

RANDOM SYMMETRIZATIONS OF CONVEX BODIES

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Abstract

In this paper we investigate the asymptotic behavior of sequences of successive Steiner and Minkowski symmetrizations. We state an equivalence result between the convergences of those sequences for Minkowski and Steiner symmetrizations. Moreover, in the case of independent (and not necessarily identically distributed) directions, we prove the almost-sure convergence of successive symmetrizations at exponential rate for Minkowski, and at rate $e^{-c\sqrt{n}}$ with $c > 0$ for Steiner.

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1. Introduction

Let A be a convex body of \mathbb{R}^d , i.e. a convex compact set with nonempty interior, and let $u \in \mathbb{S}^{d-1}$ be a unit vector. The set A can be considered as a family of line segments parallel to the direction u . Sliding these segments along u and centering them with respect to the hyperplane u^\perp gives $S_u A$, the *Steiner symmetral* of A .

Steiner symmetrization plays an important role in geometry and its applications. Indeed, this transformation possesses certain contraction properties which allow in many cases to round off the initial set after multiple applications. Moreover, the limiting ball delivers the solution of several optimization problems, such as, for instance, the isoperimetric inequality, the Brunn–Minkowski inequality, and the Blaschke–Santaló inequality (see Section 9.2 of [6]).

Another important transformation is the *Minkowski symmetrization* (sometimes called the Blaschke symmetrization). The *Minkowski symmetral* of a convex body A with direction $u \in \mathbb{S}^{d-1}$, denoted by $B_u A$, is defined as the arithmetic mean of A and $\pi_u(A)$, its orthogonal symmetric with respect to u^\perp .

Our aim is to study the asymptotic behavior of successive Steiner and Minkowski symmetrizations. Recently, this area of study has undergone considerable development. Without applying for completeness, we will note here a few works characterizing the main tendencies.

Among the works concerning deterministic sequences of directions, we mention Klain [7]. For directions chosen among a finite set, Klain gave the convergence of the sequence of successive Steiner symmetrals to a limiting set, which is symmetric under reflection in any of the directions that appear infinitely often in the sequence. Bianchi *et al.* [2] proved that, from any dense set of directions (in \mathbb{S}^{d-1}), it is always possible to extract a countable sequence

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rounding off any convex body by successive Steiner symmetrizations. They also exhibited countable dense sequences of directions and convex bodies whose corresponding sequences of Steiner symmetrals do not converge at all (the order of directions matters).

The case of random Steiner symmetrizations has also been investigated. The first result (to the authors' knowledge) is due to Mani-Levitska [9] and concerns the case of independent and identically distributed (i.i.d.) directions, chosen uniformly on the sphere \mathbb{S}^{d-1} . He established the almost-sure convergence of the sequence of successive Steiner symmetrals of any convex body to a ball. Volčič [13] extended Mani-Levitska's result to measurable sets with finite measure, and to any probability measure assigning positive mass to any open set of \mathbb{S}^{d-1} . Burchard and Fortier [4] stated that the almost-sure convergence still occurs for (independent but) nonidentically distributed directions provided they satisfy some restrictive condition (see (3.2) below). Combining a probabilistic approach and the powerful analytical device of spherical harmonics, Klartag [8] provided in his remarkable article a rate of convergence for successive Steiner symmetrizations. Specifically, for any given convex body A , there exists an (implicit) sequence of n directions such that the Hausdorff distance between the resulting sequence of successive Steiner symmetrals and the limiting ball is smaller than $e^{-c\sqrt{n}}$ with $c > 0$. As a key step, Klartag proved a similar result for successive Minkowski symmetrizations, but at an exponential rate.

Our first result (Theorem 5.1) complements and strengthens the results of [8], [9], and [13]. Indeed, it affirms that the convergence of the sequence of successive Steiner symmetrizations is almost sure on the one hand, and at rate $e^{-c\sqrt{n}}$ on the other hand. Moreover, the random directions are allowed to be nonidentically distributed and their distributions may avoid some open sets of the sphere \mathbb{S}^{d-1} , which is forbidden in [4] and [13]. The independence hypothesis of directions can also be relaxed (see Remark 5.3). The proof of Theorem 5.1 substantially follows the ideas of Klartag [8]. We first give the almost-sure convergence of successive Minkowski symmetrizations at exponential rate (Theorem 4.1). The main advantage of Minkowski symmetrization over Steiner symmetrization is that it exhibits a (strict) contraction property (see Proposition 4.1) from which Theorem 4.1 is derived. Thus, the passage from Minkowski to Steiner is based only on the inclusion $S_u A \subset B_u A$. This explains the loss in the rate of convergence between Minkowski and Steiner symmetrizations.

Our second result (Theorem 3.1) provides a surprising link between Steiner and Minkowski symmetrizations. A sequence of directions $(u_n)_{n \in \mathbb{N}}$ is said to be S -universal if, for any k , the sequence of successive Steiner symmetrizations corresponding to the shifted sequence $(u_{k+n})_{n \in \mathbb{N}}$ rounds off any convex body. The concept of an M -universal sequence (for Minkowski symmetrization) is introduced similarly. Theorem 3.1 states that the concepts of S - and M -universality coincide; thus, we omit the prefixes S and M . This allows us in many cases to use known results about Steiner symmetrization to deduce new results about Minkowski symmetrization. For example, from the aforementioned result in [9] concerning random i.i.d. Steiner symmetrizations, we immediately deduce a similar result for Minkowski symmetrizations, without the sophisticated use of spherical harmonics (Proposition 3.1). Theorem 3.1 also allows us to extend the results of [2] and [4] to Minkowski symmetrizations. In particular, any dense set of directions (in \mathbb{S}^{d-1}) contains a universal subsequence (Proposition 3.3).

The paper is organized as follows. Section 2 contains precise definitions of Steiner and Minkowski symmetrizations, and their preliminary properties. In Section 3 we introduce the concepts of S - and M -universal sequences. Theorem 3.1 is proved and applied in two different contexts; random (Propositions 3.1 and 3.2) and deterministic (Propositions 3.3 and 3.4). Sections 4 and 5 are respectively devoted to random symmetrizations, and Minkowski and

Steiner symmetrizations. The proof of Proposition 4.1, which is rather long and technical, is given in Section 4.2. Finally, some open questions are formulated in Section 6.

2. Steiner and Minkowski symmetrizations

In this section we give the definitions of Steiner and Minkowski symmetrizations, and their basic properties. Let us denote by \mathcal{K}_d the set of convex bodies in \mathbb{R}^d .

Definition 2.1. Let $A \in \mathcal{K}_d$ and $u \in \mathbb{S}^{d-1}$. The convex body A can be considered to be a family of line segments parallel to the direction u . Sliding each of these segments along u so that they become symmetrically balanced around the hyperplane u^\perp , a new set is obtained, called the *Steiner symmetral* of A with direction u , denoted by $S_u A$ (see Figure 1). The mapping S_u defined on \mathcal{K}_d is called *Steiner symmetrization* with direction u .

It follows from Definition 2.1 that Steiner symmetrization preserves the volume: for any $A \in \mathcal{K}_d$ and $u \in \mathbb{S}^{d-1}$,

$$\text{vol}(S_u A) = \text{vol}(A), \tag{2.1}$$

where $\text{vol}(A)$ denotes the d -dimensional Lebesgue measure λ^d of the measurable set A .

Let us denote by π_u the orthogonal reflection operator with respect to the hyperplane u^\perp : for all $x \in \mathbb{R}^d$,

$$\pi_u(x) = x - 2\langle x, u \rangle u.$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^d .

Definition 2.2. Let $A \in \mathcal{K}_d$ and $u \in \mathbb{S}^{d-1}$. The *Minkowski symmetral* of A with direction u , denoted by $B_u A$, is defined by

$$B_u A = \frac{1}{2}(A + \pi_u(A)),$$

where ‘+’ denotes the Minkowski sum of sets A and B . The mapping B_u defined on \mathcal{K}_d is called *Minkowski symmetrization* with direction u .

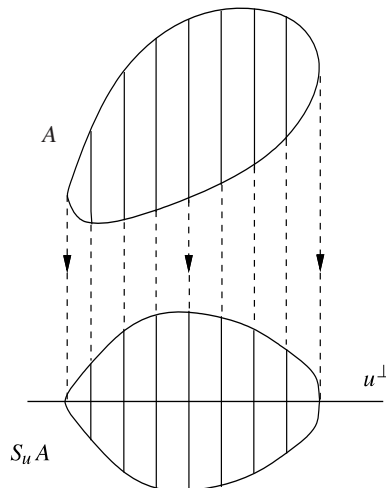


FIGURE 1: Steiner symmetrization with direction u . The dashed lines represent the sliding of orthogonal segments along u .

The support function f_A of a convex body $A \in \mathcal{K}^d$ is defined by

$$f_A(\theta) = \sup_{x \in A} \langle x, \theta \rangle \quad \text{for any } \theta \in S^{d-1}.$$

The support functions are a useful tool in convex geometry. In particular, any convex body is characterized by its support function (see Theorem 4.3 of [6, p. 57]). Let σ be the Haar probability measure on S^{d-1} . The value

$$L(A) = \int_{S^{d-1}} f_A \, d\sigma$$

is called the mean radius of A .

Minkowski symmetrization presents an advantage over Steiner symmetrization. Classical properties of support functions (see Proposition 6.2 of [6, p. 81]) allows us to express $f_{B_u A}$ as the arithmetic mean of f_A and $f_{\pi_u A}$:

$$f_{B_u A} = \frac{1}{2}(f_A + f_{\pi_u(A)}). \tag{2.2}$$

As a consequence of (2.2), Minkowski symmetrization preserves the mean radius: for any $A \in \mathcal{K}_d$ and $u \in S^{d-1}$,

$$L(B_u A) = L(A). \tag{2.3}$$

Let $B(x, r)$ be the Euclidean closed ball with center x and radius r . Let $D = B(0, 1)$ be the unit ball, and let v_d be its volume. We refer the reader to [6] for details about the following properties.

Lemma 2.1. *Let $A \in \mathcal{K}_d$ and $u \in S^{d-1}$.*

- (i) $S_u A$ and $B_u A$ are convex bodies, symmetric with respect to u^\perp .
- (ii) Let $A' \in \mathcal{K}_d$ and contain A . Then $S_u A \subset S_u A'$ and $B_u A \subset B_u A'$. In particular, if $R(A)$ denotes the circumradius of A , i.e.

$$R(A) = \inf\{R > 0, A \subset B(0, R)\},$$

then $S_u A$ and $B_u A$ are included in the centered ball $R(A)D$.

- (iii) $S_u A$ is included in $B_u A$.

The inclusion $S_u A \subset B_u A$ can be understood as follows. Let Δ be one of the orthogonal segments to u^\perp which compose $S_u A$, i.e. Δ is obtained by translation of a segment Δ' composing A (see Definition 2.1). Then

$$\Delta = \frac{1}{2}(\Delta' + \pi_u(\Delta')) \subset \frac{1}{2}(A + \pi_u(A)) = B_u A$$

and Lemma 2.1(iii) follows.

We deduce immediately from (2.1), (2.3), and Lemma 2.1(iii) that Steiner symmetrization decreases the mean radius whereas Minkowski symmetrization increases the volume:

$$L(S_u A) \leq L(A) \quad \text{and} \quad \text{vol}(B_u A) \geq \text{vol}(A). \tag{2.4}$$

Two classical metrics on the set \mathcal{K}_d are involved in our proofs: the Hausdorff distance

$$d_H(A, B) = \max\{\inf\{\varepsilon > 0 \mid A \subset B + B(0, \varepsilon)\}, \inf\{\varepsilon > 0 \mid B \subset A + B(0, \varepsilon)\}\}$$

and the Nikodým distance

$$d_N(A, B) = \lambda^d(A \Delta B) = \lambda^d(A \setminus B) + \lambda^d(B \setminus A).$$

These distances generate the same topology on \mathcal{K}_d . Hence, all the convergences stated in the sequel correspond to this topology. Inequalities that hold for the Hausdorff and Nikodým distances used in our proofs are given in Appendix A.

3. Theorem of equivalence

Let $(u_n)_{n \geq 1}$ be a sequence of elements of \mathbb{S}^{d-1} . For integers $n \geq k \geq 1$, we denote by $S_{k,n}$ the sequence of $n - k + 1$ consecutive Steiner symmetrizations from u_k to u_n , i.e.

$$S_{k,n}A = S_{u_n}(\cdots S_{u_{k+1}}(S_{u_k}A) \cdots),$$

where A is a convex body. When $k = 1$, $S_{1,n}A$ is denoted by S_nA . For Minkowski symmetrizations, the notation $B_{k,n}A$ and B_nA are as defined above.

Let $r(A)$ be the real number such that the ball $r(A)D$ has the same volume as A . Recall that the set \mathcal{K}_d of convex bodies is endowed with the Hausdorff distance. A sequence $(u_n)_{n \geq 1}$ *S-rounds* the set $A \in \mathcal{K}_d$ if

$$S_nA \rightarrow r(A)D$$

and *M-rounds* A if

$$B_nA \rightarrow L(A)D$$

as n tends to ∞ . A sequence $(u_n)_{n \geq 1}$ *strongly S-rounds* the convex body A if, for any k ,

$$S_{k,n}A \rightarrow r(A)D$$

as n tends to ∞ . The same terminology holds for Minkowski symmetrizations: $(u_n)_{n \geq 1}$ *strongly M-rounds* A if, for any k ,

$$B_{k,n}A \rightarrow L(A)D$$

as n tends to ∞ . Finally, $(u_n)_{n \geq 1}$ is said to be *S-universal* (or *M-universal*) if it strongly *S-rounds* or, respectively, strongly *M-rounds* any A of \mathcal{K}_d .

The next result shows that the notions of *S*- and *M*-universality coincide. As such, a sequence will be merely called *universal*.

Theorem 3.1. *A sequence $(u_n)_{n \geq 1}$ of \mathbb{S}^{d-1} is S-universal if and only if it is M-universal.*

Proof. We focus only on the sufficient condition because the necessary condition is proved similarly. Let A be a convex body. Since Minkowski symmetrization increases the volume, the sequence $(\text{vol}(B_nA))_{n \geq 1}$ is nondecreasing. It is also bounded from above since $B_nA \subset R(A)D$. Let V be its limit. The sets B_nA for $n \geq 1$ are all included in the compact set $\{K \in \mathcal{K}_d, K \subset R(A)D\}$ of (\mathcal{K}_d, d_H) (see Theorem 1.8.4 of [12, p. 49] for details). So, $(B_nA)_{n \geq 1}$ admits a convergent subsequence $(B_{n_k}A)_{k \geq 1}$. Let E be its limit. Since the volume is a continuous function on (\mathcal{K}_d, d_H) , the volume of E equals V .

For any $m > k$, Lemma 2.1(iii) implies that

$$S_{n_k+1, n_m}(B_{n_k}A) \subset B_{n_k+1, n_m}(B_{n_k}A) = B_{n_m}A.$$

By *S*-universality, the left-hand side of the above inclusion converges to the ball $r(B_{n_k}A)D$, whereas the right-hand side converges to E . Hence, the set E contains $r(B_{n_k}A)D$, whose volume tends to V as k tends to ∞ . This forces E to be the ball of volume V .

As a result, any convergent subsequence of $(B_n A)_{n \geq 1}$ has the same limit as $r(V)D$. By compactness, this also holds for the sequence $(B_n A)_{n \geq 1}$ itself. Thus, we use Lemma A.1 to relate $r(V)$ to $L(A)$: as $n \rightarrow \infty$,

$$L(A) = \int_{\mathbb{S}^{d-1}} f_{B_n A} \, d\sigma \rightarrow \int_{\mathbb{S}^{d-1}} f_{r(V)D} \, d\sigma = r(V).$$

Finally, for any k , applying the same strategy to the S -universal sequence $(u_{k+n})_{n \geq 1}$, $B_{k+1,n} A$ tends to $L(A)D$. The M -universal character of $(u_n)_{n \geq 1}$ follows.

In what follows, Theorem 3.1 is applied in two different contexts: random (Propositions 3.1 and 3.2) and deterministic (Propositions 3.3 and 3.4). In the next three results, a sufficient condition for the sequence of directions is given that ensures its universal character. The fourth result focuses on dimension 2: there exists a uniformly distributed sequence on \mathbb{S}^1 which is not universal.

Proposition 3.1. *Let $(U_n)_{n \geq 1}$ be a stationary sequence of random variables of \mathbb{S}^{d-1} , i.e. for any k , the sequences $(U_n)_{n \geq 1}$ and $(U_{k+n})_{n \geq 1}$ are identically distributed. Assume that, for any convex body A , $(U_n)_{n \geq 1}$ almost surely (a.s.) S -rounds A . Then $(U_n)_{n \geq 1}$ is a.s. universal. The same conclusion holds when the S -rounding hypothesis is replaced with the M -rounding hypothesis.*

Proof. We only check the result under the S -rounding hypothesis, as the proof under the M -rounding hypothesis is similar.

Let $(C_j)_{j \geq 1}$ be a countable dense subset of the separable set (\mathcal{K}_d, d_H) . By hypothesis, for any index j and any positive rational number ε , there exists an event of probability 1 on which $(U_n)_{n \geq 1}$ S -rounds $C_j^\varepsilon := C_j + B(0, \varepsilon)$. Let Ω_0 be the intersection of these events. We are going to prove that on Ω_0 , $(U_n)_{n \geq 1}$ S -rounds any convex body.

Let $A \in \mathcal{K}_d$. By compactness, let us consider a convergent subsequence $(S_{n_k} A)_{k \geq 1}$ of $(S_n A)_{n \geq 1}$ whose limit is denoted by E . Since the volume is a continuous function on (\mathcal{K}_d, d_H) ,

$$\text{vol}(E) = \lim_{k \rightarrow \infty} \text{vol}(S_{n_k} A) = \text{vol}(A). \tag{3.1}$$

Let $\varepsilon > 0$ be a rational number. There exists an index $j = j(\varepsilon)$ such that A is included in C_j^ε . Hence, for any k ,

$$S_{n_k} A \subset S_{n_k}(C_j^\varepsilon).$$

When k tends to ∞ and on Ω_0 , the above inclusion becomes $E \subset r(C_j^\varepsilon)D$. Taking $\varepsilon \searrow 0$, it follows that $E \subset r(A)D$. By (3.1), this is possible only if $E = r(A)D$. We conclude by compactness that $(U_n)_{n \geq 1}$ S -rounds any $A \in \mathcal{K}_d$ on the event Ω_0 of probability 1. By stationarity, for any k , this proof applies to $(U_{k+n})_{n \geq 1}$: there exists an event Ω_k of probability 1 on which the sequence $(U_{k+n})_{n \geq 1}$ S -rounds any $A \in \mathcal{K}_d$. Hence, by Theorem 3.1, $(U_n)_{n \geq 1}$ is universal on $\bigcap_k \Omega_k$.

When the random variables U_n , $n \geq 1$, are independent, the hypothesis of stationarity on the sequence $(U_n)_{n \geq 1}$ can be weakened. The following condition was introduced in [4]: for any $r > 0$ and any sequence $(v_n)_{n \geq 1}$ in \mathbb{S}^{d-1} ,

$$\sum_{n=1}^{\infty} \mathbb{P}(U_n \in B(v_n, r)) = \infty. \tag{3.2}$$

Thanks to the Borel–Cantelli lemma, condition (3.2) implies that each open ball V of the sphere \mathbb{S}^{d-1} with positive radius is a.s. infinitely often visited by the U_n . Burchard and Fortier [4, Corollary 1] stated that a sequence $(U_n)_{n \geq 1}$ of independent random variables satisfying (3.2) a.s. S -rounds any convex body A . Theorem 3.1 extends their result to Minkowski symmetrizations.

Proposition 3.2. *Let $(U_n)_{n \geq 1}$ be a sequence of independent random variables in \mathbb{S}^{d-1} satisfying condition (3.2). Then $(U_n)_{n \geq 1}$ is a.s. universal.*

In the case of i.i.d. directions, Theorems 4.1 and 5.1 therein give some rates of convergence. But their proofs are more complicated.

Bianchi *et al.* [2] proved that each countable dense subset $T \subset \mathbb{S}^{d-1}$ of directions contains a (deterministic) sequence $(u_n)_{n \geq 1}$ S -rounding any given convex body A . This result is strengthened here and, using Theorem 3.1, it is extended to Minkowski symmetrizations.

Proposition 3.3. *Every countable dense subset $T \subset \mathbb{S}^{d-1}$ contains a universal sequence.*

Proof. Let $R > 0$. The set $\mathcal{K}_d(R)$ of convex bodies having the same volume as the unit ball D and whose circumradius is smaller than R is compact in (\mathcal{K}_d, d_H) . Given $\varepsilon > 0$, we consider a finite ε -net of $\mathcal{K}_d(R)$, say A_1, \dots, A_m with $m = m(\varepsilon, R)$. The result of [2] applied to A_1 ensures the existence of directions u_1, \dots, u_{n_1} of T such that

$$(1 - \varepsilon)D \subset S_{n_1}A_1 \subset (1 + \varepsilon)D. \tag{3.3}$$

Applied to $S_{n_1}A_2$, it provides directions $u_{n_1+1}, \dots, u_{n_2}$ of T such that

$$(1 - \varepsilon)D \subset S_{n_1+1, n_2}(S_{n_1}A_2) = S_{n_2}A_2 \subset (1 + \varepsilon)D.$$

Steiner symmetrization increases the inradius and decreases the circumradius. So (3.3) becomes

$$(1 - \varepsilon)D \subset S_{n_2}A_1 \subset (1 + \varepsilon)D.$$

Hence, we obtain by induction a sequence of $n = n(\varepsilon, R)$ directions $\{u_1, \dots, u_n\}$ of T satisfying, for $i = 1, \dots, m$,

$$(1 - \varepsilon)D \subset S_n A_i \subset (1 + \varepsilon)D. \tag{3.4}$$

Let A be a convex body belonging to $\mathcal{K}_d(R)$. Let A_{i_0} be an element of the ε -net of $\mathcal{K}_d(R)$ such that $d_H(A, A_{i_0}) < \varepsilon$. Recall that on \mathcal{K}_d the Nikodým distance d_N generates the same topology as d_H (see Appendix A). Inclusions (3.4) and Lemmas A.2 and A.3 imply that

$$\begin{aligned} d_N(S_n A, D) &\leq d_N(S_n A, S_n A_{i_0}) + d_N(S_n A_{i_0}, D) \\ &\leq d_N(A, A_{i_0}) + C d_H(S_n A_{i_0}, D) \\ &\leq C\varepsilon, \end{aligned} \tag{3.5}$$

where $C = C(d, R)$ is a positive constant. Now, given a decreasing sequence $(\varepsilon_k)_{k \geq 1}$ tending to 0, we apply the previous strategy to each term ε_k in order to obtain some directions, say $u_{n_k+1}, \dots, u_{n_{k+1}}$, satisfying

$$d_N(S_{n_k+1, n_{k+1}} A, D) \leq C\varepsilon_k.$$

Note that this inequality holds for any $A \in \mathcal{K}_d(R)$ and that the above constant C is the same as in (3.5). Concatenating the blocks $\{u_{n_k+1}, \dots, u_{n_{k+1}}\}$, $k \geq 1$, we build a sequence $(u_n)_{n \geq 1}$ strongly S -rounding any convex bodies with the same volume as D , i.e.

$$d_N(S_{l,m}A, D) \leq d_N(S_{n_k+1, n_{k+1}}(S_{l, n_k}A), D) \leq C(d, R(A))\varepsilon_k$$

whenever $l \geq n_k$ and $m \geq n_{k+1}$. Finally, we can affirm that $(u_n)_{n \geq 1}$ is universal thanks to the identity $S_u(rA) = rS_uA$ and Theorem 3.1.

A sequence $(u_n)_{n \geq 1}$ of \mathbb{S}^1 is said to be *uniformly distributed* on \mathbb{S}^1 if, for any arc I of the unit disc,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \text{card}\{n \leq m, u_n \in I\} = \sigma(I),$$

where σ denotes the Haar probability measure on \mathbb{S}^1 . In [1], a uniformly distributed sequence $(u_n)_{n \geq 1}$ on \mathbb{S}^1 was given (see Section 5) which does not S -round a certain convex body (see Example 2.1). By Theorem 3.1, this sequence is not universal.

Proposition 3.4. *There exists a uniformly distributed sequence $(u_n)_{n \geq 1}$ on \mathbb{S}^1 and a convex body A such that $(u_n)_{n \geq 1}$ does not M -round A .*

4. Random Minkowski symmetrizations

Let A be a convex compact set in \mathbb{R}^d . The goal of this section is to give the rate of convergence of

$$B_nA = B_{U_n}(\dots B_{U_2}(B_{U_1}A)\dots)$$

to $L(A)D$ when the random directions $U_k \in \mathbb{S}^{d-1}$, $k \geq 1$, are independent.

4.1. Rate of convergence

Let σ be the Haar probability measure on \mathbb{S}^{d-1} .

Theorem 4.1. *Assume that, for any $k \geq 1$, the distribution ν_k of U_k is absolutely continuous with respect to σ , and that its density satisfies*

$$\frac{d\nu_k}{d\sigma}(u) \leq \alpha < \frac{d}{d-1} \tag{4.1}$$

for some $\alpha > 0$ and σ -almost every $u \in \mathbb{S}^{d-1}$. Then there exists a constant $c > 0$ such that, with probability 1, there exists an $n_0(\omega)$ for all $n \geq n_0$ such that

$$d_H(B_nA, L(A)D) \leq e^{-cn}. \tag{4.2}$$

Furthermore, the first random integer n_0 from which the above inequality holds admits exponential moments.

Remark 4.1. Let us compare our result with Klartag’s [8] result. Theorem 1.3 of [8] states that, for any n , there exists n Minkowski symmetrizations transforming any convex body A into a convex body A_n whose distance to $L(A)D$ is smaller than $e^{-\delta n}$ (where δ is a positive constant). Theorem 4.1 offers an advantage over Klartag’s result: whereas only one (implicit) sequence of n directions satisfies Theorem 1.3 of [8], almost every realization of (U_1, \dots, U_n) satisfies (4.2).

The exponential decay holds for any n in Theorem 1.3 of [8] and for only a random integer in Theorem 4.1. However, this latter condition admits exponential moments.

Remark 4.2. It is worth pointing out here that any real number c such that

$$0 < c < -\frac{1}{2d} \log \frac{\alpha(d-1)}{d}$$

satisfies (4.2). See the proof of Theorem 4.1 below for details.

Remark 4.3. We note that Theorem 4.1 still holds when the volume of A is null.

Let h_A be the centered support function of A :

$$h_A = f_A - L(A).$$

Proposition 4.1 below is the basis of the proof of Theorem 4.1. It essentially states that $h_{B_U A}$ is a contraction when the random direction U is uniformly distributed on \mathbb{S}^{d-1} . The proof of Proposition 4.1 is rather technical and is given in Section 4.2.

Proposition 4.1. *Let U be a random variable in \mathbb{S}^{d-1} with distribution σ . Then*

$$\mathbb{E} \|h_{B_U A}\|_2^2 \leq \frac{d-1}{d} \|h_A\|_2^2. \tag{4.3}$$

Inequality (4.3) is actually an equality when $d = 2$ and $d \rightarrow \infty$. The case $d = 2$ is treated at the beginning of Section 4.2. In higher dimensions, a vector U chosen uniformly on \mathbb{S}^{d-1} is (almost) orthogonal to a given v with large probability:

$$\pi_U(v) = v - 2\langle v, U \rangle U$$

is close to v with a probability tending to 1. We can then check that $\mathbb{E} \|h_{B_U A}\|_2^2$ is larger than $\|h_A\|_2^2 - o(1)$ as $d \rightarrow \infty$.

Theorem 4.1 follows from Proposition 4.1 and the Borel–Cantelli lemma.

Proof of Theorem 4.1. Let ρ_k be the probability density function of v_k with respect to σ , and let $\alpha_d = \alpha(d-1)/d$. Hypothesis (4.1) and Proposition 4.1 applied to $B_{U_1} A$ and, thus, A imply that

$$\begin{aligned} \mathbb{E} \|h_{B_2 A}\|_2^2 &= \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{S}^{d-1}} \|h_{B_{u_2}(B_{u_1} A)}\|_2^2 \rho_2(u_2) \, d\sigma(u_2) \right) \rho_1(u_1) \, d\sigma(u_1) \\ &\leq \alpha_d \int_{\mathbb{S}^{d-1}} \|h_{B_{u_1} A}\|_2^2 \rho_1(u_1) \, d\sigma(u_1) \\ &\leq \alpha_d^2 \|h_A\|_2^2. \end{aligned}$$

By induction, it follows that, for any integer n ,

$$\mathbb{E} \|h_{B_n A}\|_2^2 \leq \alpha_d^n \|h_A\|_2^2.$$

In order to optimize the rate of convergence with respect to dimension d in Theorem 1.3 of [8], Klartag used technical lemmas to go from the L_2 -norm to the L_∞ -norm (see Section 4 of [8]). Here, we focus only on the parameter n . So, the following basic result proved at the end of this subsection will be sufficient.

Lemma 4.1. *Recall that $R(A)$ denotes the circumradius of A . Then, for any integer $n \geq 1$,*

$$\|h_{B_n A}\|_\infty^d \leq z_d \|h_{B_n A}\|_2, \tag{4.4}$$

where $z_d = (c_d)^{-1} 2^d R(A)^{d-1}$.

Lemma 4.1 allows us to bound the expectation of the L_∞ -norm of $h_{B_n A}$. Indeed,

$$\mathbb{E}\|h_{B_n A}\|_\infty^d \leq z_d \mathbb{E}\|h_{B_n A}\|_2 \leq z_d \sqrt{\mathbb{E}\|h_{B_n A}\|_2^2} \leq z_d \alpha_d^{n/2} \|h_A\|_2.$$

Hence,

$$\mathbb{E}\|h_{B_n A}\|_\infty \leq (z_d \|h_A\|_2 \alpha_d^{n/2})^{1/d}. \tag{4.5}$$

Markov’s inequality and (4.5) give

$$\mathbb{P}(\|h_{B_n A}\|_\infty > r^n) \leq r^{-n} \mathbb{E}\|h_{B_n A}\|_\infty \leq (z_d \|h_A\|_2)^{1/d} (r^{-1} \alpha_d^{1/2d})^n. \tag{4.6}$$

The real number $r > 0$ can be chosen such that $\alpha_d^{1/2d} < r < 1$ by hypothesis (4.1). Then the Borel–Cantelli lemma applies and we find that, a.s. for large enough n , $\|h_{B_n A}\|_\infty$ is smaller than r^n . Statement (4.2) follows from the identity

$$d_H(B_n A, L(A)D) = \|h_{B_n A}\|_\infty.$$

Finally, let us denote by n_0 the first (random) integer from which the Hausdorff distance between $B_n A$ and $L(A)D$ is smaller than r^n . It follows from (4.6) that n_0 admits exponential moments:

$$\mathbb{P}(n_0 > m) \leq (z_d \|h_A\|_2)^{1/d} (r^{-1} \alpha_d^{1/2d})^m.$$

Proof of Lemma 4.1. Classical properties of support functions (namely positive homogeneity of degree 1 and subadditivity; see [6, p. 57]) imply that $f_{B_n A}$ can be extended to a Lipschitz function defined on the whole space $(\mathbb{R}^d, \|\cdot\|_2)$. Its Lipschitz constant equals $\|f_{B_n A}\|_\infty$, i.e. its supremum over \mathbb{S}^{d-1} . Since all the $B_n A$ are included in $R(A)D$, the $f_{B_n A}$ are $R(A)$ -Lipschitz functions. So do the functions $h_{B_n A}$, $n \geq 1$.

To conclude, it suffices to note that the L_1 -norm of an $R(A)$ -Lipschitz function can be compared to its L_∞ -norm. Let $u_0 \in \mathbb{S}^{d-1}$ such that $|f(u_0)| = \|f\|_\infty$, and assume that $f(u_0) \geq 0$. Let $\mathcal{U} \subset \mathbb{S}^{d-1}$ defined by

$$\mathcal{U} = \left\{ u \in \mathbb{S}^{d-1}, \|u - u_0\|_2 \leq \frac{\|f\|_\infty}{2R(A)} \right\}.$$

On the one hand, there exists a constant $c_d > 0$ such that the Haar probability measure of \mathcal{U} satisfies

$$\sigma(\mathcal{U}) \geq c_d \left(\frac{\|f\|_\infty}{2R(A)} \right)^{d-1}.$$

On the other hand, for any point $u \in \mathcal{U}$, $f(u)$ is larger than $\frac{1}{2} \|f\|_\infty$. Henceforth,

$$\|f\|_2 \geq \|f\|_1 \geq \int_{\mathcal{U}} f(u) \, d\sigma \geq \frac{c_d \|f\|_\infty^d}{2^d R(A)^{d-1}}.$$

We can treat the case in which $f(u_0)$ is negative in the same way.

4.2. Proof of Proposition 4.1

Recall that the support function $f_{B_u A}$ can be expressed as the arithmetic mean of f_A and $f_{\pi_u A}$ (see (2.2)). Then, using the invariance of the Haar probability measure σ under the map $v \mapsto \pi_u(v)$ for any $u \in \mathbb{S}^{d-1}$, the L_2 -norm of $h_{B_u A}$ satisfies

$$\|h_{B_u A}\|_2^2 = \frac{1}{2} \|h_A\|_2^2 + \frac{1}{2} \langle h_A, h_{\pi_u A} \rangle.$$

Assume that U is distributed according to σ . By Fubini's theorem,

$$\mathbb{E}\langle h_A, h_{\pi_U A} \rangle = \int_{\mathbb{S}^{d-1}} h_A(v) \left(\int_{\mathbb{S}^{d-1}} h_A(\pi_u v) \, d\sigma(u) \right) \, d\sigma(v)$$

(indeed $f_{\pi_u A} = f_A \circ \pi_u$). Now, when $d = 2$, the probability measure σ is also invariant under the map $J_v : u \mapsto \pi_u(v)$ for any $v \in \mathbb{S}^1$. So, the integral

$$\int_{\mathbb{S}^1} h_A(\pi_u v) \, d\sigma(u)$$

is null and so is $\mathbb{E}\langle h_A, h_{\pi_U A} \rangle$. To summarize, Proposition 4.1 is easily proved in dimension $d = 2$ and

$$\mathbb{E}\|h_{B_U A}\|_2^2 = \frac{1}{2}\|h_A\|_2^2.$$

However, the above strategy does not hold whenever $d > 2$, since in this case the image measure σJ_v^{-1} admits a probability density function with respect to σ which is unbounded in the vicinity of v . Consequently, in order to prove Proposition 4.1, we follow the ideas of Klartag [8] based on spherical harmonics.

In the rest of this section we assume that $d > 2$. A polynomial P defined on \mathbb{R}^d is a *homogeneous harmonic of degree k* if P is a homogeneous polynomial of degree k and is harmonic (i.e. $\Delta P = 0$). Let \mathfrak{H}_k be the linear space

$$\mathfrak{H}_k = \{P|_{\mathbb{S}^{d-1}}, P \text{ is a homogeneous harmonic of degree } k\},$$

where $P|_{\mathbb{S}^{d-1}}$ denotes the restriction of the polynomial P to the sphere \mathbb{S}^{d-1} . The elements of \mathfrak{H}_k are called *spherical harmonics of degree k* . We refer the reader to [11] for complete references about spherical harmonics.

The linear space $L_2(\mathbb{S}^{d-1})$ admits the following orthogonal direct sum decomposition:

$$L_2(\mathbb{S}^{d-1}) = \bigoplus_{k \geq 0} \mathfrak{H}_k. \tag{4.7}$$

Let us write the centered support function h_A according to (4.7): $h_A = \sum g_k$. Thus,

$$h_{B_U A} = \frac{1}{2}(h_A + h_A \circ \pi_U) = \sum_{k \geq 0} B_U g_k, \tag{4.8}$$

where

$$B_U g_k = \frac{1}{2}(g_k + g_k \circ \pi_U).$$

First, it is clear that h_A is orthogonal to \mathfrak{H}_0 . So g_0 is null. Moreover, since $g_k \in \mathfrak{H}_k$, some elementary computations give $g_k \circ \pi_U \in \mathfrak{H}_k$. So does $B_U g_k$. Hence, (4.8) is the expansion of $h_{B_U A}$ into spherical harmonics, i.e. according to (4.7). Assume that U is distributed according to the Haar probability measure σ . Then, the result follows from Lemma 4.2 below and Pythagoras' theorem. That is,

$$\mathbb{E}\|h_{B_U A}\|_2^2 = \sum_{k \geq 1} \mathbb{E}\|B_U g_k\|_2^2 = \sum_{k \geq 1} \frac{d-2+k}{d-2+2k} \|g_k\|_2^2 \leq \frac{d-1}{d} \|h_A\|_2^2$$

since $(d-2+k)/(d-2+2k)$ is smaller than $(d-1)/d$ for any $k \geq 1$.

Lemma 4.2. *Let U be a random variable distributed according to σ . Let $k \geq 1$ and $g \in \mathfrak{S}_k$. Then*

$$\mathbb{E} \|B_U g\|_2^2 = \frac{d - 2 + k}{d - 2 + 2k} \|g\|_2^2,$$

where $B_u g = \frac{1}{2}(g + g \circ \pi_u)$.

The above identity is given but not proved in [8], and so we devote the rest of this section to its proof. For any $v \in \mathbb{R}^d$, \mathfrak{S}_k^v is defined as the set of elements $g \in \mathfrak{S}_k$ symmetric with respect to the hyperplan v^\perp :

$$\mathfrak{S}_k^v = \{g \in \mathfrak{S}_k, g \circ \pi_v = g\}.$$

Let us denote by $\text{Proj}_{\mathfrak{S}_k^v}$ the orthogonal projection onto \mathfrak{S}_k^v . Then the orthogonal projection of $g \in \mathfrak{S}_k$ is actually equal to $B_v g$.

Lemma 4.3. *For any $v \in \mathbb{R}^d$ and any $g \in \mathfrak{S}_k$, $B_v g = \text{Proj}_{\mathfrak{S}_k^v}(g)$.*

Let us consider the two orthonormal bases (e_1, \dots, e_d) and (v_1, \dots, v_d) in \mathbb{R}^d , and the isometry ψ mapping e_i to v_i for any $1 \leq i \leq d$.

Lemma 4.4. *For any $g \in \mathfrak{S}_k$, $\text{Proj}_{\mathfrak{S}_k^{v_1}}(g) = \text{Proj}_{\mathfrak{S}_k^{e_1}}(g \circ \psi) \circ \psi^{-1}$.*

Let $g \in \mathfrak{S}_k$. By Lemmas 4.3 and 4.4,

$$\|B_{v_1} g\|_2^2 = \|\text{Proj}_{\mathfrak{S}_k^{v_1}}(g)\|_2^2 = \|\text{Proj}_{\mathfrak{S}_k^{e_1}}(g \circ \psi)\|_2^2 = \sum_{i=1}^{\ell(k)} \left(\int_{\mathbb{S}^{d-1}} g \circ \psi(x) S_i(x) d\sigma(x) \right)^2, \tag{4.9}$$

where $\ell(k)$ and $(S_1, \dots, S_{\ell(k)})$ respectively denote the dimension and an orthonormal basis of $\mathfrak{S}_k^{e_1}$.

Furthermore, assume that an orthonormal basis (v_1, \dots, v_d) is chosen uniformly on the orthogonal group $\mathcal{O}(d)$. Then its first vector v_1 is distributed uniformly on the sphere \mathbb{S}^{d-1} , i.e. according to σ . Specifically, let μ be the Haar probability measure on $\mathcal{O}(d)$. Let us denote by Ψ the map from $\mathcal{O}(d)$ to \mathbb{S}^{d-1} defined by $\Psi(\psi) = \psi(e_1)$.

Lemma 4.5. *The image measure $\mu\Psi^{-1}$ is equal to σ .*

Assume that U is distributed according to σ . By Lemma 4.5,

$$\mathbb{E} \|B_U g\|_2^2 = \int_{\mathcal{O}(d)} \|B_{\Psi(\psi)} g\|_2^2 d\mu(\psi).$$

For any element ψ of $\mathcal{O}(d)$, set $v_1 = \psi(e_1)$. Hence, we replace $\|B_{\Psi(\psi)} g\|_2^2$ with (4.9):

$$\mathbb{E} \|B_U g\|_2^2 = \sum_{i=1}^{\ell(k)} \int_{\mathcal{O}(d)} \left(\int_{\mathbb{S}^{d-1}} g \circ \psi(x) S_i(x) d\sigma(x) \right)^2 d\mu(\psi).$$

It now suffices to apply Lemma 2.2 of [8] to ensure that each term of the above sum is equal to $\|g\|_2^2$ divided by the dimension of \mathfrak{S}_k . So,

$$\mathbb{E} \|B_U g\|_2^2 = \frac{\ell(k)}{\dim \mathfrak{S}_k} \|g\|_2^2.$$

We complete the proof of Proposition 4.1 by applying the following identities. The first identity is well known while the second identity easily follows from the proof of Lemma 3.1 of [8]:

$$\dim \mathfrak{S}_k = \frac{d - 2 + 2k}{d - 2 + k} \binom{d + k - 2}{d - 2} \quad \text{and} \quad \ell(k) = \dim \mathfrak{S}_k^{e_1} = \binom{d + k - 2}{d - 2}.$$

Proof of Lemma 4.3. Let $v \in \mathbb{R}^d$ and $g \in \mathfrak{S}_k$. Using $\sigma\pi_v^{-1} = \sigma$ and $f \in \mathfrak{S}_k^v$, we can write

$$\int_{\mathbb{S}^{d-1}} g(\pi_v x) f(x) \, d\sigma(x) = \int_{\mathbb{S}^{d-1}} g(\pi_v x) f(\pi_v x) \, d\sigma(x) = \int_{\mathbb{S}^{d-1}} g(x) f(x) \, d\sigma(x),$$

from which $\langle g - B_v g, f \rangle = 0$ follows.

Proof of Lemma 4.4. Previous notation leads to the identity $\psi \circ \pi_{e_1} \circ \psi^{-1} = \pi_{v_1}$. Thus, Lemma 4.3 gives the result:

$$\begin{aligned} \text{Proj}_{\mathfrak{S}_k^{e_1}}(g \circ \psi) \circ \psi^{-1}(x) &= \frac{1}{2}(g(x) + g(\psi \circ \pi_{e_1} \circ \psi^{-1}(x))) \\ &= \frac{1}{2}(g(x) + g(\pi_{v_1}(x))) \\ &= \text{Proj}_{\mathfrak{S}_k^{v_1}}(g). \end{aligned}$$

Lemma 4.5 is certainly known, but we have not found it in the literature.

Proof of Lemma 4.5. Let $U_1 \in \mathcal{O}(d)$, and consider the endomorphism \bar{U}_1 of the orthogonal group $\mathcal{O}(d)$ defined by $\bar{U}_1(V) = U_1 V$. It is then easy to see that $U_1 \circ \Psi = \Psi \circ \bar{U}_1$. Since the Haar probability measure μ is invariant under \bar{U}_1 , it follows that the image measure $\mu\Psi^{-1}$ is invariant under U_1 . This holds for any $U_1 \in \mathcal{O}(d)$: only the Haar probability measure σ can do it.

5. Random Steiner symmetrizations

Let A be a convex body in \mathbb{R}^d having the same volume as the unit ball D . The main result of this section gives the rate of convergence of

$$S_n A = S_{U_n}(\cdots S_{U_2}(S_{U_1} A) \cdots)$$

to D when the random directions $U_k \in \mathbb{S}^{d-1}$, $k \geq 1$, are independent. Recall that σ denotes the Haar probability measure on \mathbb{S}^{d-1} .

Theorem 5.1. *Assume that, for any $k \geq 1$, the distribution ν_k of U_k is absolutely continuous with respect to σ and that its density satisfies*

$$\frac{d\nu_k}{d\sigma}(u) \leq \alpha < \frac{d}{d-1} \tag{5.1}$$

for some $\alpha > 0$ and σ -almost every $u \in \mathbb{S}^{d-1}$. Then there exist two positive constants c and c' which depend only on d , A , and α such that, with probability 1, there exists an $n_0(\omega)$ for all $n \geq n_0$ such that

$$d_H(S_n A, D) \leq ce^{-c'\sqrt{n}}. \tag{5.2}$$

Furthermore, the first random integer n_0 from which the above inequality holds satisfies

$$\mathbb{P}(n_0 > m) \leq ce^{-c'\sqrt{m}}. \tag{5.3}$$

Remark 5.1. The comparison between Theorem 5.1 and Klartag’s [8] result (Theorem 1.5 of [8]) states that an implicit sequence of n Steiner symmetrizations transforms A into a new

convex body a distance smaller than $e^{-\delta\sqrt{n}}$ from D) is the same as that between Theorem 4.1 and Theorem 1.3 of [8]. See the first paragraph just after Theorem 4.1.

Remark 5.2. The almost-sure convergence (but without rate of convergence) of $S_n A$ to D in the case $v_k = \sigma$ was first proved in [9]. Volčič [13] recently extended this result to any probability measure assigning positive mass to any open subset of \mathbb{S}^{d-1} . Theorem 5.1 improves Volčič’s result in two directions. First, Theorem 5.1 does not require that the random directions are identically distributed. Second, the positivity hypothesis is relaxed here, since (5.1) allows the v_k to avoid some open subsets of \mathbb{S}^{d-1} . However, let us point out here that Volčič’s result concerns much more general sets (measurable or compact) than in our case. In the same way, (5.1) completes Condition (3.2) of [4].

Remark 5.3. We note that the independence hypothesis between random directions can be slightly weakened. Indeed, Theorem 5.1 still holds when the sequence $(U_n)_{n \geq 1}$ is a time-homogeneous Markov chain on \mathbb{S}^{d-1} whose transition probability kernel P is such that, for any $v \in \mathbb{S}^{d-1}$, the probability measure $P(v, \cdot)$ satisfies condition (5.1). The same is true for Theorem 4.1. See [10] for a general reference on Markov chains with continuous state space.

Remark 5.4. The identity $S_u(rA) = rS_u A$ for $r > 0$ allows us to extend Theorem 5.1 to convex bodies with any positive volume. When the volume of A is null, A lies in a proper subspace of \mathbb{R}^d . In this case, the Steiner symmetrization S_u and the orthogonal projection onto u^\perp coincide. Then it is not difficult to prove that the rate of convergence of $S_n A$ to the origin is exponential.

Remark 5.5. To obtain Theorem 3.4 of [13], Volčič proved that the *moment of inertia* of $S_n A$, i.e.

$$I(S_n A) = \int_{S_n A} \|z\|^2 d\lambda_d(z),$$

converges to the moment of inertia of D (where $\|\cdot\|$ denotes the Euclidean norm). We have

$$|I(S_n A) - I(D)| \leq R(A)^2 d_N(S_n A, D) \leq R(A)^2 e^{-c_7\sqrt{n}} \quad \text{a.s.,}$$

where we have used the rate of convergence given in (5.7) below.

As recalled in Section 2, the sequence $(L(S_n A))_n$ is nonincreasing. Hence, the sequence of corresponding expectations converges. The following proposition gives its limit and rate of convergence.

Proposition 5.1. *There exist two positive constants c_1 and c_2 which depend only on d , A , and α such that, for any n ,*

$$0 \leq \mathbb{E}L(S_n A) - 1 \leq c_1 e^{-c_2\sqrt{n}}. \tag{5.4}$$

Before proving Proposition 5.1 we first use it to prove Theorem 5.1.

Proof of Theorem 5.1. Recall that d_N denotes the Nikodým distance. Since $S_{2n} A$ and the unit ball D have the same volume, we can write

$$\begin{aligned} \frac{1}{2}d_N(S_{2n} A, D) &= \lambda_d(S_{2n} A \setminus D) \\ &\leq \lambda_d(B_{2n,n+1}(S_n A) \setminus D) \\ &\leq d_N(B_{2n,n+1}(S_n A), D) \\ &\leq d_N(B_{2n,n+1}(S_n A), L(S_n A)D) + d_N(L(S_n A)D, D). \end{aligned} \tag{5.5}$$

Now, let us bound the two terms of the sum in (5.5). If X_n denotes the Hausdorff distance between $B_{2n,n+1}(S_n A)$ and $L(S_n A)D$, then $B_{2n,n+1}(S_n A)$ contains the centered ball with radius $L(S_n A) - X_n$ and is contained in the ball with radius $L(S_n A) + X_n$. Hence, the first term of (5.5) is smaller than

$$\kappa_d((L(S_n A) + X_n)^d - (L(S_n A) - X_n)^d). \tag{5.6}$$

The inequalities $L(S_n A) \leq L(A)$ and $X_n \leq R(A) + L(A)$ allow us to bound (5.6) by $c_3 X_n$ for a suitable constant $c_3 = c_3(d, A) > 0$. The second term of (5.5) is treated in the same way:

$$d_N(L(S_n A)D, D) = \lambda_d(L(S_n A)D \setminus D) = \kappa_d(L(S_n A)^d - 1) \leq c_4(L(S_n A) - 1)$$

for a suitable constant $c_4 = c_4(d, A) > 0$. Combining the previous inequalities with Proposition 5.1 and (5.8) below we obtain

$$\mathbb{E}d_N(S_{2n}A, D) \leq 2c_3 a_1 a_2^n + 2c_4 c_1 e^{-c_2 \sqrt{n}}$$

(a_1 and a_2 are two positive constants depending on d, A , and α , and $a_2 < 1$). The same upper bound holds for the expectation of $d_N(S_{2n+1}A, D)$ since the Steiner symmetrization is a 1-Lipschitz function with respect to the Nikodým distance (see Lemma A.2). To summarize, there exist $c_5, c_6 > 0$ such that, for any n ,

$$\mathbb{E}d_N(S_n A, D) \leq c_5 e^{-c_6 \sqrt{n}}.$$

By Markov’s inequality and the Borel–Cantelli lemma, we deduce that there exists $0 < c_7 < c_6$ such that, with probability 1, for large enough n ,

$$d_N(S_n A, D) \leq e^{-c_7 \sqrt{n}}. \tag{5.7}$$

Finally, the passage from the Nikodým distance to the Hausdorff distance is ensured by Lemma A.4. With $r = (2R(A))^{-1}$, the quantity $d_H(S_n(rA), rD)$ is smaller than $\frac{1}{2}$ for any integer n . So, Lemma A.4 applies; thus, with probability 1,

$$\begin{aligned} d_H(S_n A, D) &= r^{-1} d_H(S_n(rA), rD) \\ &\leq C r^{-1} d_N(S_n(rA), rD)^{2/(d+1)} \\ &\leq C r^{2/(d+1)-1} e^{-2c_7 \sqrt{n}/(d+1)}. \end{aligned}$$

Statement (5.2) follows. To obtain (5.3), we proceed as in the proof of Theorem 4.1.

5.1. Proof of Proposition 5.1

Assume that there exists n such that $\beta := L(S_n A) < 1$. By Theorem 4.1, conditionally to $S_n A$,

$$B_{m,n+1}(S_n A) = B_{U_m}(\cdots B_{U_{n+1}}(S_n A) \cdots)$$

converges a.s. to βD as m tends to ∞ . Combining this with the fact that the Minkowski symmetrization of a given set increases its volume (recall (2.4)), it follows that

$$\text{vol}(D) > \beta^d \text{vol}(D) \geq \text{vol}(B_{m,n+1}(S_n A)) \geq \text{vol}(S_n A) = \text{vol}(A).$$

This contradicts the hypothesis that $\text{vol}(A) = \text{vol}(D)$ and gives the lower bound of (5.4).

The proof of the upper bound of (5.4) requires more work. First, we require the next lemma, which is a particular case of a result on quermassintegrals due to Bokowski and Heil [3, Theorem 2].

Lemma 5.1. (Corollary 6.2 of [8].) *Let $\varepsilon > 0$ and $K \subset (1 + \varepsilon)D$ be a convex body having the same volume as D . Then*

$$L(K) - 1 \leq r_d \varepsilon,$$

where $r_d = 1 - 1/d^2 < 1$.

Second, we need to check the expectation of $\|h_{B_{n+m,n+1}(S_n A)}\|_\infty$. Since $R(S_n A)$ is smaller than $R(A)$, Lemma 4.1 applies to $S_n A$ instead of A , but with the same constant as in (4.4), denoted by z_d . Thus, the following analogue of inequality (4.5) is obtained. For any integers m, n ,

$$\mathbb{E}\|h_{B_{n+m,n+1}(S_n A)}\|_\infty \leq a_1 a_2^m, \tag{5.8}$$

where $a_1 = (z_d R(A))^{1/d}$ and $a_2 = (\alpha(d - 1)/d)^{1/2d}$. This latter quantity being strictly smaller than 1 thanks to hypothesis (5.1).

Some additional constants have to be introduced. We set

$$\gamma = \frac{r_d + 1}{2} < 1, \quad b = \frac{\log \gamma}{\log a_2} > 0,$$

and $m \in \mathbb{N}$ such that

$$a_1 a_2^m \leq \frac{1 - r_d}{2r_d} (L(A) - 1).$$

Thus, by induction, we define a sequence of integers $(m_k)_{k \geq 0}$ by

$$m_0 = m \quad \text{and} \quad m_{k+1} = \lfloor m_k + b \rfloor + 1 \quad \text{for all } k \in \mathbb{N}$$

(where $\lfloor x \rfloor$ denotes the integer part of x), and a sequence of convex bodies $(A_k)_{k \geq 0}$ by $A_0 = A$, $A_1 = S_{m_0} A$, and, for any $k \geq 1$,

$$A_{k+1} = S_{\hat{m}_k, \hat{m}_{k-1} + 1} A_k, \quad \text{where} \quad \hat{m}_k = \sum_{i=0}^k m_i.$$

Roughly speaking, the passage from A_k to A_{k+1} is obtained after m_k Steiner symmetrizations. This process actually reduces the mean radius L . Specifically, we are going to prove that, for any $k \in \mathbb{N}$,

$$\mathbb{E}L(A_k) - 1 \leq \gamma^k (L(A) - 1). \tag{5.9}$$

The case $k = 0$ is obvious. Assume that (5.9) holds for a given $k \in \mathbb{N}$. Let us denote by X_k the Hausdorff distance between $B_{\hat{m}_k, \hat{m}_{k-1} + 1} A_k$ and $L(A_k)D$. Thanks to (5.8), the expectation of X_k is bounded by $a_1 a_2^{m_k}$. Furthermore, A_{k+1} is included in $B_{\hat{m}_k, \hat{m}_{k-1} + 1} A_k$, itself included in $(X_k + L(A_k))D$. So, we can apply Lemma 5.1 to A_{k+1} whose volume equals that of D :

$$L(A_{k+1}) - 1 \leq r_d (L(A_k) - 1 + X_k).$$

The induction hypothesis then gives

$$\mathbb{E}L(A_{k+1}) - 1 \leq r_d (\gamma^k (L(A) - 1) + a_1 a_2^{m_k}).$$

Now the sequence $(m_k)_{k \geq 0}$ has been built so that

$$a_1 a_2^{m_k} \leq \gamma a_1 a_2^{m_{k-1}} \leq \dots \leq \gamma^k a_1 a_2^m \leq \gamma^k \frac{1 - r_d}{2r_d} (L(A) - 1),$$

which finally provides

$$\mathbb{E}L(A_{k+1}) - 1 \leq \gamma^k(L(A) - 1) \left(r_d + \frac{1 - r_d}{2r_d} \right) = \gamma^{k+1}(L(A) - 1).$$

To conclude, it suffices to extend inequality (5.9) from A_k to $S_n A$. So, let $n \in \mathbb{N}$ be larger than m . Let us introduce the integer $k \geq 0$ satisfying

$$\hat{m}_k \leq n < \hat{m}_{k+1}.$$

The choice of k implies, on the one hand, that

$$\mathbb{E}L(S_n A) - 1 \leq \mathbb{E}L(S_{\hat{m}_k} A) - 1 = \mathbb{E}L(A_k) - 1 \leq \gamma^k(L(A) - 1)$$

by (5.9). On the other hand, it allows us to compare k and \sqrt{n} . Indeed,

$$n < \hat{m}_{k+1} \leq (k + 2)m + \frac{(k + 2)(k + 1)}{2}(b + 1) \leq c(k + 2)^2$$

for a suitable constant $c > 0$, depending only on m and b . This proves the upper bound of (5.4) for any $n \geq m$, with $c_1 = \gamma^{-2}(L(A) - 1)$ and $c_2 = -(\log \gamma)/\sqrt{c}$. Finally, it suffices to increase c_1 in order to obtain (5.4) for any n .

6. Open questions

The first open question concerns the rate of convergence of the random sequence $(S_n A)_{n \geq 1}$ to the corresponding ball: how far from optimal is the rate given in Theorem 5.1? However, no (strict) contraction property for Steiner symmetrization is available—one may expect an exponential rate.

Corollary 2 and Lemma 3.4 of [4] suggest that the almost-sure convergence of $(S_n A)_{n \geq 1}$ takes place for i.i.d. directions, provided the support of the common distribution contains a nonempty open set of the sphere \mathbb{S}^{d-1} . Is this condition sufficient to receive an assessment of the speed of convergence?

What about the rate of convergence of $(S_n A)_{n \geq 1}$ and $(B_n A)_{n \geq 1}$ when A is only assumed to be a compact set, or a set of finite measures?

Does there exist a stronger theorem of equivalence ensuring some relation between the rates of convergence of both sequences $(S_n A)_{n \geq 1}$ and $(B_n A)_{n \geq 1}$?

The counterexample given in [1] proves that an asymptotically uniformly distributed non-random sequence $(u_n)_{n \geq 1}$ on \mathbb{S}^1 does not always round off any given convex body. It would be interesting to find a reasonable strengthening of this condition which implies an asymptotic rounding of any convex body.

Appendix A. Metrics on \mathcal{K}_d

The Hausdorff distance provides a bridge between convex bodies and their support functions. Specifically, the mapping $\phi: A \mapsto f_A$ is an isometry from (\mathcal{K}_d, d_H) onto the subset $\phi(\mathcal{K}_d)$ of the space of continuous functions on \mathbb{S}^{d-1} endowed with the L^∞ -norm. See [6, p. 84].

Lemma A.1. *Let $A, B \in \mathcal{K}_d$. Then*

$$d_H(A, B) = \|f_A - f_B\|_\infty.$$

The reason we use the Nikodým distance in this paper is because the Steiner symmetrization is 1-Lipschitz with respect to it. See Lemma 2.2 of [13].

Lemma A.2. *Let $A, B \in \mathcal{K}_d$ and $u \in \mathbb{S}^{d-1}$. Then*

$$d_N(S_u A, S_u B) \leq d_N(A, B).$$

We complete this section by presenting two inequalities that compare the Hausdorff and Nikodým distances. The first inequality is Theorem (i) of [5], and so we omit its proof.

Lemma A.3. *There exists a positive constant $C = C(d, R)$ such that, for all $A, B \in \mathcal{K}_d$ included in the ball $B(0, R)$,*

$$d_N(A, B) \leq C d_H(A, B).$$

The following lemma is very similar to Theorem (iii) of [5].

Lemma A.4. *Let A be a convex body having the same volume as D and such that $d_H(A, D) \leq \frac{1}{2}$. Then there exists a positive constant $C = C(d)$ such that*

$$d_H(A, D) \leq C d_N(A, D)^{2/(d+1)}.$$

Proof. Let $r = d_H(A, D)$. There exists a vector $a \in A$ such that $\|a\|_2 = 1 \pm r$. We only treat the case $\|a\|_2 = 1 + r$ since the case $\|a\|_2 = 1 - r$ is similar. Let us consider the semi-infinite cone K_1 formed by all rays emanating from a and intersecting the ball $(1 - r)D$, and the outer half-space K_2 which is tangent to D at $a/\|a\|_2$. An elementary calculation shows that the set $K_1 \cap K_2$ is a right cone with height r over a $(d - 1)$ -dimensional ball with radius larger than $\sqrt{r}/4$ (because $r \leq \frac{1}{2}$). Hence, the volume of $A \setminus D$ which contains $K_1 \cap K_2$ is larger than $C r^{(d+1)/2}$, where $C = C(d)$ is a positive constant. To conclude, we use the identity

$$d_N(A, D) = 2\lambda^d(A \setminus D)$$

since A and D have the same volume.

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