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NIL AND s-PRIME Ω-GROUPS

A. BUYS and G. K. GERBER

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Abstract

The concepts nilpotent element, s-prime ideal and s-semi-prime ideal are defined for Ω-groups. The class $\{G|G$ is a nil Ω-group$\}$ is a Kurosh-Amitsur radical class. The nil radical of an Ω-group coincides with the intersection of all the s-prime ideals. Furthermore an ideal P of G is an s-semi-prime ideal if and only if G/P has no non-zero nil ideals.

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1. Notation and Definitions

Throughout this paper we shall use the definitions of Higgins [4]. Whenever we refer to G it is meant to be an Ω-group. By $P \lhd G$ we mean that P is an ideal of G, and $\mathbf{g} = (g_1, g_2, \ldots, g_n) \in G$ means that $g_i \in G$ for $i = 1, 2, \ldots, n$. Higgins [4] called words which involve only the operations $\omega \in \Omega$ monomials. We shall call monomials Ω-words.

1.1 Definition. Let Ω be a fixed set of operations; $\omega \in \Omega$ will be called a trivial operation in the variety K of Ω-groups if $\mathbf{x}\omega = 0$ is satisfied in K, that is, if for all $G \in K$ and for all $\mathbf{a} \in G$, $\mathbf{a}\omega = 0$ holds. We call $\omega \in \Omega$ a non-trivial operation if it is not trivial.

An Ω-word involving only non-trivial operations will be called a non-trivial Ω-word. For example, in the variety of R-modules 0 is a trivial operation. In the

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variety of rings the binary operation multiplication is non-trivial although it may sometimes be zero. If $f(\mathbf{x})$ is any word, then $f(x, x, \ldots, x)$ will be denoted by $f(x)$. Note that if $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are non-trivial Ω-words then so is $f_2(f_1(\mathbf{x}))$. If $S \subseteq G$ and $f(\mathbf{x})$ is an Ω-word then $f(S) = \{ f(\mathbf{s}) \mid \mathbf{s} \in S \}$.

2. Nil Ω-groups

2.1 Definition. An element $a \in G$ will be called nilpotent if there exists a non-trivial Ω-word $f(\mathbf{x})$ such that $f(a) = 0$. We call $S \subseteq G$ a nil subset of G if each element of S is nilpotent, and $S \subseteq G$ will be called a nilpotent subset of G if there exists a non-trivial Ω-word $f(\mathbf{x})$ such that $f(S) = 0$.

2.2 Corollary. 1. *If $S \subseteq G$ is nilpotent then S is a nil subset of G.*

2. *If $S \subseteq T \subseteq G$ and T is a nil (nilpotent) subset of G then S is a nil (nilpotent) subset of G.*

3. *If $f(a) = 0$ for $a \in G$ and an Ω-word $f(\mathbf{x})$ then $g(f(a)) = 0$ for all Ω-words $g(\mathbf{y})$.*

Definition 2.1 generalizes the definition of nilpotent elements for rings and near-rings. Coppage and Luh [2] defined nilpotent elements for Γ-rings. A nilpotent element according to their definition will always be nilpotent in the sense of 2.1. The converse is not true as can be verified by looking at the residue classes modulo 6 considered as a Γ-ring. An element of a non-associative ring is nilpotent if there exists a product of a with itself, taken n times for a certain arrangement of the parentheses, which is zero. An element m of the R-module M is nilpotent if there exist $0 \neq r_i \in R$ such that $mr_1r_2 \cdots r_n = 0$. Thus an R-module which is not faithful is nilpotent. Conversely, an R-module which is nilpotent is not faithful if R has no zero-divisors. An element a of the algebra A over the ring R will be nilpotent if $a^n = 0$ or if $a^k r_1 r_2 \cdots r_m = 0$ where $0 \neq r_i \in R$, $i = 1, 2, \ldots, m$ and n and k are natural numbers.

2.3 Lemma. *Let $I \lhd G$. Then G is nil (nilpotent) if and only if I and G/I are nil (nilpotent).*

Proof (for nilpotent). Suppose G is nilpotent. From 2.2 it follows that I is nilpotent. Since G is nilpotent there is a non-trivial Ω-word $f(\mathbf{x})$ such that $f(G) = 0$. From Higgins [4], Theorem 3A, it follows that $f(G/I) = 0$. Conversely, if I and G/I are nilpotent, there exist non-trivial Ω-words $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ such

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that $f_1(I) = 0$ and $f_2(G/I) = 0$. But $f_2(G/I) = \{f_2(\mathbf{a}) + I | \mathbf{a} \in G\}$ and therefore $f_2(\mathbf{a}) \in I$ for all $\mathbf{a} \in G$, that is $f_2(G) \subseteq I$. From 2.2 it follows that $f_1(f_2(G)) = 0$. Since $f_1(f_2(\mathbf{x}))$ is a non-trivial Ω-word, G is nilpotent. The proof for nil is similar.

2.4 LEMMA. *Let I, $J \triangleleft G$ be nil (nilpotent). Then $I + J$ is a nil (nilpotent) ideal of G.*

PROOF. The proof follows from $I + J/I \cong J/J \cap I$ (Higgins [4], Theorem 3C) and 2.3.

2.5 COROLLARY. *A finite sum of nil (nilpotent) ideals is a nil (nilpotent) ideal. The sum of all nil ideals of an Ω-group G is a nil ideal.*

Rjabuhin ([7], Definition p. 151) called a radical class $\mathscr{R}$ absolutely hereditary if for every Ω-sub-group A of G, $G \in \mathscr{R}$, it follows that $A \in \mathscr{R}$.

2.6 THEOREM. *The class $\mathscr{G} = \{G | G$ is a nil Ω-group$\}$ is an absolutely hereditary radical class.*

PROOF. Properties R3, R5 and R7 of Rjabuhin [7] respectively follow from 2.3, 2.5 and 2.3. From Rjabuhin [7], Theorem 1.2 it follows that $\mathscr{G}$ is a radical class. From 2.2 it follows that $\mathscr{G}$ is an absolutely hereditary class.

3. s-prime and s-semi-prime ideals

3.1 DEFINITION. $U^* \subseteq G$ is called a complete system if for each $a \in U^*$ it follows that $f(a) \in U^*$ for all non-trivial Ω-words $f(\mathbf{x})$.

$S^* \subseteq G$ is called an Ω-system if S^* is closed with respect to all non-trivial $\omega \in \Omega$.

$S \subseteq G$ will be called an s-system (a u-system) if S contains an Ω-system (a complete system) S^*, called the kernel of S, such that for each $s \in S$ it follows that $s^G \cap S^* \neq \varnothing$ where s^G is the ideal generated by s in G. $S(S^*)$ denotes an s-system S with kernel S^*. Similarly for u-systems, $U(U^*)$.

3.2 COROLLARY 1. *Each Ω-system is a complete system.*

2. *Each s-system is a u-system.*

3. *Each Ω-system (complete system) is an s-system (a u-system).*

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The concept u-system generalizes the concept u-system as defined by Le Roux [6], for rings and generalized by Groenewald [3], for near-rings. s-systems for rings and near-rings have been defined by Van der Walt [8], [9].

3.3 DEFINITION. An ideal P of G is called an s-prime (s-semi-prime) ideal if $\mathscr{C}(P)$ is an s-system (a u-system). G is an s-prime (s-semi-prime) Ω-group if 0 is an s-prime (s-semi-prime) ideal.

It immediately follows that every s-prime ideal is also an s-semi-prime ideal.

3.4 LEMMA. *Let $P \lhd G$. For any Ω-word $f(\mathbf{x})$ and for any $\mathbf{p} \in P$, $\mathbf{g} \in G$, there exists a $p^* \in P$ such that $f(\mathbf{p} + \mathbf{g}) = p^* + f(\mathbf{g})$.*

PROOF. The proof will be by mathematical induction on n, the number of operations involved in $f(\mathbf{x})$. If $n = 1$, then $f(\mathbf{x}) = \mathbf{x}\omega$ for some $\omega \in \Omega$. Then

$$\begin{aligned}(\mathbf{p} + \mathbf{g})\omega &= (\mathbf{p} + \mathbf{g})\omega - \mathbf{g}\omega + \mathbf{g}\omega \\ &= p^* + \mathbf{g}\omega \quad \text{where } p^* = (\mathbf{p} + \mathbf{g})\omega - \mathbf{g}\omega \\ &\in P \quad (\text{Kurosh [5], Section 8.4}).\end{aligned}$$

Suppose the lemma is true for all Ω-words involving k or less than k operations. Let $f(\mathbf{x})$ be any Ω-word involving $k + 1$ operations. Then $f(\mathbf{x})$ must be of the form $(f_1(\mathbf{x})f_2(\mathbf{x}) \cdots f_n(\mathbf{x}))\omega$ where each $f_i(\mathbf{x})$ involves k or less than k operations. Note that some of the $f_i(\mathbf{x})$ could be identity words. Then

$$\begin{aligned}f(\mathbf{p} + \mathbf{g}) &= (f_1(\mathbf{p} + \mathbf{g})f_2(\mathbf{p} + \mathbf{g}) \cdots f_n(\mathbf{p} + \mathbf{g}))\omega \\ &= (p_1 + f_1(\mathbf{g}))(p_2 + f_2(\mathbf{g})) \cdots (p_n + f_n(\mathbf{g}))\omega \\ &= p^* + (f_1(\mathbf{g})f_2(\mathbf{g}) \cdots f_n(\mathbf{g}))\omega \\ &= p^* + f(\mathbf{g})\end{aligned}$$

where p_i, $p^* \in P$, $i = 1, 2, \ldots, n$. Thus the lemma follows.

Note that the order is not important in $f(\mathbf{p} + \mathbf{g}) = p^* + f(\mathbf{g})$ since P is a normal divisor of G.

3.5 LEMMA. *Let $A \lhd G$ and $S(S^*)$ be a non-empty s-system (u-system) such that $A \cap S = \varnothing$. Then there exists a maximal s-prime (s-semi-prime) ideal P such that $A \subseteq P$ and $P \cap S = \varnothing$.*

PROOF (for s-systems). Let $\mathscr{I} = \{I \lhd G \mid A \subseteq I \text{ and } I \cap S = \varnothing\}$. Applying Zorn's lemma it follows that $\mathscr{I}$ has a maximal element P (say). We show that P is an s-prime ideal. Let $S_1^* = S^* + P = \{s + p \mid s \in S^*, p \in P\}$. We first show that S_i^*

is an Ω-system. Let $\omega \in \Omega$ be non-trivial and $a_1, a_2, \ldots, a_n \in S_1^*$. Then $a_i = s_i + p_i$ where $s_i \in S^*$ and $p_i \in P$, $i = 1, 2, \ldots, n$. From 3.4 it follows that $a_1 a_2 \cdots a_n \omega \in S_1^*$. $S_1^* \cap P = \varnothing$ since if $a \in S_1^* \cap P$ then $a = s + p$, $s \in S^*$, $p \in P$. Thus $s = a - p \in P$ which is a contradiction. Therefore, $S_1^* \subseteq \mathscr{C}(P)$. Let $S_1 = \mathscr{C}(P)$ and $a \in S_1$. Then $a \notin P$ and thus a $a^G + P$ is an ideal such that $(a^G + P) \cap S(S^*) \neq \varnothing$. It follows that we can find $p \in P$, $a' \in a^G$ such that $s = a' + p \in S$. Then there exists an $s^* \in s^G \cap S^*$ and $s^* \in s^G \subseteq a^G + P$. Thus $s^* \in (a^G + P) \cap S^*$, that is $s^* = a^* + p_1$ where $p \in P$ and $a^* \in a^G$. But then $a^* = s^* - p_1 \in S_1^*$. It follows that $S_1(S_1^*)$ is an s-system and thus that P is an s-prime ideal. The proof for u-systems is similar.

3.6 Lemma. *If $\{P_\alpha | \alpha \in A\}$ is a family of s-semi-prime ideals then $\bigcap_{\alpha \in A} P_\alpha$ is an s-semi-prime ideal of G.*

Proof. We have to show that $\mathscr{C}(\bigcap_{\alpha \in A} P_\alpha) = \bigcup_{\alpha \in A} \mathscr{C}(P_\alpha)$ is a u-system. $\mathscr{C}(P_\alpha)$ is a u-system with kernel U_α (say). Let $U^* = \bigcup_{\alpha \in A} U_\alpha$. Then U^* is a complete system and $U^* \subseteq U = \bigcup_{\alpha \in A} \mathscr{C}(P_\alpha)$. It easily follows that $U(U^*)$ is a u-system.

3.7 Corollary. *An intersection of s-prime ideals is an s-semi-prime ideal.*

3.8 Lemma. *$I \triangleleft G$ is an s-prime (s-semi-prime) ideal if and only if G/I is an s-prime (s-semi-prime) Ω-group.*

3.9 Lemma. *If $U(U^*)$ is a u-system and $s \in U(U^*)$ then there exists an s-system $S(S^*)$ such that $s \in S(S^*) \subseteq U(U^*)$.*

Proof. Since $s \in U(U^*)$ there exists an $s^* \in s^G \cap U^*$. Let $S = \{s, s^*\} \cup \{f(s^*) | f(\mathbf{x})$ a non-trivial Ω-word$\}$ and $S^* = \{s^*\} \cup \{f(s^*) | f(\mathbf{x})$ a non-trivial Ω-word$\}$. $S(S^*)$ is an s-system such that $s \in S(S^*) \subseteq U(U^*)$.

3.10 Theorem. *Let $P \triangleleft G$. P is an s-semi-prime ideal in G if and only if G/P contains no non-zero nil ideals.*

Proof. Suppose G/P contains no non-zero nil ideals. Let $U^* = \{r \in \mathscr{C}(P) | f(r) \in \mathscr{C}(P)$ for all non-trivial Ω-words $f(\mathbf{x})\}$. U^* is a complete system by definition. Put $U = \mathscr{C}(P)$. We shall show that $U(U^*)$ is a u-system.

Let $a \in U$. Since G/P has no non-zero nil ideals $a^G + P/P$ is not a nil ideal. Thus there exists an $a_1 \in a^G$ such that $f(a_1) \notin P$ for all non-trivial Ω-words. Furthermore $a_1 \notin P$, for if $a_1 \in P$ then $f(a_1) \in P$ for all Ω-words $f(\mathbf{x})$ since P is an ideal. Thus $a_1 \in U^*$ and we have shown that $U(U^*)$ is a u-system. Thus P is an s-semi-prime ideal.

Suppose P is an s-semi-prime ideal. From 3.8 it follows that G/P is an s-semi-prime Ω-group. Thus $\mathscr{C}(0)$ is a u-system with kernel U^* (say) in G/P. Suppose A/P is a non-zero nil ideal of G/P. For each non-zero element $a + P$ in A/P there exists a non-trivial Ω-word $f(\mathrm{x})$ such that $f(a + P) = 0$. But $a + P \neq 0$ and therefore $(a + P)^{G/P} \cap U^* \neq \varnothing$. But

$$(a + P)^{G/P} = (a\theta)^{G\theta} = (a^G)\theta \quad \text{(Higgins [4], Lemma 3.1)}$$
$$= a^G + P/P \quad \text{where } \theta \text{ is the natural homomorphism.}$$

Therefore there exists a $b \in (a^G + P)/P$ such that $b \in U^*$ and also $f(b) \in U^*$ for all non-trivial Ω-words $f(\mathrm{x})$. Thus $f(b) \neq 0$ for all non-trivial Ω-words $f(\mathrm{x})$. It follows that A/P is not a nil ideal, which is a contradiction. Therefore G/P has no non-zero nil ideals

3.11 Definition. Let $A \triangleleft G$. The s-radical of A, written $s(A)$, is the set of all $g \in G$ with the property that each s-system S with $g \in S$ has non-empty intersection with A. $s(0)$ will be called the s-radical of G. In particular, $s(0) = \{g \in G|$ $0 \in S$ for each s-system S with $g \in S\}$.

3.12 Corollary 1. $A \subseteq s(A)$.
2. *A and $s(A)$ are contained in precisely the same s-prime ideals.*

The proof is similar to that of Van der Walt ([9], Theorem 3).

3.13 Theorem. *$s(A) = \bigcap\{P \triangleleft G | P \supseteq A$ and P is an s-prime ideal$\}$ and in particular $s(0) = \bigcap\{P \triangleleft G | P$ is an s-prime ideal$\}$.*

Proof. The proof is similar to that of Van der Walt ([9, Theorem 3]).

Using 3.7 we get:

3.14 Corollary. *$s(A)$ is an s-semi-prime ideal.*

3.15 Theorem. *$s(A)/A$ is a nil ideal in G/A and $s(A)$ is precisely the set of $a \in G$ such that $a^G + A/A$ is a nil ideal in G/A.*

Proof. Let $a \in s(A)$ such that $a \notin A$ (that is $a + A$ is a non-zero element of $s(A)/A$). Now $K = \{a\} \cup \{f(a) | f(\mathrm{x})$ a non-trivial Ω-word$\}$ is an Ω-system and thus also an s-system with $a \in K$. It follows that $A \cap K \neq \varnothing$. Since $a \notin A$ there exists a non-trivial Ω-word $f(\mathrm{x})$ such that $f(a) \in A$. But then $a + A$ is nilpotent in G/A and therefore $s(A)/A$ is a nil ideal of G/A. By 2.2, each element of $s(A)/A$ generates a nil ideal in G/A.

Let $a \notin s(A)$. Then there exists an s-system $S(S^*)$ such that $S \cap A = \varnothing$ and $a \in S$. But then there exists an $a^* \in a^G \cap S^*$ and therefore $f(a^*) \in S^*$ for all non-trivial Ω-words $f(\mathbf{x})$. Since $A \cap S^* = \varnothing$ it follows that $a^G + A/A$ is not a nil ideal in G/A.

3.16 COROLLARY 1. *$s(0)$ is a nil ideal.*
2. *$s(A)$ equals the sum of all the ideals B such that B/A is a nil ideal in G/A.*

In particular $s(0)$ concides with the sum of all the nil ideals of G, that is with the nil radical of G. Thus the property "The Ω-group G is equal to its s-radical $s(0)$" is a radical property.

3.17 THEOREM. *An ideal P of G is an s-semi-prime ideal if and only if $s(P) = P$.*

PROOF. If $s(P) = P$ the result follows from 3.7 and 3.13. Conversely, suppose P is an s-semi-prime ideal. If $P = G$ then the result follows. If $P \neq G$ suppose $P \subset s(P)$. Then there exists an $a \in s(P)$ such that $a \notin P$. But $\mathscr{C}(P)$ is a non-empty u-system and by 3.9 there exists an s-system S such that $a \in S \subseteq \mathscr{C}(P)$. Since $a \in s(P)$ we have $P \cap S \neq \varnothing$ which is a contradiction. Therefore $s(P) = P$.

3.18 COROLLARY. *$P \triangleleft G$ is an s-semi-prime ideal if and only if P is an intersection of s-prime ideals.*

3.19 DEFINITION. An s-prime ideal P is a quasi-minimal s-prime ideal belonging to the ideal A if $A \subseteq P$ and there exists a kernel S^* for the s-system $S = \mathscr{C}(P)$ such that if S_1^* is any Ω-system properly containing S^* then $S_1^* \cap A \neq \varnothing$.

3.20 THEOREM. *$s(A)$, the s-radical of $A \triangleleft G$, is the intersection of all the quasi-minimal s-prime ideals belonging to A.*

PROOF. The theorem will follow from 3.13 if we can show that each s-prime ideal P containing A also contains a quasi-minimal s-prime ideal belonging to A.

Let P be an s-prime ideal containing A. By definition $\mathscr{C}(P)$ is an s-system, $S(S^*)$ say, such that $S \cap A = \varnothing$. Consider

$$\mathscr{S} = \{T^* | T^* \supseteq S^* \text{ and } T^* \cap A = \varnothing \text{ where } T^* \text{ is an } \Omega\text{-system}\}.$$

By applying Zorn's lemma to $\mathscr{S}$ it easily follows that there exists a maximal Ω-system S_1^* such that $S_1^* \cap A = \varnothing$ and $S_1^* \supseteq S^*$. Define $S_1 = \{a \in G | a^G \cap S_1^* \neq \varnothing\}$. Then $S_1^* \subseteq S_1$ and $S_1(S_1^*)$ is an s-system. $S_1 \cap A = \varnothing$ because if $a \in S_1 \cap A$ then there exists an $a^* \in a^G \cap S_1^*$ and $a^* \in a^G \subseteq A$ which is a contradiction. We now show that $S_1(S_1^*)$ is the complement of an s-prime ideal containing

A. From 3.5 it follows that there exists a maximal s-prime ideal B such that $A \subseteq B$ and $B \cap S_1 = \varnothing$. It follows that $S_2 = \mathscr{C}(B)$ is an s-system with $S_1 \subseteq S_2$. From the proof of 3.5 it follows that $S_1^* \subseteq S_2^*$ (where $S_2^* = S_1^* + B$). But S_1^* is a maximal Ω-system such that $A \cap S_1^* = \varnothing$. Therefore $S_1^* = S_2^*$. From the definition of S_1 it follows that $S_1 = S_2 = \mathscr{C}(B)$. Thus $B = \mathscr{C}(S_1)$ is a quasi-minimal s-prime ideal belonging to A such that $B \subseteq P$.

3.21 COROLLARY. *$P \lhd G$ is an s-semi-prime ideal if and only if P is the intersection of quasi-minimal s-prime ideals belonging to P.*

3.22 COROLLARY. *If $P \lhd G$ then $s(P)$ is the smallest s-semi-prime ideal in G (in the set theoretic sense) which contains P.*

From Rjabuhin ([7], 3.13 and 3.8) it follows

3.23 THEOREM. *The s-radical of G, $s(0)$, is zero (that is G is nil semi-simple) if and only if G is the subdirect sum of s-prime Ω-groups.*

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Department of Mathematics
University of Port Elizabeth
6000 Port Elizabeth
Republic of South Africa