \check{\mathfrak{t}}/W$ makes $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}}$ a sheaf of abelian Lie algebras. Ginzburg identifies $H_{G(\mathbf{O})}^{\bullet}(\text{Gr}_G)$ with the global sections of the relative universal enveloping algebra $U_{\check{\mathfrak{t}}/W}(\mathfrak{Z}_{\mathfrak{g}}^{\check{G}})$. One can easily check that this result is compatible

with our Theorem 2.12, part (b) as follows. For a group scheme A over a base S one has a natural pairing $U(\mathfrak{a}) \times \mathcal{O}(A) \rightarrow \mathcal{O}(S)$ where $U(\mathfrak{a})$ is the enveloping (over $\mathcal{O}(S)$) of the Lie algebra of A ; the pairing sends (ξ, f) to $\xi(f)$ restricted to the identity of A . On the other hand, for a compact (or ind-compact) H -space X we have a pairing $H_H^\bullet(X) \times H_H^\bullet(X) \rightarrow H_H^\bullet(pt)$ induced by the action of cohomology on homology, and the push-forward map in Borel–Moore homology $H_H^\bullet(X) \rightarrow H_H^\bullet(pt)$. The isomorphisms of [Gin95] and of Theorem 2.12 take the first pairing into the second one.

2.14 Equivariant K -theory. For the definition of convolution in equivariant K -theory we refer the reader to [CG97, ch. 5].

We have $K^{G(\mathbf{O})}(pt) = \mathbb{C}[T/W]$ and $K^{G(\mathbf{O}) \times \mathbb{G}_m}(pt) = \mathbb{C}[T/W][q^{\pm 1}]$. We will consider the $\mathbb{C}[T/W]$ -algebra (respectively $\mathbb{C}[T/W][q^{\pm 1}]$ -algebra) (with respect to convolution) $K^{G(\mathbf{O})}(\text{Gr}_G)$ (respectively $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$). Note that setting $q = 1$ in $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ we obtain $K^{G(\mathbf{O})}(\text{Gr}_G)$.

THEOREM 2.15.

- (a) *The algebra $K^{G(\mathbf{O})}(\text{Gr}_G)$ is commutative.*
- (b) *Its spectrum together with the projection onto T/W is naturally isomorphic to $\mathfrak{B}_G^{\check{G}} \xrightarrow{\cong} T/W$.*
- (c) *The Poisson structure on $K^{G(\mathbf{O})}(\text{Gr}_G)$ arising from the q -deformation $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ corresponds under the above identification to the Poisson structure of Proposition 2.7 on $\mathfrak{B}_G^{\check{G}}$ when the latter variety is smooth, i.e. G is simply connected.*

3. Calculations in rank 1

In this section $G = SL_2$, and $\check{G} = PGL_2$. The Weyl group $W = \mathbb{Z}/2\mathbb{Z}$, the Cartan torus $T = \mathbb{G}_m = \mathbb{C}^*$ with a coordinate z , and the only simple root $\alpha(z) = z^2$. The dual torus $\check{T} = \mathbb{G}_m = \mathbb{C}^*$ with a coordinate t , and $\check{\alpha}(t) = t$. The Cartan Lie algebra $\mathfrak{t} = \mathbb{C}$ with a coordinate $x = \alpha(x)$. We fix a $\sqrt{-1}$.

3.1 Calculating \mathfrak{Z}_G^G and \mathfrak{B}_G^G . We choose the standard \mathfrak{sl}_2 -triple

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the Kostant slice

$$\Sigma_G = \left\{ \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}, a \in \mathbb{C} \right\}.$$

One checks that a matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ commutes with $\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}$ if and only if

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \sqrt{-1} \begin{pmatrix} (1-a)c+b & (2-a)c \\ -c & b-c \end{pmatrix} \quad \text{for } b, c \in \mathbb{C}.$$

Then the condition $\det = 1$ reads as

$$1 = abc - b^2 - c^2. \tag{4}$$

Thus, \mathfrak{Z}_G^G is identified with a hypersurface \mathcal{S} in \mathbb{A}^3 given by (4). The left (respectively right) multiplication by $-1 \in Z(SL_2)$ is an involution ι (respectively j) on \mathcal{S} given by $\iota(a, b, c) = (a, -b, -c)$ (respectively $j(a, b, c) = (-a, b, -c)$). Hence, $\mathfrak{Z}_G^{\check{G}} = \iota \backslash \mathcal{S} / j$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that

$$g\sqrt{-1} \begin{pmatrix} (1-a)c+b & (2-a)c \\ -c & b-c \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$$

and

$$g \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

for some $y, z \in \mathbb{G}_m = \mathbb{C}^* = T$ defined up to simultaneous inversion. Then we have

$$a = z + z^{-1}, \quad b = \frac{-\sqrt{-1}}{2} \left(y + y^{-1} + \frac{(y - y^{-1})(z + z^{-1})}{z - z^{-1}} \right), \quad c = -\sqrt{-1} \frac{y - y^{-1}}{z - z^{-1}}. \tag{5}$$

We conclude that $\mathbb{C}[\mathcal{S}] = \mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y - y^{-1})/(z - z^{-1})]^W$ where the nontrivial element $w \in W$ acts by $w(y, z) = (y^{-1}, z^{-1})$. We can rewrite $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y - y^{-1})/(z - z^{-1})]^W$ as $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y^2 - 1)/(z^2 - 1)]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}_G^G]$. All in all, we have $\mathfrak{B}_G^G \simeq \mathcal{S} \simeq \mathfrak{Z}_G^G$. Since we can identify \check{T} with $T/Z(G)$, the identifications $\mathfrak{B}_G^G \simeq \mathcal{S}/j$, $\mathfrak{B}_G^{\check{G}} \simeq \iota \backslash \mathcal{S}$, $\mathfrak{B}_G^{\check{G}} \simeq \iota \backslash \mathcal{S}/j \simeq \mathfrak{Z}_G^{\check{G}}$ follow immediately.

3.2 Calculating \mathfrak{Z}_g^G and \mathfrak{B}_g^G . The Kostant slice

$$\Sigma_g = \left\{ \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}, \delta \in \mathbb{C} \right\}.$$

One checks that a matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ commutes with $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ if and only if

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix} \quad \text{for } \xi, \eta \in \mathbb{C}.$$

Then the condition $\det = 1$ reads as

$$1 = \xi^2 - \delta\eta^2. \tag{6}$$

Thus, \mathfrak{Z}_g^G is identified with a hypersurface \mathcal{S}' in \mathbb{A}^3 given by (6). The action of $-1 \in Z(SL_2)$ is an involution ι on \mathcal{S}' given by $\iota(\delta, \xi, \eta) = (\delta, -\xi, -\eta)$. Hence, $\mathfrak{Z}_g^{\check{G}} = \iota \backslash \mathcal{S}'$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that

$$g \begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$$

and

$$g \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$$

for some $y \in \mathbb{G}_m = \mathbb{C}^* = T$, $x \in \mathbb{C} = \mathfrak{t}$, defined up to $(y, x) \mapsto (y^{-1}, -x)$. Then we have

$$\delta = x^2, \quad \xi = \frac{y + y^{-1}}{2}, \quad \eta = \frac{y - y^{-1}}{2x}.$$

We conclude that $\mathbb{C}[\mathcal{S}'] = \mathbb{C}[y^{\pm 1}, x, (y - y^{-1})/x]^W$ where the nontrivial element $w \in W$ acts by $w(y, x) = (y^{-1}, -x)$. We can rewrite $\mathbb{C}[y^{\pm 1}, x, (y - y^{-1})/x]^W$ as $\mathbb{C}[y^{\pm 1}, x, (y^2 - 1)/x]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}_g^G]$. All in all, we have $\mathfrak{B}_g^G \simeq \mathcal{S}' \simeq \mathfrak{Z}_g^G$. Since we can identify \check{T} with $T/Z(G)$, the identification $\mathfrak{B}_g^{\check{G}} \simeq \iota \backslash \mathcal{S}' \simeq \mathfrak{Z}_g^{\check{G}}$ follows immediately.

3.3 Calculating \mathfrak{Z}_G^g and \mathfrak{B}_G^g . Recall the Kostant slice

$$\Sigma_G = \left\{ \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}, a \in \mathbb{C} \right\}.$$

One checks that a traceless matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$ commutes with $\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}$ if and only if

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \zeta \begin{pmatrix} 2-a & 4-2a \\ -2 & a-2 \end{pmatrix} \quad \text{for } \zeta \in \mathbb{C}.$$

Thus, $\mathfrak{Z}_G^{\mathfrak{g}}$ is identified with \mathbb{A}^2 with coordinates a, ζ . The action of $-1 \in Z(SL_2)$ is an involution j on \mathbb{A}^2 given by $j(a, \zeta) = (-a, -\zeta)$. Hence, $\mathfrak{Z}_{\check{G}}^{\mathfrak{g}} = \mathbb{A}^2/j$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that

$$g\zeta \begin{pmatrix} 2-a & 4-2a \\ -2 & a-2 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$$

and

$$g \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

for some $x \in \mathbb{C} = \mathfrak{t}$, $z \in \mathbb{G}_m = \mathbb{C}^* = T$ defined up to $(x, z) \mapsto (-x, z^{-1})$. Then we have

$$a = z + z^{-1}, \quad \zeta = \frac{x}{z - z^{-1}}.$$

We conclude that $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[x, z^{\pm 1}, x/(z - z^{-1})]^W$ where the nontrivial element $w \in W$ acts by $w(x, z) = (-x, z^{-1})$. We can rewrite $\mathbb{C}[x, z^{\pm 1}, x/(z - z^{-1})]^W$ as $\mathbb{C}[x, z^{\pm 1}, x/(z^2 - 1)]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}_G^{\mathfrak{g}}]$. All in all, we have $\mathfrak{B}_G^{\mathfrak{g}} \simeq \mathbb{A}^2 \simeq \mathfrak{Z}_G^{\mathfrak{g}}$. Since we can identify \check{T} with $T/Z(G)$, the identification $\mathfrak{B}_{\check{G}}^{\mathfrak{g}} \simeq \mathbb{A}^2/j \simeq \mathfrak{Z}_{\check{G}}^{\mathfrak{g}}$ follows immediately.

3.4 Calculating $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$ and $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}}$. Recall the Kostant slice

$$\Sigma_{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}, \delta \in \mathbb{C} \right\}.$$

One checks that a traceless matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$ commutes with $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ if and only if

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \begin{pmatrix} 0 & \delta\theta \\ \theta & 0 \end{pmatrix} \quad \text{for } \theta \in \mathbb{C}.$$

Thus, $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$ is identified with \mathbb{A}^2 with coordinates δ, θ .

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that

$$g \begin{pmatrix} 0 & \delta\theta \\ \theta & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$$

and

$$g \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$$

for some $u, x \in \mathbb{C} = \mathfrak{t}$, defined up to $(u, x) \mapsto (-u, -x)$. Then we have

$$\delta = x^2, \quad \theta = \frac{u}{x}.$$

We conclude that $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[u, x, u/x]^W$ where the nontrivial element $w \in W$ acts by $w(u, x) = (-u, -x)$. Hence we get an identification $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} \simeq \mathbb{A}^2 \simeq \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$.

3.5 Calculating $\mathfrak{B}_{\mathfrak{g}}^G$ and $\mathfrak{F}_{\mathfrak{g}}^G$. Recall the setup of Proposition 2.10. We will prove that the functor $\mathfrak{F}_{\mathfrak{g}}^G$ is representable by the scheme $\mathfrak{B}_{\mathfrak{g}}^G$; the other parts of Proposition 2.10 are proved absolutely similarly, as well as the Proposition 2.10 for G replaced by \check{G} . For a scheme S flat over \mathfrak{t}/W

we will denote by $S_{\mathfrak{t}}$ the cartesian product $S \times_{\mathfrak{t}/W} \mathfrak{t}$. Our usual coordinate x on \mathfrak{t} gives rise to the same named function on $S_{\mathfrak{t}}$. The nontrivial element $w \in W$ acts by the involution of $S_{\mathfrak{t}}$. Finally, we denote by $(\mathfrak{B}_{\mathfrak{g}}^G)_{S_{\mathfrak{t}}}$ the affine blow-up of $S_{\mathfrak{t}} \times T$, that is $S_{\mathfrak{t}} \times_{\mathfrak{t}} \mathfrak{B}_{\mathfrak{g}}^G$. Clearly, w acts as an involution of $(\mathfrak{B}_{\mathfrak{g}}^G)_{S_{\mathfrak{t}}}$.

Note that the condition (1) is void in the case under consideration. Given a w -equivariant morphism $f : S_{\mathfrak{t}} \rightarrow T = \mathbb{G}_m$ we see that $f^2 - 1$ is divisible by x ; hence f lifts uniquely to a section \hat{f} of $(\mathfrak{B}_{\mathfrak{g}}^G)_{S_{\mathfrak{t}}}$ over $S_{\mathfrak{t}}$. Evidently, \hat{f} is w -invariant. If we consider \hat{f} as a closed subscheme of $(\mathfrak{B}_{\mathfrak{g}}^G)_{S_{\mathfrak{t}}}$, then \hat{f}/W is a closed subscheme of $(\mathfrak{B}_{\mathfrak{g}}^G)_{S_{\mathfrak{t}}}/W = S \times_{\mathfrak{t}/W} \mathfrak{B}_{\mathfrak{g}}^G$ which is the graph of a morphism $\tilde{f} : S \rightarrow \mathfrak{B}_{\mathfrak{g}}^G$.

Conversely, given a morphism $\tilde{f} : S \rightarrow \mathfrak{B}_{\mathfrak{g}}^G$ we consider its graph $\Gamma_{\tilde{f}}$ as a closed subscheme of $S \times_{\mathfrak{t}/W} \mathfrak{B}_{\mathfrak{g}}^G$, and then the cartesian product $\Gamma_{\tilde{f}} \times_{S \times_{\mathfrak{t}/W} \mathfrak{B}_{\mathfrak{g}}^G} (\mathfrak{B}_{\mathfrak{g}}^G)_{S_{\mathfrak{t}}}$ is a section \hat{f} of $(\mathfrak{B}_{\mathfrak{g}}^G)_{S_{\mathfrak{t}}}$ over $S_{\mathfrak{t}}$. Evidently, \hat{f} gives rise to a w -equivariant function $f : S_{\mathfrak{t}} \rightarrow T$.

3.6 A basis in equivariant K-theory. We recall a few standard facts about the affine Grassmannians Gr_G and $\text{Gr}_{\check{G}}$. The $G(\mathbf{O})$ -orbits (equivalently, $\check{G}(\mathbf{O})$ -orbits) on $\text{Gr}_{\check{G}}$ are numbered by nonnegative integers and denoted by $\text{Gr}_{\check{G},n}$, $n \in \mathbb{N}$. The orbits $\text{Gr}_{\check{G},2n}$, $n \in \mathbb{N}$, form a connected component of $\text{Gr}_{\check{G}}$ equal to Gr_G . The open embedding of an orbit into its closure will be denoted by $j_n : \text{Gr}_{\check{G},n} \hookrightarrow \overline{\text{Gr}}_{\check{G},n}$ or simply by j if no confusion is likely. We have $\dim \text{Gr}_{\check{G},n} = n$; in particular, $\text{Gr}_{\check{G},0}$ is a point.

We have $K^{G(\mathbf{O})}(\text{Gr}_{\check{G},0}) = \text{Rep}(G)$ with a basis $\mathbf{v}(n)$, $n \in \mathbb{N}$, formed by the classes of irreducible G -modules $\mathcal{V}(n)$. Also, $K^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},0}) = \text{Rep}(\check{G}) \subset \text{Rep}(G)$ has a basis $\mathbf{v}(2n)$, $n \in \mathbb{N}$.

For $m > 0$ the $G(\mathbf{O})$ -equivariant line bundles in $\text{Gr}_{\check{G},m}$ are numbered by integers and denoted by $\mathcal{L}(n)_m$. Among them, the $\check{G}(\mathbf{O})$ -equivariant line bundles are exactly $\mathcal{L}(2n)_m$, $n \in \mathbb{Z}$. We define $\mathcal{V}(n)_m$ as $j_*\mathcal{L}(n)_m[m/2]$, that is, the (nonderived) direct image to the orbit closure placed in the homological degree $-m/2$. Note that, since the complement $\overline{\text{Gr}}_{\check{G},m} - \text{Gr}_{\check{G},m}$ has codimension 2, the above direct image is a coherent sheaf. The degree shift will become clear later. The class $[\mathcal{L}(n)_m]$ in $K^{G(\mathbf{O})}(\text{Gr}_{\check{G}})$ will be denoted by $\mathbf{v}(n)_m$. Thus, it is natural to denote $\mathbf{v}(n)$ above by $\mathbf{v}(n)_0$; we will keep both names.

The collection $\{\mathbf{v}(n)_m : n \in \mathbb{N} \text{ if } m = 0; n \in \mathbb{Z} \text{ if } m \in \mathbb{N} - 0\}$ forms a basis in $K^{G(\mathbf{O})}(\text{Gr}_{\check{G}})$. Among this collection, all the $\mathbf{v}(n)_m$ with n even (respectively m even) form a basis in $K^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ (respectively $K^{G(\mathbf{O})}(\text{Gr}_G)$).

3.7 Convolution: commutativity. In this subsection G is an arbitrary semisimple group. We prove Theorem 2.15, part (a). We refer the reader to [Gai01] for the basics of the Beilinson–Drinfeld Grassmannian. Recall that $\text{Gr}_G^{\text{BD}} \xrightarrow{\pi} \mathbb{A}^1$ is a flat ind-scheme such that $\pi^{-1}(\mathbb{A}^1 - 0) = (\mathbb{A}^1 - 0) \times \text{Gr}_G \times \text{Gr}_G$, while $\pi^{-1}(0) = \text{Gr}_G$. We also have the deformed convolution diagram $\text{Gr}_G^{\text{BD,conv}} \xrightarrow{\Pi} \text{Gr}_G^{\text{BD}}$ such that Π is an isomorphism over $\mathbb{A}^1 - 0$, while over $0 \in \mathbb{A}^1$ our Π is the usual convolution diagram $G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G \xrightarrow{\Pi_0} \text{Gr}_G$.

Given two $G(\mathbf{O})$ -equivariant complexes of coherent sheaves \mathcal{A}, \mathcal{B} on Gr_G , we can form their ‘deformed convolution’ complex $\mathcal{A} \tilde{\star} \mathcal{B}$ on $\text{Gr}_G^{\text{BD,conv}}$ such that over $\mathbb{A}^1 - 0$ it is isomorphic to $\mathcal{O}_{\mathbb{A}^1 - 0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}$, while over $0 \in \mathbb{A}^1$ it is isomorphic to the usual twisted product $\mathcal{A} \star \mathcal{B}$ on the convolution diagram $G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G$. In addition, if \mathcal{A}, \mathcal{B} are coherent sheaves, then $\mathcal{A} \tilde{\star} \mathcal{B}$ is flat over \mathbb{A}^1 . It implies that in the K -group the class $[\mathcal{A} \star \mathcal{B}]$ is the *specialization* (see [CG97, 5.3]) of the class $[\mathcal{O}_{\mathbb{A}^1 - 0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\text{Gr}_G^{\text{BD,conv}} \xrightarrow{\pi \circ \Pi} \mathbb{A}^1$, and also the class $[\mathcal{A} \star \mathcal{B}] = [\Pi_{0*}(\mathcal{A} \star \mathcal{B})]$ is the specialization of the class $[\mathcal{O}_{\mathbb{A}^1 - 0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\text{Gr}_G^{\text{BD}} \xrightarrow{\pi} \mathbb{A}^1$. Hence we have the desired commutativity.

3.8 Convolution: relations. We return to the setup of § 3.6. Note that $\text{Gr}_{\check{G},1} \simeq \mathbb{P}^1$, and $\mathcal{V}(n)_1$ is the line bundle $\mathcal{O}(n)$ on \mathbb{P}^1 . The twisted product $\mathcal{V}(n)_1 \times \mathcal{V}(l)_1$ is the line bundle $\mathcal{O}(n, l)$ on the two-dimensional subvariety $\mathcal{H}_2 \subset \check{G}(\mathbf{F}) \times_{\check{G}(\mathbf{O})} \text{Gr}_{\check{G}}$ isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O})$ over \mathbb{P}^1 . The projection $\Pi_0 : \mathcal{H}_2 \rightarrow \text{Gr}_{\check{G},2}$ is the contraction of the -2 -section $\mathbb{P}^1 \hookrightarrow \mathcal{H}_2$.

Now it is easy to compute $\mathbf{v}(n)_1 \star \mathbf{v}(n)_1 = \mathbf{v}(2n)_2$, $\mathbf{v}(1)_1 \star \mathbf{v}(-1)_1 = \mathbf{v}(0)_2 + 1$. Taking into account the evident relation $\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 = \mathbf{v}(1)_1 + \mathbf{v}(-1)_1$ we arrive at

$$\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 \star \mathbf{v}(1)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1 + \mathbf{v}(0)_1 \star \mathbf{v}(0)_1 + 1. \tag{7}$$

A moment of reflection shows that $K^{G(\mathbf{O})}(\text{Gr}_G)$ is generated as algebra by $\mathbf{v}(1)_0$, $\mathbf{v}(0)_2 = \mathbf{v}(0)_1 \star \mathbf{v}(0)_1$, $\mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1$, $\mathbf{v}(1)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(0)_1$ (one has to use that $\mathbf{v}(k)_{2l} \star \mathbf{v}(n)_{2m} = \mathbf{v}(k+n)_{2l+2m}$ plus the terms supported on the smaller orbits). Similarly, $K^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ is generated as algebra by $\mathbf{v}(2)_0 = \mathbf{v}(1)_0 \star \mathbf{v}(1)_0 - 1$, $\mathbf{v}(0)_1$, $\mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1$, $\mathbf{v}(2)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_0 - \mathbf{v}(0)_1$.

Note that both algebras $K^{G(\mathbf{O})}(\text{Gr}_G)$ and $K^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ lie in the vector space $K^{G(\mathbf{O})}(\text{Gr}_{\check{G}})$, and their intersection is the common subalgebra $K^{\check{G}(\mathbf{O})}(\text{Gr}_G)$. The tensor product algebra $K^{G(\mathbf{O})}(\text{Gr}_G) \otimes_{K^{\check{G}(\mathbf{O})}(\text{Gr}_G)} K^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ can be identified as a vector space with $K^{G(\mathbf{O})}(\text{Gr}_{\check{G}})$, and then it is generated by the three basic elements $\mathbf{v}(1)_0, \mathbf{v}(0)_1$ and $\mathbf{v}(1)_1$ subject to the only relation (7).

The comparison of (7) and (4) shows that the assignments $a \mapsto \mathbf{v}(1)_0$, $b \mapsto \mathbf{v}(0)_1$, $c \mapsto \mathbf{v}(1)_1$ establish an isomorphism $\mathbb{C}[\mathcal{S}] \simeq K^{G(\mathbf{O})}(\text{Gr}_{\check{G}})$. It identifies the spectrum of $K^{G(\mathbf{O})}(\text{Gr}_G)$ with $\iota \backslash \mathcal{S} \simeq \mathfrak{B}_{\check{G}}^{\check{G}}$, and the spectrum of $K^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ with $\mathcal{S}/j \simeq \mathfrak{B}_{\check{G}}^G$.

3.9 Iwahori-equivariant K -theory. Let $I \subset G(\mathbf{O})$ be the Iwahori subgroup. The space $K^I(\text{Gr}_G) = K^T(\text{Gr}_G) = K^{T(\mathbf{O})}(\text{Gr}_G) = K(T(\mathbf{O}) \backslash G(\mathbf{F})/G(\mathbf{O}))$ is equipped with the two commuting actions: $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$ acts by convolutions on the left, and $K^G(\text{Gr}_G) = K^{G(\mathbf{O})}(\text{Gr}_G) = K(G(\mathbf{O}) \backslash G(\mathbf{F})/G(\mathbf{O}))$ acts by convolutions on the right. Also, W acts on $K^T(\text{Gr}_G)$ commuting with the right action of $K^G(\text{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$ is isomorphic to $\mathbb{C}[\check{T} \times T]$. The action of W on $K^T(\text{Gr}_G)$ normalizes the action of $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$ and induces the natural (diagonal) action of W on $\mathbb{C}[\check{T} \times T]$.

Our aim in this subsection is to identify the $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O})) \times W - K^G(\text{Gr}_G)$ -bimodule $K^T(\text{Gr}_G)$ with the $\mathbb{C}[\check{T} \times T] \times W - \mathbb{C}[\mathfrak{B}_{\check{G}}^{\check{G}}]$ -bimodule $\mathbb{C}[\mathfrak{B}_{\check{G}}^{\check{G}}]$ (and similarly for G replaced by \check{G}). As in § 3.8, it suffices to identify the $K(T(\mathbf{O}) \backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \times W - K^G(\text{Gr}_{\check{G}})$ -bimodule $K^T(\text{Gr}_{\check{G}})$ with the $\mathbb{C}[T \times T] \times W - \mathbb{C}[\mathfrak{B}_{\check{G}}^G]$ -bimodule $\mathbb{C}[\mathfrak{B}_{\check{G}}^G]$.

Note that $K^G(\text{Gr}_{\check{G}}) \subset K^T(\text{Gr}_{\check{G}})$, and the $K^G(\text{Gr}_{\check{G}})$ -module $K^T(\text{Gr}_{\check{G}})$ is free of rank 2 with the generators $1, z$, where z is the generator of $K^T(pt) = \mathbb{C}[T]$ (so that, for example, $\mathbf{v}(1)_0 = z + z^{-1}$). Furthermore, $\mathbb{C}[y^{\pm 1}, z^{\pm 1}] = \mathbb{C}[T \times T] = K(T(\mathbf{O}) \backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \subset K^T(\text{Gr}_{\check{G}})$, and one can check that

$$y + y^{-1} = \sqrt{-1}(2\mathbf{v}(0)_1 - \mathbf{v}(1)_0 \star \mathbf{v}(1)_1), \quad y - y^{-1} = \sqrt{-1}(z - z^{-1})\mathbf{v}(1)_1. \tag{8}$$

Comparing (8) with (5) we get the desired identification of the $K(T(\mathbf{O}) \backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \times W - K^G(\text{Gr}_{\check{G}})$ -bimodule $K^T(\text{Gr}_{\check{G}})$ with the $\mathbb{C}[y^{\pm 1}, z^{\pm 1}] \times W - \mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y - y^{-1})/(z - z^{-1})]^W$ -bimodule $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y - y^{-1})/(z - z^{-1})]$.

3.10 Borel–Moore homology. For an arbitrary semisimple G one proves the commutativity of $H^{G(\mathbf{O})}(\text{Gr}_G)$ (Theorem 2.12, part (a)) exactly as in § 3.7 using the Beilinson–Drinfeld Grassmannian and the *specialization* in Borel–Moore homology (see [CG97, 2.6.30]).

For $\check{G} = PGL_2$, let us denote by $\delta \in H^4_{\check{G}(\mathbf{O})}(pt, \mathbb{Z}) = H^4_{\check{G}(\mathbf{O})}(pt, \mathbb{Z})$ the generator of the equivariant (co)homology. Furthermore, we denote by η (respectively ξ) the generator of $H^{-2}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},1}, \mathbb{Z})$ (respectively the generator of $H^0_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},1}, \mathbb{Z})$). Then it is easy to see that δ, ξ, η generate $H^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ (while $\delta, \xi^2, \eta^2, \xi\eta$ generate the subalgebra $H^{G(\mathbf{O})}(\text{Gr}_G)$), and we claim that

$$1 = \xi^2 - \delta\eta^2. \tag{9}$$

In effect, this is an equality in $H^0_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},2})$. Since $\text{Gr}_{\check{G},2}$ is rationally smooth, $H^0_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},2}) = H^4_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},2})$. Let us denote by $\mathbf{BGr}_{\check{G},2} \xrightarrow{p} \mathbf{B}\check{G}(\mathbf{O})$ the associated fiber bundle over the classifying space of $\check{G}(\mathbf{O})$ with the fiber $\text{Gr}_{\check{G},2}$. Then $1 \in H^4_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},2}) = H^4(\mathbf{BGr}_{\check{G},2})$ is the Poincaré dual class of the codimension-2 cycle $\mathbf{B}\check{G}(\mathbf{O}) = \mathbf{BGr}_{\check{G},0} \hookrightarrow \mathbf{BGr}_{\check{G},2}$, and $\delta\eta^2 = p^*\delta$.

Recall the convolution morphism $\Pi_0 : \mathcal{H}_2 \rightarrow \text{Gr}_{\check{G},2}$ of § 3.8. This is a morphism of $\check{G}(\mathbf{O})$ -varieties, and we denote by $\Pi_0 : \mathbf{B}\mathcal{H}_2 \rightarrow \mathbf{BGr}_{\check{G},2}$ the corresponding morphism of associated fiber bundles. Note that (additively) $H^*(\mathbf{B}\mathcal{H}_2) = H^*(\mathbf{BGr}_{\check{G},1}) \otimes_{H^*(\mathbf{B}\check{G}(\mathbf{O}))} H^*(\mathbf{BGr}_{\check{G},1})$. Recall that ξ is the generator of $H^0_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},1}) = H^2_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G},1}) = H^2(\mathbf{BGr}_{\check{G},1})$. Finally, we have $\xi^2 = \Pi_{0*}(\xi \otimes \xi)$. Now (9) follows easily.

Comparing the sizes of $H^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ and $\mathbb{C}[\delta, \xi, \eta]/(\xi^2 - \delta\eta^2 - 1)$ we conclude that they are isomorphic. The comparison with (6) establishes an isomorphism $H^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}}) \simeq \mathbb{C}[\mathcal{S}']$, and identifies the spectrum of $H^{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ with $\mathcal{S}' \simeq \mathfrak{Z}_g^G$, and the spectrum of $H^{G(\mathbf{O})}(\text{Gr}_G)$ with $\iota \backslash \mathcal{S}' \simeq \mathfrak{Z}_g^G$.

4. Centralizers and blow-ups

The aim of this section is a proof of Proposition 2.8. We will consider \mathfrak{B}_G^G and \mathfrak{Z}_G^G , the other cases being similar. Till further notice, G is assumed simply connected.

LEMMA 4.1. *The projection $\varpi : \mathfrak{B}_G^G \rightarrow T/W$ is flat.*

Proof. It suffices to prove that the first projection of \mathfrak{B}_G^G to T is smooth (recall that \mathfrak{B}_G^G is defined as $\text{Spec}(\mathbb{C}[T \times T, (1^\alpha - 1)/(2^\alpha - 1), \alpha \in R])$). In effect, then $\mathbb{C}[T \times T, (1^\alpha - 1)/(2^\alpha - 1), \alpha \in R]$ is a flat $\mathbb{C}[T]$ -module; hence it is a flat $\mathbb{C}[T]^W$ -module (since $\mathbb{C}[T]$ is free over $\mathbb{C}[T]^W$; see [Ste75]). Finally, $\mathbb{C}[T \times T, (1^\alpha - 1)/(2^\alpha - 1), \alpha \in R]^W$ is a direct summand of a flat $\mathbb{C}[T]^W$ -module $\mathbb{C}[T \times T, (1^\alpha - 1)/(2^\alpha - 1), \alpha \in R]$; hence it is flat.

The affine blow-up \mathfrak{B}_G^G is the result of the following successive blow-up of $T \times T$. We choose an ordering $\alpha_1, \dots, \alpha_\nu$ of the set of positive roots R^+ . We define \mathfrak{B}_1 as the blow-up of $T \times T$ at the diagonal wall ${}^1\alpha_1 = {}^2\alpha_1 = 1$ with the proper preimage of the divisor ${}^1\alpha_1 = 1$ removed. We define \mathfrak{B}_2 as the blow-up of \mathfrak{B}_1 at the proper transform of the diagonal wall ${}^1\alpha_2 = {}^2\alpha_2 = 1$ with the proper preimage of the divisor ${}^1\alpha_2 = 1$ removed. Going on like this we construct \mathfrak{B}_ν ; evidently, it coincides with \mathfrak{B}_G^G .

Note that at each step the center of the blow-up is smooth over the corresponding wall ${}^2\alpha_i = 1$ in T by the following claim. Thus the desired flatness assertion follows inductively from the

CLAIM. Let $p : X \rightarrow Y$ be a smooth morphism of smooth varieties; let $X' \subset X$ be a subvariety such that $Y' = f(X') \subset Y$ is a smooth hypersurface, and $p : X' \rightarrow Y'$ is also smooth. Then the blow-up $\text{Bl}_{X'} X$ with the proper preimage of the divisor $p^{-1}(p(X'))$ removed is smooth over Y .

The smoothness is checked in the formal neighborhoods of points by direct calculation in coordinates. This completes the proof of the lemma. \square

4.2 The simultaneous resolution. Recall that $\{(g, B) : g \in B\} = \dot{G} \xrightarrow{p} G$ is the Grothendieck simultaneous resolution; here B is a Borel subgroup, and $p(g, B) = g$. We also have the projection $\varrho : \dot{G} \rightarrow T$ to the abstract Cartan, which we identify with T ; namely, $\varrho(g, B) = g \pmod{\text{rad}(B)}$. The preimage $p^{-1}(\Sigma_G) \subset \dot{G}$ is identified with T by ϱ . We denote by $\dot{\mathfrak{Z}}_G^G \subset G \times \dot{G}$ the subset of triples (g_1, g_2, B) such that $Ad_{g_1}(g_2) = g_2$ and $(g_2, B) \in p^{-1}(\Sigma_G)$. Note that necessarily $g_1 \in B$ (as well as $g_2 \in B$); hence we have the projections $\varrho_1, \varrho_2 : \dot{\mathfrak{Z}}_G^G \rightarrow T$; namely, $\varrho_i(g_1, g_2, B) = g_i \pmod{\text{rad}(B)}$. Equivalently, $\dot{\mathfrak{Z}}_G^G$ can be defined as the categorical quotient by the diagonal G -action of the variety of triples $\dot{C}_{G,G} = \{(g_1, g_2, B) : Ad_{g_1}(g_2) = g_2, B \ni g_1, g_2, \text{ and } g_2 \text{ is regular}\}$.

The natural projection $\dot{\mathfrak{Z}}_G^G \rightarrow \mathfrak{Z}_G^G$ (forgetting B) is a Galois W -covering. Finally, $\varrho_2 : \dot{\mathfrak{Z}}_G^G \rightarrow T$ is flat.

4.3 The proof of Proposition 2.8. In order to identify \mathfrak{Z}_G^G and \mathfrak{B}_G^G it suffices to identify their Galois W -coverings $\dot{\mathfrak{Z}}_G^G \rightarrow T$ and $\dot{\mathfrak{B}}_G^G \rightarrow T$ in an equivariant way. Let $\mathbf{D} \subset T$ denote the discriminant, so that $T - \mathbf{D} = T^{\text{reg}}$. Let $\Delta \in \mathbb{C}[T]^W$ denote the product $\prod_{\alpha \in R} (\alpha - 1)$, so that \mathbf{D} is the divisor cut out by Δ .

Evidently, both $\dot{\mathfrak{Z}}_G^G|_{T^{\text{reg}}}$ and $\dot{\mathfrak{B}}_G^G|_{T^{\text{reg}}}$ are isomorphic to $T \times T^{\text{reg}}$. Hence both $\mathbb{C}[\dot{\mathfrak{Z}}_G^G]$ and $\mathbb{C}[\dot{\mathfrak{B}}_G^G]$ are flat $\mathbb{C}[T]$ -modules embedded into $\mathbb{C}[T \times T](\Delta^{-1})$. We must prove that the identification of $\dot{\mathfrak{Z}}_G^G|_{T^{\text{reg}}}$ and $\dot{\mathfrak{B}}_G^G|_{T^{\text{reg}}}$ extends to the identification over the whole T . To this end it suffices to check that the identification extends over the codimension-1 points of T (indeed, for a flat quasi-coherent sheaf \mathcal{F} on a normal irreducible scheme we have $\mathcal{F} \xrightarrow{\sim} j_* j^* \mathcal{F}$ if j is an open embedding with complement of codimension 2). Let $g \in T$ be a regular point of \mathbf{D} ; that is, g is a semisimple element of G such that the centralizer $Z(g)$ has semisimple rank 1.

We must construct an isomorphism between localizations $(\dot{\mathfrak{Z}}_G^G)_g$ and $(\dot{\mathfrak{B}}_G^G)_g$ which is compatible with the above isomorphism at the generic point. To this end note that $Z(g)$ is connected since G is simply connected. So $Z(g)$ is a Levi subgroup of G sharing with it the Cartan torus T . We have an evident projection $\dot{\mathfrak{B}}_{Z(g)}^{Z(g)} \leftarrow \dot{\mathfrak{B}}_G^G$ which becomes an isomorphism after localization: $(\dot{\mathfrak{B}}_{Z(g)}^{Z(g)})_g \simeq (\dot{\mathfrak{B}}_G^G)_g$.

Now let us construct a projection $\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)} \leftarrow \dot{\mathfrak{Z}}_G^G$ which becomes an isomorphism after localization: $(\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)})_g \simeq (\dot{\mathfrak{Z}}_G^G)_g$. We choose a subminimal parabolic subgroup $Z(g) \subset P \subset G$. We consider the variety of triples $\dot{C}_{P,P} \subset \dot{C}_{G,G}$ cut out by the requirement $g_1, g_2 \in P, B \subset P$. The inclusion $\dot{C}_{P,P} \subset \dot{C}_{G,G}$ induces an isomorphism $\dot{\mathfrak{Z}}_P^P := \dot{C}_{P,P} // P \xrightarrow{\sim} \dot{C}_{G,G} // G = \dot{\mathfrak{Z}}_G^G$. The projection $P \rightarrow Z(g)$ (quotient by the unipotent radical) induces the morphisms $\dot{C}_{P,P} \rightarrow \dot{C}_{Z(g),Z(g)}$ and $\dot{\mathfrak{Z}}_P^P \rightarrow \dot{\mathfrak{Z}}_{Z(g)}^{Z(g)}$. Thus we have obtained the desired morphism $\dot{\mathfrak{Z}}_G^G \simeq \dot{\mathfrak{Z}}_P^P \rightarrow \dot{\mathfrak{Z}}_{Z(g)}^{Z(g)}$ which becomes an isomorphism after localization: $(\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)})_g \simeq (\dot{\mathfrak{Z}}_G^G)_g$.

Finally, the desired identification $(\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)})_g \simeq (\dot{\mathfrak{B}}_{Z(g)}^{Z(g)})_g$ follows from the calculations in § 3.1.

This completes the identification $\mathfrak{Z}_G^G \simeq \mathfrak{B}_G^G$ for a simply connected G . Evidently, this identification respects the left and right actions of the center $Z(G)$, so the isomorphism for an arbitrary G follows from that for its universal cover. The other isomorphisms in Proposition 2.8, part (b) are proved in a similar way.

To prove Proposition 2.8, parts (c) and (d) it suffices to note that the minimal level (viewed as a W -equivariant homomorphism $T \rightarrow \check{T}$) for a simply laced simply connected G identifies \check{T} with $T/Z(G)$; also, $\check{G} = G/Z(G)$.

5. *W*-invariant sections and blow-ups

The aim of this section is a proof of Proposition 2.10. We concentrate on the last statement, the other being completely similar.

Let $T^{\text{reg}} \subset T$, $T_\alpha^{\text{reg}} \subset T$ be the open subschemes defined by $T^{\text{reg}} = \{t \mid \alpha(t) \neq 1 \text{ for all roots } \alpha\}$; $T_\alpha^{\text{reg}} = \{t \mid \beta(t) \neq 1 \text{ for all roots } \beta \neq \alpha\}$; and $\mathring{T} = \bigcup_\alpha T_\alpha^{\text{reg}}$ (thus $T - \mathring{T}$ has codimension 2 in T (where the empty subscheme in a curve is considered to be of codimension 2)). Notice that since G is simply connected the action of W on T^{reg} is free.

We start with a lemma.

LEMMA 5.1. *The map $\mathfrak{B}_G^{\check{G}} \times_T \mathring{T} \rightarrow \mathfrak{B}_G^{\check{G}}/W \times_{T/W} \mathring{T}$ is an isomorphism.*

Proof. Let $X \rightarrow Y$ be a flat morphism of semiseparated (which means that the diagonal embedding is affine) schemes of finite type over a characteristic zero field, and let a finite group W act on X, Y so that the map is W -equivariant. Assume that Y is flat over Y/W . We then claim that the map $X \rightarrow X/W \times_{Y/W} Y$ is an isomorphism provided that for every Zariski point $y \in Y$ the action of $\text{Stab}_W(y)$ on the scheme-theoretic fiber X_y is trivial (here $X/W, Y/W$ stand for categorical quotients). To check this claim, we can assume X is affine: by semiseparatedness every W -invariant subset in X has a W -invariant affine neighborhood. Let us first assume also that Y/W is a point; then (by replacing Y by its connected component, and W by the stabilizer of that component) we can assume that Y is nilpotent. Then \mathcal{O}_X is free over \mathcal{O}_Y , and the generators of \mathcal{O}_X as an \mathcal{O}_Y -module can be chosen to be W -invariant (by semisimplicity of the W -action on \mathcal{O}_X , and triviality of the W -action on $\mathcal{O}_X \otimes_{\mathcal{O}_Y} k$); since $\mathcal{O}_Y^W = k$ (where k is the base field) we see that $\mathcal{O}_X^W \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X$ as claimed. Now for a general Y we see that the morphism in question is a morphism of flat schemes of finite type over Y/W , which induces an isomorphism on every fiber; and such a morphism is necessarily an isomorphism.

Now it remains to check that the above conditions hold for $X = \mathfrak{B}_G^{\check{G}} \times_T \mathring{T}$, $Y = T$. For $y \in T^{\text{reg}}$, the stabilizer of y is trivial, so there is nothing to check. Consider now $y \in T_\alpha^{\text{reg}}$, $y \notin T^{\text{reg}}$. Then the stabilizer of y is $\{1, s_\alpha\}$. The ring of functions on $\mathfrak{B}_G^{\check{G}}$ is generated by ${}^1\check{\lambda}, {}^2\mu, t_\alpha$ where $\check{\lambda}, \mu$ run over weights of \mathring{T}, T respectively, $\alpha \in R^+$, and $t_\alpha(2\alpha - 1) = {}^1\check{\alpha} - 1$. We have $s_\alpha^*({}^1\check{\lambda}) = {}^1\check{\lambda} \cdot ({}^1\check{\alpha})^{\langle -\alpha, \check{\lambda} \rangle}$, $s_\alpha^*({}^2\mu) = {}^2\mu \cdot ({}^2\alpha)^{\langle -\mu, \check{\alpha} \rangle}$, and $s_\alpha^*(t_\alpha) = t_\alpha \cdot ({}^2\alpha/{}^1\check{\alpha})$. On the fiber we have ${}^2\alpha = 1$, hence ${}^1\check{\alpha} = 1$, so the action of s_α on the fiber is trivial. □

Proposition 2.10 clearly follows from the (ii) \iff (iv) part of the next proposition.

PROPOSITION 5.2. *Let $S \rightarrow T/W$ be a flat morphism, and set $\phi : S \times_{T/W} T^{\text{reg}}/W \rightarrow (\mathring{T} \times T)/W$ to be a T^{reg}/W -morphism. Then the following are equivalent:*

- (i) ϕ extends to a morphism $S \times_{T/W} \mathring{T}/W \rightarrow \mathfrak{B}_G^{\check{G}} \times_{T/W} \mathring{T}$;
- (ii) ϕ extends to a morphism $S \rightarrow \mathfrak{B}_G^{\check{G}}$;
- (iii) for every $\alpha \in R$ the morphism $\phi \times \text{id}_{T^{\text{reg}}} : S \times_{T/W} T^{\text{reg}} \rightarrow \mathring{T} \times T^{\text{reg}}$ extends to a morphism $S \times_{T/W} T_\alpha^{\text{reg}} \rightarrow \mathring{T} \times T_\alpha^{\text{reg}}$ such that (3) holds;
- (iv) $\phi \times \text{id}_{T^{\text{reg}}} : S \times_{T/W} T^{\text{reg}} \rightarrow \mathring{T} \times T^{\text{reg}}$ extends to a morphism $S \times_{T/W} T \rightarrow \mathring{T} \times T$, such that (3) holds for every $\alpha \in R$.

Proof. It is enough to assume that S is affine. Indeed, a morphism from S extends if and only if its restriction to every affine open in S does, because compatibility on intersections follows from uniqueness of such an extension. This uniqueness follows from flatness: if S is flat affine, then tensoring the injection $\mathcal{O} \rightarrow j_*\mathcal{O}$ with \mathcal{O}_S we get an embedding $\mathcal{O}_S \hookrightarrow j_*j^*\mathcal{O}_S$, where j stands for the embedding $T^{\text{reg}}/W \rightarrow T/W$, or $T^{\text{reg}} \rightarrow T$. So we will assume S affine from now on.

That (iv) \Rightarrow (iii) and (ii) \Rightarrow (i) is obvious.

To check that (iii) \Rightarrow (iv) we tensor (over $\mathcal{O}_{T/W}$) the exact sequence of \mathcal{O}_T -modules

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{T^{\text{reg}}} \rightarrow \bigoplus_{\alpha} (\mathcal{O}_{T^{\text{reg}}}/\mathcal{O}_T) \tag{10}$$

with \mathcal{O}_S . The resulting exact sequence shows that a regular function on $S \times_{T/W} T^{\text{reg}}$ extends to a regular function on $S \times_{T/W} T$ if and only if it extends to $S \times_T T^{\text{reg}}_{\alpha}$ for all α . Applying this observation to $(\phi \times \text{id})^*(f|_{\check{T} \times T^{\text{reg}}})$ for each regular function f on $\check{T} \times T$, we see that (iii) implies extendability of $\phi \times \text{id}$ to $S \times_{T/W} T$. It is also clear that (3) holds if it holds on \check{T} .

Verification of (i) \Rightarrow (ii) is similar (with (10) replaced by the W -invariant part of (10)).

It remains to check (i) \iff (iii). If (i) holds, i.e. ϕ extends to a map $S \times_{T/W} \check{T}/W \rightarrow \mathfrak{B}^{\check{G}}_{\check{G}} \times_{T/W} \check{T}$, then we can take the fiber product of this map with $\text{id}_{\check{T}}$ over T/W . By Lemma 5.1 it yields a map $S \times_{T/W} \check{T} \rightarrow \mathfrak{B}^{\check{G}}_{\check{G}} \times_T \check{T}$, which can be composed with the projection $\mathfrak{B}^{\check{G}}_{\check{G}} \rightarrow \check{T} \times T$ to produce a map $S \times_{T/W} \check{T} \rightarrow \check{T} \times \check{T}$. It is clear that this map satisfies (3), because the image of the map $\mathfrak{B}^{\check{G}}_{\check{G}} \rightarrow \check{T} \times T$ intersected with $\check{T} \times \text{Ker}(^2\alpha)$ is contained in $\text{Ker}(^1\check{\alpha}) \times T$.

Conversely, if (iii) holds, then restricting the given map $S \times_{T/W} \check{T} \rightarrow \check{T} \times \check{T}$ to $S \times_{T/W} (\text{Ker}(\alpha) \cap \check{T})$ we get a map into $\text{Ker}(\check{\alpha}) \times T$ (this is immediate from (3)). This means that the map lifts to a map into $\mathfrak{B}^{\check{G}}_{\check{G}}$. Replacing both the source and the target by their quotients by W we get the map required in (i). \square

6. K -theory and blow-ups

The aim of this section is a proof of Theorem 2.15. Recall that Theorem 2.15, part (a) was already proved in § 3.7. Here G is assumed simply connected till further notice.

6.1 Reminder on affine Grassmannians. Let $X = X_G$ be the lattice of characters of T , and let $Y = Y_G$ be the lattice of cocharacters of G . Note that $X_G = Y_{\check{G}}$, $Y_G = X_{\check{G}}$. Let $X^+ \subset X$ (respectively $Y^+ \subset Y$) be the cone of dominant weights (respectively dominant coweights). It is well known that the $G(\mathbf{O})$ -orbits in Gr_G are numbered by the dominant coweights: $\text{Gr}_G = \bigsqcup_{\check{\lambda} \in Y^+} \text{Gr}_{G, \check{\lambda}}$. The adjacency relation of orbits corresponds to the standard partial order on coweights: $\overline{\text{Gr}}_{G, \check{\lambda}} = \bigsqcup_{\check{\mu} \leq \check{\lambda}} \text{Gr}_{G, \check{\mu}}$. The open embedding $\text{Gr}_{G, \check{\lambda}} \hookrightarrow \overline{\text{Gr}}_{G, \check{\lambda}}$ will be denoted by $j_{\check{\lambda}}$ or simply by j if no confusion is likely. The dimension $\dim(\text{Gr}_{G, \check{\lambda}}) = \langle 2\rho, \check{\lambda} \rangle$ where $2\rho = \sum_{\alpha \in R^+} \alpha$, and $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{Z}$ is the canonical perfect pairing.

Recall that the T -fixed points in Gr_G are naturally numbered by Y ; a point $\check{\mu}$ lies in an orbit $\text{Gr}_{G, \check{\lambda}}$ if and only if $\check{\mu}$ lies in the W -orbit of $\check{\lambda}$. Each $G(\mathbf{O})$ -orbit $\text{Gr}_{G, \check{\lambda}}$ is partitioned into Iwahori orbits isomorphic to affine spaces and numbered by $\check{\mu} \in W\check{\lambda}$. Hence the basics of [CG97, Chapter 5] are applicable in our situation.

In particular, $K^T(\text{Gr}_{G, \check{\lambda}})$ is a free $K^T(pt)$ -module, and $K^{G(\mathbf{O})}(\text{Gr}_{G, \check{\lambda}}) = K^G(\text{Gr}_{G, \check{\lambda}})$ is a free $K^G(pt)$ -module (recall that $K^T(pt) = \mathbb{C}[T]$, and $K^G(pt) = \mathbb{C}[T/W]$). Moreover, the natural map $K^T(pt) \otimes_{K^G(pt)} K^G(\text{Gr}_{G, \check{\lambda}}) \rightarrow K^T(\text{Gr}_{G, \check{\lambda}})$ is an isomorphism, and $K^G(\text{Gr}_{G, \check{\lambda}}) = K^T(\text{Gr}_{G, \check{\lambda}})^W$ (cf. [CG97, 6.1.22]).

Since $K^{T(\mathbf{O})}(\text{Gr}_G) = K^T(\text{Gr}_G)$ (respectively $K^{G(\mathbf{O})}(\text{Gr}_G) = K^G(\text{Gr}_G)$) is filtered by the support in $G(\mathbf{O})$ -orbit closures, with the associated graded $\bigoplus_{\check{\lambda} \in Y^+} K^T(\text{Gr}_{G, \check{\lambda}})$ (respectively $\bigoplus_{\check{\lambda} \in Y^+} K^G(\text{Gr}_{G, \check{\lambda}})$), we arrive at the following lemma.

LEMMA 6.2. *We have that $K^{T(\mathbf{O})}(\text{Gr}_G) = K^T(\text{Gr}_G)$ is a flat $K^T(pt)$ -module, and $K^{G(\mathbf{O})}(\text{Gr}_G) = K^G(\text{Gr}_G)$ is a flat $K^G(pt)$ -module. Moreover, the natural map $K^T(pt) \otimes_{K^G(pt)} K^G(\text{Gr}_G) \rightarrow K^T(\text{Gr}_G)$ is an isomorphism, and $K^G(\text{Gr}_G) = (K^T(\text{Gr}_G))^W$.*

6.3 Localization. The space $K^T(\mathrm{Gr}_G) = K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K(T(\mathbf{O}) \backslash G(\mathbf{F}) / G(\mathbf{O}))$ is equipped with the two commuting actions: $K(T(\mathbf{O}) \backslash T(\mathbf{F}) / T(\mathbf{O}))$ acts by convolutions on the left, and $K^G(\mathrm{Gr}_G) = K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K(G(\mathbf{O}) \backslash G(\mathbf{F}) / G(\mathbf{O}))$ acts by convolutions on the right. Also, W acts on $K^T(\mathrm{Gr}_G)$ commuting with the right action of $K^G(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O}) \backslash T(\mathbf{F}) / T(\mathbf{O}))$ is isomorphic to $\mathbb{C}[\check{T} \times T]$. The action of W on $K^T(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O}) \backslash T(\mathbf{F}) / T(\mathbf{O}))$ and induces the natural (diagonal) action of W on $\mathbb{C}[\check{T} \times T]$.

Let g be a general (regular) element of T . Then the fixed point set $(\mathrm{Gr}_G)^g = (\mathrm{Gr}_G)^T = Y$ coincides with the image of the embedding $\mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$. According to the Thomason localization theorem (see e.g. [CG97, 5.10]), after localization, $(K^T(\mathrm{Gr}_G))_g$ becomes a free rank-one $(K(T(\mathbf{O}) \backslash T(\mathbf{F}) / T(\mathbf{O})))_g$ -module. This means that after restriction to $T^{\mathrm{reg}} \subset T = \mathrm{Spec}(K^T(pt))$ we have an isomorphism $K^T(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}} \simeq \mathbb{C}[\check{T} \times T]|_{T^{\mathrm{reg}}}$ compatible with the natural W -actions. The localized algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W}$ is embedded into $(\mathrm{End}_{K(T(\mathbf{O}) \backslash T(\mathbf{F}) / T(\mathbf{O}))|_{T^{\mathrm{reg}}}}(K^T(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}}))^W$. According to Lemma 6.2, $K^G(\mathrm{Gr}_G) = (K^T(\mathrm{Gr}_G))^W$; hence this embedding is an isomorphism, and we have $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W} \simeq \mathbb{C}[\check{T} \times T]^W|_{T^{\mathrm{reg}}/W}$.

Hence both $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$ and $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ are flat $\mathbb{C}[T]^W$ -modules embedded into $\mathbb{C}[\check{T} \times T](\Delta^{-1})$ (see § 4.3). We must prove that the identification of $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{\mathrm{reg}}/W}$ and $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W}$ extends to the identification over the whole T/W . To this end it suffices to check that the identification extends over the codimension-1 points of T/W . Let $g \in T/W$ be a regular point of \mathbf{D} ; that is, g is represented by a semisimple element of G such that the centralizer $Z(g)$ has semisimple rank 1.

We must prove that the localizations $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$ and $(K^{G(\mathbf{O})}(\mathrm{Gr}_G))_g$ are isomorphic. To this end it suffices to identify $\mathbb{C}[\check{T} \times T, ({}^1\check{\alpha} - 1)/({}^2\alpha - 1), \alpha \in R]_g$ (which we denote by $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$ for short) and $(K^T(\mathrm{Gr}_G))_g$. Note that the embedding of reductive groups $Z(g) \hookrightarrow G$ induces the isomorphism $\mathrm{Gr}_{Z(g)} = (\mathrm{Gr}_G)^g \hookrightarrow \mathrm{Gr}_G$. According to the Thomason localization theorem, we have an isomorphism of localizations $(K^T(\mathrm{Gr}_{Z(g)}))_g \simeq (K^T(\mathrm{Gr}_G))_g$. Finally, the isomorphism $K^T(\mathrm{Gr}_{Z(g)}) \simeq \mathbb{C}[\mathfrak{B}_{Z(g)}^{\check{Z}(g)}]$ follows from the calculations in §§ 3.8 and 3.9, and together with the evident isomorphism of localizations $\mathbb{C}[\mathfrak{B}_{Z(g)}^{\check{Z}(g)}]_g \simeq \mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$ establishes the desired isomorphism $(K^T(\mathrm{Gr}_G))_g \simeq \mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$.

This completes the proof of Theorem 2.15, part (b).

6.4 Comparison of Poisson structures. In order to compare the Poisson structures on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ and $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$ it suffices to identify them on the open subset $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W} = \mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{\mathrm{reg}}/W} = \mathbb{C}[\check{T} \times T^{\mathrm{reg}}]^W$. The space

$$K^{T \times \mathbb{G}_m}(\mathrm{Gr}_G) = K^{T(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G) = K(T(\mathbf{O}) \times \mathbb{G}_m \backslash G(\mathbf{F}) \times \mathbb{G}_m / G(\mathbf{O}) \times \mathbb{G}_m)$$

is equipped with the two commuting actions: $K(T(\mathbf{O}) \times \mathbb{G}_m \backslash T(\mathbf{F}) \times \mathbb{G}_m / T(\mathbf{O}) \times \mathbb{G}_m)$ acts by convolutions on the left, and

$$K^{G \times \mathbb{G}_m}(\mathrm{Gr}_G) = K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G) = K(G(\mathbf{O}) \times \mathbb{G}_m \backslash G(\mathbf{F}) \times \mathbb{G}_m / G(\mathbf{O}) \times \mathbb{G}_m)$$

acts by convolutions on the right. Also, W acts on $K^{T(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ commuting with the right action of $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O}) \times \mathbb{G}_m \backslash T(\mathbf{F}) \times \mathbb{G}_m / T(\mathbf{O}) \times \mathbb{G}_m)$ is isomorphic to the group algebra $\mathbb{C}[\Gamma]$ of the following Heisenberg group Γ .

It is a \mathbb{Z} -central extension of $Y \times X$ with the multiplication (written multiplicatively)

$$(q^{n_1}, e^{\check{\lambda}_1}, e^{\mu_1}) \cdot (q^{n_2}, e^{\check{\lambda}_2}, e^{\mu_2}) = (q^{n_1+n_2+\langle \mu_1, \check{\lambda}_2 \rangle}, e^{\check{\lambda}_1+\check{\lambda}_2}, e^{\mu_1+\mu_2})$$

where $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{Z}$ is the canonical perfect pairing.

Finally, the action of the Weyl group W on $K^{T(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O}) \times \mathbb{G}_m \backslash T(\mathbf{F}) \times \mathbb{G}_m / T(\mathbf{O}) \times \mathbb{G}_m)$ and induces the natural (diagonal) action of W on $\mathbb{C}[\Gamma]$. From this we

deduce, exactly as in § 6.3, that $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W} \simeq \mathbb{C}[\Gamma]|_{T^{\mathrm{reg}}/W}$. It follows that the Poisson structure on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W}$ coincides with the standard Poisson structure on $\mathbb{C}[\check{T} \times T^{\mathrm{reg}}]^W$.

This completes the proof of Theorem 2.15, part (c).

6.5 The case of nonsimply connected G . For general G , let \tilde{G} denote its universal cover, and let \check{T} stand for the Cartan of \tilde{G} . Note that the dual torus is $\check{T}/\pi_1(G)$. As in § 6.3, we have $K^G(\mathrm{Gr}_G) = (\mathrm{End}_{K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))}(K^T(\mathrm{Gr}_G)))^W$, so it suffices to identify the $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O})) \times W = \mathbb{C}[\check{T} \times T] \times W$ -module $K^T(\mathrm{Gr}_G)$ with $\mathbb{C}[\check{T} \times T, ({}^1\check{\alpha} - 1)/({}^2\alpha - 1), \alpha \in R] = \mathrm{Spec} \mathbb{C}[\check{\mathfrak{B}}_{\tilde{G}}^{\check{G}}]$. We do this by reduction to the known case of \tilde{G} .

Evidently, the $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O})) \times W = \mathbb{C}[\check{T} \times T] \times W$ -module $K^T(\mathrm{Gr}_G)$ equals $\mathbb{C}[\check{T} \times T] \times W \otimes_{\mathbb{C}[(\check{T}/\pi_1(G)) \times T] \times W} K^T(\mathrm{Gr}_{\tilde{G}})$. On the other hand, it follows from § 6.3 that the $K(T(\mathbf{O}) \backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \times W = \mathbb{C}[(\check{T}/\pi_1(G)) \times T] \times W$ -module $K^T(\mathrm{Gr}_{\tilde{G}})$ equals the invariants of $\pi_1(G)$ in $K^{\check{T}}(\mathrm{Gr}_{\tilde{G}})$, that is $\mathbb{C}[(\check{T}/\pi_1(G)) \times \check{T}, ({}^1\check{\alpha} - 1)/({}^2\alpha - 1), \alpha \in R]^{\pi_1(G)} = \mathbb{C}[(\check{T}/\pi_1(G)) \times T, ({}^1\check{\alpha} - 1)/({}^2\alpha - 1), \alpha \in R]$.

This completes the proof of Theorem 2.15 for general G .

6.6 Borel–Moore homology and blow-ups. Theorem 2.12 is proved in absolute parallel fashion to the proof of Theorem 2.15.

7. Computation of $K_{G(\mathbf{O})}(\Lambda)$

7.1 The affine Grassmannian Steinberg variety. We denote by $\mathfrak{u} \subset \mathfrak{g}(\mathbf{O})$ (respectively $U \subset G(\mathbf{O})$) the nilpotent (respectively unipotent) radical. It has a filtration $\mathfrak{u} = \mathfrak{u}^{(0)} \supset \mathfrak{u}^{(1)} \supset \dots$ by congruence subalgebras. The trivial (Tate) vector bundle $\mathfrak{g}(\mathbf{F})$ with the fiber $\mathfrak{g}(\mathbf{F})$ over Gr_G has a structure of an ind-scheme. It contains a profinite dimensional vector subbundle $\underline{\mathfrak{u}}$ whose fiber over a point $g \in \mathrm{Gr}_G$ represented by a compact subalgebra in $\mathfrak{g}(\mathbf{F})$ is the pronilpotent radical of this subalgebra. The trivial vector bundle $\underline{\mathfrak{g}}(\mathbf{F}) = \mathfrak{g}(\mathbf{F}) \times \mathrm{Gr}_G$ also contains a trivial vector subbundle $\mathfrak{u} \times \mathrm{Gr}_G$.

We will call $\underline{\mathfrak{u}}$ the *cotangent bundle* of Gr_G , and we will call the intersection $\Lambda := \underline{\mathfrak{u}} \cap (\mathfrak{u} \times \mathrm{Gr}_G)$ the *affine Grassmannian Steinberg variety*. It has a structure of an ind-scheme of ind-infinite type. Namely, if p stands for the natural projection $\Lambda \rightarrow \mathrm{Gr}_G$, then $\Lambda_{\leq \check{\lambda}} := p^{-1}(\overline{\mathrm{Gr}}_{G, \check{\lambda}})$ is a scheme of infinite type, and $\Lambda = \bigcup \Lambda_{\leq \check{\lambda}}$.

Note that for a fixed $\check{\lambda}$ and $l \gg 0$ the intersection of fibers of $\underline{\mathfrak{u}}$ over all points of $\overline{\mathrm{Gr}}_{G, \check{\lambda}}$ (as vector subspaces of $\mathfrak{g}(\mathbf{F})$) contains $\mathfrak{u}^{(l)}$. Thus $\mathfrak{u}^{(l)}$ acts freely (by fiberwise translations) on $\Lambda_{\leq \check{\lambda}}$, and the quotient is a scheme of finite type, to be denoted by $\Lambda_{\leq \check{\lambda}}^l$. For $k > l$ we have evident affine fibrations $p_l^k : \Lambda_{\leq \check{\lambda}}^k \rightarrow \Lambda_{\leq \check{\lambda}}^l$, and $\Lambda_{\leq \check{\lambda}}$ coincides with the inverse limit of this system.

Similarly, the total space of the vector bundle $\underline{\mathfrak{u}}$ (to be denoted by the same symbol) is a union of infinite type schemes $\underline{\mathfrak{u}}_{\leq \check{\lambda}}$, and for fixed $\check{\lambda}$ and $l \gg 0$, the scheme $\underline{\mathfrak{u}}_{\leq \check{\lambda}}^l$ is the inverse limit of affine fibrations $p_l^k : \underline{\mathfrak{u}}_{\leq \check{\lambda}}^k \rightarrow \underline{\mathfrak{u}}_{\leq \check{\lambda}}^l$ ($k > l$). Note that the proalgebraic group $G(\mathbf{O})$ acts on all the above schemes, and the fibrations p_l^k are $G(\mathbf{O})$ -equivariant.

A $G(\mathbf{O})$ -equivariant coherent sheaf \mathcal{F} on $\underline{\mathfrak{u}}$ is by definition supported on some $\underline{\mathfrak{u}}_{\leq \check{\lambda}}$. There, it is defined as a collection of $G(\mathbf{O})$ -equivariant sheaves \mathcal{F}^l on $\underline{\mathfrak{u}}_{\leq \check{\lambda}}^l$ for $l \gg 0$ together with isomorphisms $(p_l^k)^* \mathcal{F}^l \simeq \mathcal{F}^k$. We will consider the $G(\mathbf{O})$ -equivariant coherent sheaves on $\underline{\mathfrak{u}}$ supported on Λ , and $D^b \mathrm{Coh}_{\Lambda}^{G(\mathbf{O})}(\underline{\mathfrak{u}})$ stands for the derived category of such sheaves, and $K^{G(\mathbf{O})}(\Lambda)$ stands for the K -group of such sheaves.

7.2 Convolution in $D^b Coh_{\Lambda}^{G(\mathbf{O})}(\underline{\mathbf{u}})$. We have a principal $G(\mathbf{O})$ -bundle $G(\mathbf{F}) \rightarrow \text{Gr}_G$. Given a $G(\mathbf{O})$ -(ind)-scheme A we can form an associated bundle $\tilde{A} = G(\mathbf{F}) \times_{G(\mathbf{O})} A \rightarrow \text{Gr}_G$. Given a coherent $G(\mathbf{O})$ -equivariant sheaf \mathcal{F} on A we can form an associated sheaf $\tilde{\mathcal{F}}$ on \tilde{A} as $G(\mathbf{O})$ -invariants in the direct image of $\mathcal{O}_{G(\mathbf{F})} \boxtimes \mathcal{F}$ from $G(\mathbf{F}) \times A$ to $G(\mathbf{F}) \times_{G(\mathbf{O})} A$. If $A = \text{Gr}_G$, apart from the natural projection $p_1 : \tilde{A} \rightarrow \text{Gr}_G$, we have a multiplication map $G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G \rightarrow \text{Gr}_G$, to be denoted p_2 . Then (p_1, p_2) identifies $\widetilde{\text{Gr}}_G$ with $\text{Gr}_G \times \text{Gr}_G$. Furthermore, $\tilde{\underline{\mathbf{u}}}$ is a vector bundle over $\widetilde{\text{Gr}}_G = \text{Gr}_G \times \text{Gr}_G$ which is naturally identified with $p_2^* \underline{\mathbf{u}}$. Thus we have an ind-proper morphism $p_2 : \tilde{\underline{\mathbf{u}}} \rightarrow \underline{\mathbf{u}}$.

Note that both $\tilde{\underline{\mathbf{u}}} = p_2^* \underline{\mathbf{u}}$ and $p_1^* \underline{\mathbf{u}}$ are subbundles in the trivial (Tate) vector bundle $\underline{\mathfrak{g}}(\mathbf{F})$ over $\text{Gr}_G \times \text{Gr}_G$ with the fiber $\mathfrak{g}(\mathbf{F})$. Their intersection is naturally identified with $\tilde{\Lambda}$. In particular, we have an embedding $\tilde{\Lambda} \subset p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}}$, and an ind-proper morphism $p_2 : \tilde{\Lambda} \rightarrow \underline{\mathbf{u}}$.

Hence given $G(\mathbf{O})$ -equivariant coherent sheaves \mathcal{F}, \mathcal{G} on Λ we can consider the $G(\mathbf{O})$ -equivariant complex $\mathcal{F} \star \mathcal{G} := (p_2)_*(p_1^* \mathcal{F} \overset{L}{\otimes} \tilde{\mathcal{G}})$ (tensor product over the structure sheaf of the profinite dimensional vector bundle $p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}}$). Clearly, $\mathcal{F} \star \mathcal{G}$ is supported on Λ . Hence we get a convolution operation on $D^b Coh_{\Lambda}^{G(\mathbf{O})}(\underline{\mathbf{u}})$ and on $K^{G(\mathbf{O})}(\Lambda)$ once we check that $p_1^* \mathcal{F} \overset{L}{\otimes} \tilde{\mathcal{G}}$ is bounded.

To this end, note that $\tilde{\mathcal{G}}$ is flat over the first copy of Gr_G , and for some $\check{\lambda}$ the sheaf \mathcal{F} is supported on $\Lambda_{\leq \check{\lambda}}$, so the tensor product $p_1^* \mathcal{F} \overset{L}{\otimes} \tilde{\mathcal{G}}$ can actually be computed over the structure sheaf of $(p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}})|_{\overline{\text{Gr}}_{G, \check{\lambda}} \times \text{Gr}_G} = \underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}} \subset \underline{\mathbf{u}} \times \underline{\mathbf{u}} = p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}}$. That is, $p_1^* \mathcal{F} \overset{L}{\otimes} \tilde{\mathcal{G}}$ is the direct image of $p_1^* \mathcal{F}|_{\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}} \overset{L}{\otimes}_{\mathcal{O}_{\underline{\mathbf{u}}_{\check{\lambda}} \times \underline{\mathbf{u}}}} \tilde{\mathcal{G}}|_{\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}}$ under the closed embedding $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}} \hookrightarrow \underline{\mathbf{u}} \times \underline{\mathbf{u}}$. On the other hand, $p_1^* \mathcal{F}$ is flat over the second copy of Gr_G , while the support of $\tilde{\mathcal{G}}$ intersected with $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}$ is contained in $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}_{\leq \check{\mu}}$ for some $\check{\mu}$. Hence the tensor product $p_1^* \mathcal{F} \overset{L}{\otimes} \tilde{\mathcal{G}}$ can actually be computed over the structure sheaf of $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}_{\leq \check{\mu}}$. There exists $l \gg 0$ such that the diagonal fiberwise action of $\mathfrak{u}^{(l)}$ on $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}_{\leq \check{\mu}}$ is free, and both $p_1^* \mathcal{F}$ and $\tilde{\mathcal{G}}$ restricted to $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}_{\leq \check{\mu}}$ are $\mathfrak{u}^{(l)}$ -equivariant, that is, they are lifted from the sheaves on $(\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}_{\leq \check{\mu}})/\mathfrak{u}^{(l)} =: V$; we abuse notation by keeping the same names for these sheaves. So the tensor product $p_1^* \mathcal{F} \overset{L}{\otimes} \tilde{\mathcal{G}}$ can actually be computed as the tensor product of coherent sheaves over the structure sheaf of the profinite dimensional vector bundle V over the finite dimensional scheme $\overline{\text{Gr}}_{G, \check{\lambda}} \times \overline{\text{Gr}}_{G, \check{\mu}}$.

Now there exists a vector subbundle $V' \subset V$ such that the quotient $\overline{V} := V/V'$ is a finite dimensional vector bundle, $p_1^* \mathcal{F}$ is lifted from \overline{V} , and the support of $\tilde{\mathcal{G}}$ in V projects isomorphically onto its image in \overline{V} . Moreover, recall that $p_1^* \mathcal{F}$ is flat over $\overline{\text{Gr}}_{G, \check{\mu}}$, while $\tilde{\mathcal{G}}$ is flat over $\overline{\text{Gr}}_{G, \check{\lambda}}$. Clearly, in this situation $p_1^* \mathcal{F} \overset{L}{\otimes} \tilde{\mathcal{G}} \in D^b(V)$. This explains why for $G(\mathbf{O})$ -equivariant coherent sheaves \mathcal{F}, \mathcal{G} on Λ the tensor product $p_1^* \mathcal{F} \overset{L}{\otimes} \tilde{\mathcal{G}}$ is a bounded complex of coherent sheaves on $p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}}$ supported on $\tilde{\Lambda}$. Hence the same is true for the bounded complexes of $G(\mathbf{O})$ -equivariant coherent sheaves \mathcal{F}, \mathcal{G} on $\underline{\mathbf{u}}$ supported on Λ . Thus, $D^b Coh_{\Lambda}^{G(\mathbf{O})}(\underline{\mathbf{u}})$ is closed with respect to convolution.

THEOREM 7.3. *We have that $K^{G(\mathbf{O})}(\Lambda)$ is a commutative algebra isomorphic to $\mathbb{C}[\check{T} \times T]^W$.*

Remark 7.4. Since Λ_G is an affine Grassmannian analogue of the classical Steinberg variety, this result agrees well with the geometric realization of the Cherednik double affine Hecke algebra in [GG95] and [Vas02]. In effect, $K^{G(\mathbf{O})}(\Lambda_G)$ is the spherical subalgebra of the Cherednik algebra with both parameters trivial: $q = t = 1$.

7.5 Bialynicki–Birula stratifications. The proof of Theorem 7.3 uses the following lemma on K -theory of cellular spaces. Let M be a normal quasi-projective variety equipped with a torus H -action with finitely many fixed points. We assume that M is equipped with an H -invariant stratification $M = \bigsqcup_{\mu \in M^H} M_\mu$ such that each stratum M_μ contains exactly one H -fixed point μ , and M_μ is isomorphic to an affine space. For $\mu \in M^H$ we denote by $j_\mu : M_\mu \hookrightarrow M$ the locally closed embedding of the corresponding stratum. We denote by $i_\mu : \mu \hookrightarrow M_\mu$ the closed embedding of an H -fixed point in the corresponding stratum, or in the whole of M when no confusion is likely. We denote by $\mu \leq \nu$ the closure relation of strata. We denote by $M_{\leq \mu} \subset M$ the union $\bigcup_{\nu \leq \mu} M_\nu$.

Given an H -equivariant closed embedding of M into a smooth H -variety M' (for the existence see [Sum74]) we denote by \mathbb{T}^*M the restriction of the cotangent bundle \mathbb{T}^*M' to $M \subset M'$. We denote by $\iota : M \hookrightarrow \mathbb{T}^*M$ the embedding of the zero section. We also denote by i_μ the closed embedding of the conormal bundle $\mathbb{T}^*_\mu M' \hookrightarrow \mathbb{T}^*M$ when no confusion is likely. Finally, we denote by \mathcal{L}' the union of conormal bundles $\bigcup_\mu \mathbb{T}^*_{M_\mu} M'$, and j stands for the closed embedding $\mathcal{L}' \hookrightarrow \mathbb{T}^*M$. We denote by $\mathcal{L}'_{\leq \mu} \subset \mathcal{L}'$ the union $\bigcup_{\nu \leq \mu} \mathbb{T}^*_{M_\nu} M'$; it is a closed subvariety of \mathcal{L}' . It has a closed subvariety $\mathcal{L}'_{< \mu} := \bigcup_{\nu < \mu} \mathbb{T}^*_{M_\nu} M'$.

For $\mu \in M^H$ we have an embedding $i_{\mu*} : K^H(\mu) \hookrightarrow K^H(M)$. We have an embedding $j_* : K^H(\mathcal{L}') \hookrightarrow K^H(\mathbb{T}^*M) \simeq^* K^H(M)$. Indeed, the exact sequences (see [CG97, Chapter 5])

$$\begin{aligned} 0 \rightarrow K^H(\mathcal{L}'_{< \mu}) \rightarrow K^H(\mathcal{L}'_{\leq \mu}) \rightarrow K^H(\mathbb{T}^*_{M_\mu} M') \rightarrow 0, \\ 0 \rightarrow K^H(\mathbb{T}^*M'|_{M_{< \mu}}) \rightarrow K^H(\mathbb{T}^*M'|_{M_{\leq \mu}}) \rightarrow K^H(\mathbb{T}^*M'|_{M_\mu}) \end{aligned}$$

give rise to the support filtrations on $K^H(\mathcal{L}')$ and $K^H(\mathbb{T}^*M)$ with associated graded groups $\bigoplus_{\mu \in M^H} K^H(\mathbb{T}^*_{M_\mu} M')$ and $\bigoplus_{\mu \in M^H} K^H(\mathbb{T}^*M'|_{M_\mu})$, respectively. Now j_* is strictly compatible with the support filtrations and clearly injective on the associated graded groups.

Note that the image $j_*(K^H(\mathcal{L}')) \subset K^H(M)$ is independent of the choice of the closed embedding $M \hookrightarrow M'$. In effect, given another embedding $M \hookrightarrow \widetilde{M}$, we can consider the diagonal embedding $M \hookrightarrow M'' := M' \times \widetilde{M}$. Clearly, we have a projection $p : \mathbb{T}^*M''|_M \rightarrow \mathbb{T}^*M'|_M$ which realizes $\mathbb{T}^*M''|_M$ as a vector bundle over $\mathbb{T}^*M'|_M$. Moreover, if we denote by \mathcal{L}'' the union of conormal bundles $\bigcup_\mu \mathbb{T}^*_{M_\mu} M'' \subset \mathbb{T}^*M''|_M$ then $\mathcal{L}'' = p^{-1}\mathcal{L}'$. This shows that the images of $K^H(\mathcal{L}')$ and $K^H(\mathcal{L}'')$ in $K^H(M)$ coincide, and thus $j_*(K^H(\mathcal{L}')) \subset K^H(M)$ is well defined.

LEMMA 7.6. *In $K^H(M)$ we have an equality $j_*(K^H(\mathcal{L}')) = \bigoplus_\mu i_{\mu*}(K^H(\mu))$.*

Proof. Let $K^H(D_M)$ stand for the K -group of weakly H -equivariant D -modules on M' supported on $M \subset M'$. Given such a D -module and passing to the associated graded sheaf with respect to a good filtration, we obtain an H -equivariant coherent sheaf on \mathbb{T}^*M , and in this way one obtains a homomorphism $SS : K^H(D_M) \rightarrow K^H(\mathbb{T}^*M) \simeq^* K^H(M)$ (see e.g. [Gin86]). Let δ_μ stand for a δ -function D -module at the point $\mu \in M^H$ with its obvious H -equivariance. Then, evidently, $SS(\delta_\mu)$ generates $i_{\mu*}(K^H(\mu))$ as a module over $K^H(pt)$. Moreover, $\{SS(j_{\mu!}\mathcal{O}_{M_\mu}), \mu \in M^H\}$ forms a basis of $j_*(K^H(\mathcal{L}'))$.

In effect, the closed embedding $\mathcal{L}'_{< \mu} \hookrightarrow \mathcal{L}'_{\leq \mu}$ gives rise to the exact sequence

$$0 \rightarrow K^H(\mathcal{L}'_{< \mu}) \rightarrow K^H(\mathcal{L}'_{\leq \mu}) \rightarrow K^H(\mathbb{T}^*_{M_\mu} M') \rightarrow 0$$

(see [CG97, Chapter 5]), and the image of $SS(j_{\mu!}\mathcal{O}_{M_\mu})$ in $K^H(\mathbb{T}^*_{M_\mu} M')$ clearly generates it.

So it is enough to check the equality in $K^H(\mathbb{T}^*M)$:

$$SS(\delta_\mu) = SS(j_{\mu!}\mathcal{O}_{M_\mu}) \cdot (-1)^{\dim M_\mu} \det(\mathbb{T}_\mu M_\mu), \tag{11}$$

where $\det(\mathbb{T}_\mu M_\mu)$ is the character of H (thus an invertible element of $K^H(pt) = \mathbb{C}[H]$) acting in the determinant of the tangent bundle of M_μ at μ .

To this end note that restriction to the H -fixed points gives rise to an embedding $\bigoplus_{\nu} i_{\nu}^* i^* : K^H(\mathbb{T}^*M) \hookrightarrow \bigoplus_{\nu} K^H(\nu)$. This is checked by induction in ν using the exact sequences

$$0 \rightarrow K^H(\mathbb{T}^*M'|_{M_{<\nu}}) \rightarrow K^H(\mathbb{T}^*M'|_{M_{\leq\nu}}) \rightarrow K^H(\mathbb{T}^*M'|_{M_{\nu}}) \rightarrow 0.$$

It is clear that for $\nu = \mu$ the restrictions $i_{\mu}^* i^*$ of the left- and right-hand sides of (11) coincide. We are going to check that for $\nu \neq \mu$ the restrictions $i_{\nu}^* i^*$ of the left- and right-hand sides of (11) both vanish. Evidently, $i_{\nu}^* i^* SS(\delta_{\mu}) = 0$.

Recall that i_{ν} also stands for the closed embedding $\mathbb{T}_{\nu}^*M' \hookrightarrow \mathbb{T}^*M$, so we just have to check that $i_{\nu}^* SS(j_{\mu!} \mathcal{O}_{M_{\mu}}) = 0 \in K^H(\mathbb{T}_{\nu}^*M')$. Note that the functor of global sections of H -equivariant coherent sheaves on the vector space \mathbb{T}_{ν}^*M' gives rise to an embedding $\Gamma : K^H(\mathbb{T}_{\nu}^*M') \hookrightarrow \mathbb{Z}^{X^*(H)}$ where $X^*(H)$ stands for the lattice of characters of H . Now for a D -module \mathcal{F} we have $\Gamma(i_{\nu}^* SS\mathcal{F}) = \mathbf{i}_{\nu}^* \mathcal{F}$ where $\mathbf{i}_{\nu}^* \mathcal{F}$ stands for the fiber at $\nu \in M$ of the H -equivariant quasi-coherent $\mathcal{O}_{M'}$ -module \mathcal{F} . Finally, for $\mathcal{F} = j_{\mu!} \mathcal{O}_{M_{\mu}}$ and $\nu \neq \mu$ we have $\mathbf{i}_{\nu}^* j_{\mu!} \mathcal{O}_{M_{\mu}} = 0$. This completes the proof of the lemma. \square

7.7 Bialynicki–Birula stratification of Gr_G . We consider the stratification of Gr_G by the Iwahori orbits $\text{Gr}_G = \bigsqcup_{\check{\mu} \in Y} \text{Gr}_G^{\check{\mu}}$. This is a refinement of the stratification by the $G(\mathbf{O})$ -orbits: $\text{Gr}_{G, \check{\lambda}} = \bigsqcup_{\check{\mu} \in W\check{\lambda}} \text{Gr}_G^{\check{\mu}}$. Let us denote by $\mathfrak{n} \supset \mathfrak{u}$ the nilpotent radical of the Iwahori subalgebra in $\mathfrak{g}(\mathbf{F})$. The union of conormal bundles to the Iwahori orbits is the following subvariety Λ_I of the cotangent bundle \underline{u} : by definition, $\Lambda_I := \underline{u} \cap (\mathfrak{n} \times \text{Gr}_G)$. We have a closed embedding $\Lambda \subset \Lambda_I$.

Lemma 7.6 allows us to compute $K^T(\Lambda_I) = \bigoplus_{\check{\mu} \in Y} K^T(\check{\mu}) \subset K^T(\text{Gr}_G)$, i.e. $K^T(\Lambda_I) \simeq \mathbb{C}[\check{T} \times T]$ (note that the natural W -action on $K^T(\text{Gr}_G)$ induces the diagonal W -action on $\mathbb{C}[\check{T} \times T] \simeq K^T(\Lambda_I) \subset K^T(\text{Gr}_G)$). Although Lemma 7.6 was formulated for finite dimensional varieties M , its proof goes through for Gr_G without changes: we only need to have the singular support map $SS : K^T(D_{\text{Gr}_G}) \rightarrow K^T(\underline{u}) \simeq K^T(\text{Gr}_G)$. For this see [KT95], [BD00, Chapter 15] and [GG95].

The embedding $\Lambda \hookrightarrow \Lambda_I$ gives rise to the embedding $K^T(\Lambda) \hookrightarrow K^T(\Lambda_I) \hookrightarrow K^T(\underline{u}) = K^T(\text{Gr}_G)$. Note that W acts naturally on both $K^T(\Lambda)$ and $K^T(\text{Gr}_G)$, and the embedding $K^T(\Lambda) \hookrightarrow K^T(\text{Gr}_G)$ is W -equivariant. Also, $(K^T(\Lambda))^W = K^G(\Lambda) = K^G(\mathbf{O})(\Lambda)$. Hence, the image of the embedding $K^G(\mathbf{O})(\Lambda) \hookrightarrow K^T(\Lambda_I) \simeq \mathbb{C}[\check{T} \times T] \subset K^T(\text{Gr}_G)$ lies in the invariants of the diagonal W -action on $\mathbb{C}[\check{T} \times T]$. Thus to prove Theorem 7.3 we must check that the image of this embedding contains $\mathbb{C}[\check{T} \times T]^W$.

We have projections $\pi : \Lambda \rightarrow \text{Gr}_G$ and $\pi_I : \Lambda_I \rightarrow \text{Gr}_G$. For $\check{\lambda} \in Y^+$ we denote by $\Lambda_{\check{\lambda}}$ (respectively $\Lambda_{\leq \check{\lambda}}, \Lambda_{< \check{\lambda}}$) the preimage $\pi^{-1}(\text{Gr}_{G, \check{\lambda}})$ (respectively $\pi^{-1}(\overline{\text{Gr}}_{G, \check{\lambda}})$, $\pi^{-1}(\overline{\text{Gr}}_{G, \check{\lambda}} - \text{Gr}_{G, \check{\lambda}})$). For $\check{\lambda} \in Y^+$ we denote by $\Lambda_{I, \check{\lambda}}$ (respectively $\Lambda_{I, \leq \check{\lambda}}, \Lambda_{I, < \check{\lambda}}$) the preimage $\pi_I^{-1}(\text{Gr}_{G, \check{\lambda}})$ (respectively $\pi_I^{-1}(\overline{\text{Gr}}_{G, \check{\lambda}})$, $\pi_I^{-1}(\overline{\text{Gr}}_{G, \check{\lambda}} - \text{Gr}_{G, \check{\lambda}})$). Clearly, $\Lambda_{< \check{\lambda}}$ (respectively $\Lambda_{I, < \check{\lambda}}$) is closed in $\Lambda_{\leq \check{\lambda}}$ (respectively $\Lambda_{I, \leq \check{\lambda}}$), with the open complement $\Lambda_{\check{\lambda}}$ (respectively $\Lambda_{I, \check{\lambda}}$). In K -groups we have exact sequences (see [CG97, Chapter 5])

$$\begin{aligned} 0 \rightarrow K^T(\Lambda_{< \check{\lambda}}) \rightarrow K^T(\Lambda_{\leq \check{\lambda}}) \rightarrow K^T(\Lambda_{\check{\lambda}}) \rightarrow 0, \\ 0 \rightarrow K^T(\Lambda_{I, < \check{\lambda}}) \rightarrow K^T(\Lambda_{I, \leq \check{\lambda}}) \rightarrow K^T(\Lambda_{I, \check{\lambda}}) \rightarrow 0. \end{aligned}$$

Thus we obtain a support filtration on $K^T(\Lambda_I)$ (respectively $K^T(\Lambda)$) with associated graded $\bigoplus_{\check{\lambda} \in Y^+} K^T(\Lambda_{I, \check{\lambda}})$ (respectively $\bigoplus_{\check{\lambda} \in Y^+} K^T(\Lambda_{\check{\lambda}})$).

We have the embeddings $K^T(\Lambda_{\check{\lambda}}) \hookrightarrow K^T(\Lambda_{I, \check{\lambda}}) \hookrightarrow K^T(\underline{u}|_{\text{Gr}_{\check{\lambda}}}) \simeq K^T(\text{Gr}_{\check{\lambda}})$. The Weyl group W acts naturally on both $K^T(\Lambda_{\check{\lambda}})$ and $K^T(\text{Gr}_{\check{\lambda}})$, and to prove Theorem 7.3 it suffices to check that the image of $(K^T(\Lambda_{\check{\lambda}}))^W$ in $K^T(\Lambda_{I, \check{\lambda}})$ contains (equivalently, coincides with) the intersection $K^T(\Lambda_{I, \check{\lambda}}) \cap (K^T(\text{Gr}_{\check{\lambda}}))^W$.

To this end recall that $\text{Gr}_{G,\check{\lambda}}$ can be G -equivariantly identified with the total space $\tilde{\mathcal{B}}$ of a vector bundle over a certain partial flag variety \mathcal{B} of the group G (the quotient $G/P_{\check{\lambda}}$ by a parabolic subgroup depending on $\check{\lambda}$). The Borel subgroup $B \subset G$ acts on \mathcal{B} with finitely many orbits numbered by the cosets of parabolic Weyl subgroup $W^{\check{\lambda}} = W/W_{\check{\lambda}}$: we have $\mathcal{B} = \bigsqcup_{w \in W^{\check{\lambda}}} \mathcal{B}_w$. Let us denote by $\mathcal{L} \subset T^*\mathcal{B}$ the union of conormal bundles $\mathcal{L} = \bigsqcup_{w \in W^{\check{\lambda}}} T_{\mathcal{B}_w}^* \mathcal{B}$. Let us also denote by $\tilde{\mathcal{B}}_w$ the preimage of \mathcal{B}_w in $\tilde{\mathcal{B}}$ (it coincides with a certain Iwahori orbit $\text{Gr}_G^{\check{\mu}} \subset \text{Gr}_{G,\check{\lambda}} = \tilde{\mathcal{B}}$). We define $\tilde{\mathcal{L}} := \bigsqcup_{w \in W^{\check{\lambda}}} T_{\tilde{\mathcal{B}}_w}^* \tilde{\mathcal{B}} \subset T^*\tilde{\mathcal{B}}$. Then there exists a G -equivariant profinite dimensional vector bundle $\mathcal{V} \xrightarrow{p} T^*\tilde{\mathcal{B}}$ such that $\mathcal{V} \simeq \underline{\mathbf{u}}|_{\text{Gr}_{\check{\lambda}}}$, and under this isomorphism we have $\mathcal{V}|_{\tilde{\mathcal{L}}} \simeq \Lambda_{I,\check{\lambda}}$, $\mathcal{V}|_{\tilde{\mathcal{B}} \hookrightarrow T^*\tilde{\mathcal{B}}} \simeq \Lambda_{\check{\lambda}}$. Thus to prove Theorem 7.3 it is enough to check that the image of $(K^T(\tilde{\mathcal{B}}))^W$ in $K^T(T^*\tilde{\mathcal{B}})$ contains the intersection $K^T(\tilde{\mathcal{L}}) \cap (K^T(T^*\tilde{\mathcal{B}}))^W$. Equivalently, we have to check that the image of $(K^T(\mathcal{B}))^W$ in $K^T(T^*\mathcal{B})$ contains the intersection $K^T(\mathcal{L}) \cap (K^T(T^*\mathcal{B}))^W$. This is the subject of the following lemma.

LEMMA 7.8. *Let $\iota : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ denote the embedding of the zero section, and let $j : \mathcal{L} \hookrightarrow T^*\mathcal{B}$ denote the natural closed embedding. Then $\iota_*(K^T(\mathcal{B}))^W$ coincides with $\text{Im}(j_* : K^T(\mathcal{L}) \hookrightarrow K^T(T^*\mathcal{B})) \cap (K^T(T^*\mathcal{B}))^W$.*

Proof. For $w \in W^{\check{\lambda}}$ we denote by $w \in \mathcal{B}_w \subset \mathcal{B}$ the corresponding T -fixed point. We denote by i_w the closed embedding $T_w^*\mathcal{B} \hookrightarrow T^*\mathcal{B}$ (and also the closed embedding $w \hookrightarrow \mathcal{B}$, when confusion is unlikely), and we denote by i_w the closed embedding $w \hookrightarrow T^*\mathcal{B}$. According to Lemma 7.6, the image of $j_* : K^T(\mathcal{L}) \hookrightarrow K^T(T^*\mathcal{B})$ coincides with the image of $\bigoplus_{w \in W^{\check{\lambda}}} i_{w*} : \bigoplus_{w \in W^{\check{\lambda}}} K^T(T_w^*\mathcal{B}) \rightarrow K^T(T^*\mathcal{B})$. We have an embedding $\bigoplus_{w \in W^{\check{\lambda}}} i_w^* : K^T(T^*\mathcal{B}) \hookrightarrow \bigoplus_{w \in W^{\check{\lambda}}} K^T(w)$, and similarly an embedding $\bigoplus_{w \in W^{\check{\lambda}}} i_w^* : K^T(\mathcal{B}) \hookrightarrow \bigoplus_{w \in W^{\check{\lambda}}} K^T(w)$.

Clearly, the W -invariants project injectively into any direct summand: $K^G(\mathcal{B}) = (K^T(\mathcal{B}))^W \xrightarrow{i_w^*} K^T(w)$ (respectively $K^G(T^*\mathcal{B}) = (K^T(T^*\mathcal{B}))^W \xrightarrow{i_w^*} K^T(w)$) for any $w \in W^{\check{\lambda}}$. Thus it suffices to check that for any $w \in W^{\check{\lambda}}$ we have a coincidence $\text{Im}(i_w^* i_{w*} : K^T(T_w^*\mathcal{B})^W \rightarrow K^T(w)) = \text{Im}(i_w^* \iota_* \text{Res}_T^G : K^G(\mathcal{B}) \rightarrow K^T(w))$. Note that if $w = e$ (the identity coset of $W_{\check{\lambda}}$ in W), then the image $i_e^*(K^T(\mathcal{B}))^W \subset K^T(e)$ (respectively $i_e^*(K^T(T^*\mathcal{B}))^W \subset K^T(e)$) coincides with $(K^T(e))^{W_{\check{\lambda}}} = \mathbb{C}[T]^{W_{\check{\lambda}}}$. Moreover, under identification $K^T(T_e^*\mathcal{B}) = K^T(e) = \mathbb{C}[T]$, we have $K^T(T_e^*\mathcal{B}) \cap (K^T(T^*\mathcal{B}))^W = \mathbb{C}[T]^{W_{\check{\lambda}}}$.

Identifying both $K^T(T_e^*\mathcal{B})$ and $K^T(e)$ with $\mathbb{C}[T]$, the map $i_e^* i_{e*}$ is a multiplication by the product $\Delta_1 = \prod_{k=1}^{\dim \mathcal{B}} (1 - \chi_k)$ where the χ_k run through the characters of T in the tangent space $T_e(T_e^*\mathcal{B}) = T_e^*\mathcal{B}$. Furthermore, identifying $K^G(\mathcal{B})$ with $\mathbb{C}[T]^{W_{\check{\lambda}}}$, and $K^T(e)$ with $\mathbb{C}[T]$, the map $i_e^* \iota_* \text{Res}_T^G$ is a multiplication by the product $\Delta_2 = \prod_{k=1}^{\dim \mathcal{B}} (1 - \chi'_k)$ where the χ'_k run through the characters of T in the tangent space $T_e \mathcal{B}$. We can arrange the characters χ'_k so that we have $\chi'_k = \chi_k^{-1}$. Then we see that $\Delta_1 = \Delta_2 \prod_{k=1}^{\dim \mathcal{B}} (-\chi_k)$, so they differ by an invertible function. Hence the corresponding images coincide: $\Delta_1 \mathbb{C}[T]^{W_{\check{\lambda}}} = \Delta_2 \mathbb{C}[T]^{W_{\check{\lambda}}}$.

This completes the proof of the lemma along with Theorem 7.3.

7.9 Conjecture. In this subsection we describe (without striving for high precision) a conjectural picture motivating Theorem 7.3. We hope that the isomorphism $K^{G(\mathbf{O})}(\Lambda_G) = \mathbb{C}[\check{T} \times T]^W = \mathbb{C}[T \times \check{T}]^W = K^{\check{G}(\mathbf{O})}(\Lambda_{\check{G}})$ lifts to an equivalence of monoidal categories $F : D^b \text{Coh}_{\Lambda_G}^{G(\mathbf{O})}(\underline{\mathbf{u}}_G) \simeq D^b \text{Coh}_{\Lambda_{\check{G}}}^{\check{G}(\mathbf{O})}(\underline{\mathbf{u}}_{\check{G}})$. The conjectural equivalence F is related to the Langlands correspondence in the following way.

Recall that the conjectural (for $G = GL(n)$ mostly proven in [Gai02]) geometric Langlands correspondence is an equivalence of triangulated categories between the derived category of

D -modules on the stack Bun_G of G -bundles on a given smooth projective curve C , and the derived category of coherent sheaves on the stack of \check{G} local systems on the same curve. One might expect its ‘classical limit’ to be an equivalence between the derived categories of coherent sheaves $L : D(\mathbb{T}^* \text{Bun}_G) \simeq D(\mathbb{T}^* \text{Bun}_{\check{G}})$ where $\mathbb{T}^* \text{Bun}_G$ is the cotangent bundle to the moduli stack of G -bundles on C . Given a point $c \in C$, and identifying \mathbf{O} with the algebra of functions on the formal neighborhood of c , one gets an action of $D^b \text{Coh}_{\Lambda_G}^{G(\mathbf{O})}(\underline{\mathfrak{u}}_G)$ on $D(\mathbb{T}^* \text{Bun}_G)$. The ‘classical limit’ of the Hecke eigenproperty of geometric Langlands correspondence (see [BD00]) should be stated in terms of this action; it should say that the global equivalence L is compatible with our local equivalence F . □

8. Perverse sheaves and fusion

We refer the reader to [Bez00] for the definition of perverse equivariant coherent sheaves and related objects.

8.1 Recall the setup of § 6.1. Note that all the $G(\mathbf{O})$ -orbits in a connected component of Gr_G have dimensions of the same parity. Thus it makes sense to consider the middle perversity function $p(\text{Gr}_{G,\check{\lambda}}) = -\frac{1}{2} \dim(\text{Gr}_{G,\check{\lambda}}) = -\langle \rho, \check{\lambda} \rangle$. It is obviously strictly monotone and comonotone, but at some connected components of Gr_G it takes values in half-integers. This means that we consider equivariant complexes formally placed in half-integer homological degrees. The theory of [Bez00] defines the artinian abelian category $\mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$ of perverse $G(\mathbf{O})$ -equivariant coherent sheaves (with respect to the above middle perversity). Let $D^{b,G(\mathbf{O})}(\text{Gr}_G)$ denote the bounded derived category of $G(\mathbf{O})$ -equivariant coherent sheaves on Gr_G (with the same convention that the complexes at ‘odd’ connected components are placed in half-integer homological degrees).

Given two complexes $\mathcal{F}, \mathcal{G} \in D^{b,G(\mathbf{O})}(\text{Gr}_G)$ we have their convolution $\mathcal{F} \star \mathcal{G} \in D^{b,G(\mathbf{O})}(\text{Gr}_G)$. Recall that $\mathcal{F} \star \mathcal{G} = \Pi_{0*}(\mathcal{F} \times \mathcal{G})$ where $\Pi_0 : G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G \rightarrow \text{Gr}_G$ is the convolution diagram, and $\mathcal{F} \times \mathcal{G}$ is the twisted product of \mathcal{F} and \mathcal{G} on $G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G$.

PROPOSITION 8.2. *The convolution preserves perverse sheaves: for $\mathcal{F}, \mathcal{G} \in \mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$ we have $\mathcal{F} \star \mathcal{G} \in \mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$.*

Proof. Denote the projection $G(\mathbf{F}) \rightarrow G(\mathbf{F})/G(\mathbf{O}) = \text{Gr}_G$ by p , and consider a stratification $G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G = \bigsqcup_{\check{\lambda}, \check{\mu} \in Y^+} p^{-1}(\text{Gr}_{G,\check{\lambda}}) \times_{G(\mathbf{O})} \text{Gr}_{G,\check{\mu}}$. Clearly, $\mathcal{F} \times \mathcal{G}$ is smooth (locally free) along this stratification, and perverse (with respect to the middle perversity). According to [MV00, 2.7], the map Π_0 is stratified semismall with respect to the above stratification. Now the perversity of $\Pi_{0*}(\mathcal{F} \times \mathcal{G})$ follows in the same manner as in the constructible case (cf. [MV00]). □

8.3 The absence of commutativity constraint. According to Proposition 8.2, $\mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$ acquires the structure of abelian artinian monoidal category. Moreover, according to Theorem 2.15, part (a), its K -ring is commutative. Nevertheless, $\mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$ admits no commutativity constraint, as can be seen in the following example.

We recall the setup of § 3.6, and consider Gr_{PGL_2} . One can check that there are the nonsplit exact sequences in $\mathcal{P}^{PGL_2(\mathbf{O})}(\text{Gr}_{PGL_2})$:

$$\begin{aligned} 0 \rightarrow \mathcal{V}(0)_0 \rightarrow \mathcal{V}(0)_1 \star \mathcal{V}(-2)_1 \rightarrow \mathcal{V}(-2)_2 \rightarrow 0, \\ 0 \rightarrow \mathcal{V}(-2)_2 \rightarrow \mathcal{V}(-2)_1 \star \mathcal{V}(0)_1 \rightarrow \mathcal{V}(0)_0 \rightarrow 0. \end{aligned}$$

Thus $\mathcal{V}(0)_1 \star \mathcal{V}(-2)_1$ and $\mathcal{V}(-2)_1 \star \mathcal{V}(0)_1$ are nonisomorphic.

8.4 $G(\mathbf{O}) \rtimes \mathbb{G}_m$ -equivariant sheaves and fusion. The orbits of $G(\mathbf{O}) \rtimes \mathbb{G}_m$ on Gr_G coincide with the $G(\mathbf{O})$ -orbits, so one can consider the abelian artinian monoidal category $\mathcal{P}^{G(\mathbf{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ of $G(\mathbf{O}) \rtimes \mathbb{G}_m$ -equivariant coherent perverse sheaves on Gr_G . For $\mathcal{F} \in \mathcal{P}^{G(\mathbf{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ we have $R\Gamma(\text{Gr}_G, \mathcal{F}) \in D^b(G(\mathbf{O}) \rtimes \mathbb{G}_m\text{-mod})$.

Feigin and Loktev define (under certain restrictions) in [FL99] the *fusion product* $V_1 \star \cdots \star V_k \in G(\mathbf{O}) \rtimes \mathbb{G}_m\text{-mod}$ of $G(\mathbf{O}) \rtimes \mathbb{G}_m$ -modules V_1, \dots, V_k . We recall some of their results in the case $G = PGL_2$.

Let $V(n)$ be the $(n + 1)$ -dimensional $G(\mathbf{O}) \rtimes \mathbb{G}_m$ -module factoring through $G(\mathbf{O}) \rtimes \mathbb{G}_m \rightarrow G \times \mathbb{G}_m \rightarrow G$. Recall the irreducible $PGL_2(\mathbf{O})$ -equivariant perverse sheaf $\mathcal{V}(n)_m$ introduced in § 3.6. It can be lifted to the same named $PGL_2(\mathbf{O}) \rtimes \mathbb{G}_m$ -equivariant perverse sheaf, where the action of \mathbb{G}_m in the fiber over a \mathbb{G}_m -fixed point in the orbit $\text{Gr}_{PGL_2, m}$ is set *trivial*. In particular, $R\Gamma(\text{Gr}_{PGL_2}, \mathcal{V}(n)_1) = V(n)[\frac{1}{2}]$ for $n \geq 0$.

Now we can reformulate Theorem 2.5 of [FL99] as follows.

PROPOSITION 8.5. *Let $n_1 \geq n_2 \geq \dots \geq n_k$. Then*

- (a) $R\Gamma(\text{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \cdots \star \mathcal{V}(n_k)_1)$ is concentrated in degree $-k/2$;
- (b) $R\Gamma(\text{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \cdots \star \mathcal{V}(n_k)_1)[-k/2] \simeq V(n_k) \star \cdots \star V(n_1)$.

8.6 Multiplication table. According to Proposition 8.5, the calculation of fusion product in $K(G(\mathbf{O}) \rtimes \mathbb{G}_m\text{-mod})$ is closely related to the ring structure of $K^{G(\mathbf{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$. Let us formulate the recurrence relations in $K^{G(\mathbf{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ (compare [FL99, end of § 2.1]). So $\mathbf{v}(n)_m$ is the class of $\mathcal{V}(n)_m$ in $K^{G(\mathbf{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$. We assume that $n \geq 0$. Then we obtain

$$q^{-l}\mathbf{v}(l+n)_1 \star \mathbf{v}(l)_1 = q^{-2l}\mathbf{v}(2l+n)_2 + q^2\mathbf{v}(n-2)_0 + q^4\mathbf{v}(n-4)_0 + \dots \tag{12}$$

(the last summand being $q^n\mathbf{v}(0)_0$ if n is even, and $q^{n-1}\mathbf{v}(1)_0$ if n is odd),

$$q^{-l-2}\mathbf{v}(l-n)_1 \star \mathbf{v}(l)_1 = q^{-2l-2}\mathbf{v}(2l-n)_2 + q^{-2}\mathbf{v}(n-2)_0 + q^{-4}\mathbf{v}(n-4)_0 + \dots \tag{13}$$

(the last summand being $q^{-n}\mathbf{v}(0)_0$ if n is even, and $q^{-n+1}\mathbf{v}(1)_0$ if n is odd),

$$\mathbf{v}(l+1)_1^{\star a} \star \mathbf{v}(l)_1^{\star b} = q^{\frac{1}{2}(a(1-a)+l(a+b)(1-a-b))} \mathbf{v}(a+l(a+b))_{a+b}. \tag{14}$$

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