DIRECT EXPRESSION OF INCOMPATIBILITY IN CURVILINEAR SYSTEMS

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Abstract

We would like to present a method to compute the incompatibility operator in any system of curvilinear coordinates (components). The procedure is independent of the metric in the sense that the expression can be obtained by means of the basis vectors only, which are first defined as normal or tangential to the domain boundary, and then extended to the whole domain. It is an intrinsic method, to some extent, since the chosen curvilinear system depends solely on the geometry of the domain boundary. As an application, the in-extenso expression of incompatibility in a spherical system is given.

1. Introduction

Let us consider a smooth body $\Omega \subset \mathbb{R}^3$. The incompatibility, denoted as inc, is a well-known operator in elasticity, since, as applied to the linearized strain tensor $\boldsymbol{\varepsilon}$, it determines whether the strain is derived from a displacement field. Specifically, let the elastic strain be obtained by a constitutive law from the stress tensor $\boldsymbol{\sigma}$: that is, $\boldsymbol{\varepsilon} = \mathbb{C}\boldsymbol{\sigma}$ with $\mathbb{C}$ the compliance fourth-rank tensor. Then inc $\boldsymbol{\varepsilon} = 0$ if and only if $\boldsymbol{\varepsilon} = \boldsymbol{\nabla}^S \boldsymbol{u}$ for some displacement field $\boldsymbol{u}$. If, on the contrary, it does not vanish, then Kröner's works [10] tell us that dislocations are present, preventing the existence of a well-defined displacement field defined in the whole body. In general, a dislocation is a three-dimensional (3D) line singularity for the strain field, reducing to a straight line in some simplified cases where a two-dimensional (2D) treatment is sufficient, (see, for example, [18]). Specifically, Kröner's relation reads

$$\operatorname{curl} \boldsymbol{\kappa} = \operatorname{inc} \boldsymbol{\varepsilon}, \tag{1.1}$$

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where the contortion tensor $\boldsymbol{\kappa}$ is related to the tensor-valued dislocation density $\boldsymbol{\Lambda}$ by $\boldsymbol{\kappa} = \boldsymbol{\Lambda} - (1/2) \operatorname{tr} \boldsymbol{\Lambda} \mathbb{I}_2$. At the mesoscopic scale, the dislocation density reads $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\mathcal{L}} = \boldsymbol{\tau} \otimes \boldsymbol{b} \mathcal{H}^1_{\lfloor \mathcal{L}}$, where $\mathcal{H}^1_{\lfloor \mathcal{L}}$ stands for the one-dimensional Hausdorff measure concentrated in the dislocation loop $\mathcal{L}$. At the mesoscale, Kröner's relation also holds, as proved by the author [15, 16]. At the macro (or continuous) scale, (which is the scale considered in the present work), $\boldsymbol{\Lambda}$ is a smooth tensor obtained from its mesoscopic counterpart by some regularization. The fact that, at the mesoscale, dislocations are closed loops or end at the boundary implies that $\operatorname{div} \boldsymbol{\Lambda}^t = 0$. However, $\operatorname{div} \boldsymbol{\kappa}^t \neq 0$ except in particular cases, for instance, if one considers pure edge dislocation loops in 3D (that is, satisfying $\operatorname{tr} \boldsymbol{\Lambda} = 0$) and, therefore, the knowledge of the right-hand side of (1.1) is, in general, not sufficient to uniquely determine the field $\boldsymbol{\kappa}$. Note that, in this case, the Frank tensor $\operatorname{curl}^t \boldsymbol{\varepsilon}$ and the dislocation density are unequivocally related, since equation (1.1) reduces to $\operatorname{curl}^t \boldsymbol{\varepsilon} = \boldsymbol{\kappa}$ in Ω with $\boldsymbol{\varepsilon} \times N = 0$ on $\partial \Omega$, by virtue of (1.4) and a uniqueness result, as proved by Scala and Van Goethem [14].

Being a symmetrical tensor, the elastic strain satisfies Beltrami decomposition [12]

$$\boldsymbol{\varepsilon} = \boldsymbol{\nabla}^S \boldsymbol{u} + \boldsymbol{\varepsilon}^0, \tag{1.2}$$

where $\boldsymbol{u}$ is a vector field and where $\boldsymbol{\varepsilon}^0 = \operatorname{inc} \boldsymbol{F}$ represents, in a Cartesian system, the incompatible part of the strain for some symmetrical and solenoidal second-rank tensor $\boldsymbol{F}$. Thus, by (1.1) and (1.2), the field $\boldsymbol{F}$, related to the presence of dislocations, satisfies an equation of the form

$$\operatorname{inc} \operatorname{inc} \boldsymbol{F} = \operatorname{inc}(\operatorname{inc} \boldsymbol{F}) = \operatorname{curl} \boldsymbol{\kappa}, \tag{1.3}$$

where "inc inc" stands for the application of the "inc" operator twice. Equation (1.3) was proved by Amstutz and Van Goethem [1] to be well posed (with appropriate boundary conditions on $\boldsymbol{F}$ and $\operatorname{curl}^t \boldsymbol{F} \times N$), provided the dislocation density is given (here we assume that $\boldsymbol{\kappa}$ is known, for instance, by solving a transport–reaction–diffusion equation, as done for point defects in [17]).

The incompatibility operator on second-rank tensors is classically defined as

$$\operatorname{inc} \boldsymbol{T} = \operatorname{curl} \operatorname{curl}^t \boldsymbol{T}, \tag{1.4}$$

meaning (in a Cartesian system) that the curl is taken over the rows and the columns of a second-rank tensor $\boldsymbol{T}$, consecutively. In the present work, our concern is to compute the incompatibility in a subset of Ω, say, an inclusion, whose shape might be arbitrary or the solution of a geometric optimization problem. For this reason, there is a need to express incompatibility in curvilinear systems, chosen to fit the inclusion geometry.

In a general curvilinear system, the same definition (1.4) holds, but care must be taken, since the covariant derivatives do not, in general, commute, because the basis vectors depend on the position and hence must also be differentiated in (1.4). The general second-rank tensor $\boldsymbol{T}$, in the Cartesian basis, is written as $\boldsymbol{T} = T^{\mathrm{CART}}_{ij} \boldsymbol{e}^i \otimes \boldsymbol{e}^j$ and, in the curvilinear basis, as $\boldsymbol{T} = T^{\mathrm{CURV}}_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j$. As explained by Amstutz and Van Goethem [1], the basis $\{\boldsymbol{g}^i\}_i = \{N, \boldsymbol{\tau}^R\}$ for $R = A, B$, where $\boldsymbol{\tau}^R$ are tangential to the

boundary and N is its unit outwards normal, is first defined on the domain boundary $\partial\Omega$ and then extended to Ω, where its differentials can be computed. This latter operation gives rise to five numbers: κ^R, the two surface curvatures, γ^R, the two divergences of τ^R and ξ, the deviation with respect to the principal directions. From these five numbers, the Christoffel symbols [6] can be found and hence the covariant derivatives. Thus the expression for the curvilinear curl and, eventually, for the incompatibility can be found. An important preliminary step is to express the differentials of the basis vectors in terms of κ^R, γ^R, ξ, which is given in Theorem 3.2 (the proof of which can be found in [1]). The curvilinear coordinates are simply the abscissae of the curves with tangent vector τ^R and radial coordinate r associated to N.

Expressions for the incompatibility in a general curvilinear system are rarely found in-extenso in the literature. Let us mention Malvern's textbook, where an expression can be found [13, Appendix II], given in terms of the metric factors h_i defining the intrinsic metric. Our approach can be considered as a metric-free alternative to Malvern's approach, and we base our method on the sole geometry of the domain boundary and on the natural orthogonal basis that we may define on it. In Section 5, we apply our method to the spherical system and provide explicit expressions for all six components of inc T.

Applications of this method can be found in dislocation modelling, where the incompatibility in an inclusion is to be found in order to determine its dislocation content and to design an optimization method where inclusions are inserted so as to obtain a maximal increase (or decrease) of certain functionals.

1.1. Notation and conventions Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. By smooth we mean C^∞, but this assumption could be considerably weakened. The curl, incompatibility and cross product with second-order tensors are defined componentwise, as follows, with the summation convention on repeated indices. Here E represents a second-rank tensor, N is a unit vector and ϵ is the Levi-Civita third-rank (pseudo-)tensor [6].

$$(\text{curl}\,E)_{ij} = (\nabla \times E)_{ij} = \epsilon_{jkm}\partial_k E_{im},$$
$$(\text{inc}\,E)_{ij} = (\text{curl}\,\text{curl}^t\,E)_{ij} = \epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l E_{mn},$$
$$(E \times N)_{ij} = -(N \times E)_{ij} = -\epsilon_{jkm}N_k E_{im}.$$

Note that the transpose of curl E will be denoted by curl^t E. Moreover, the tensorial product of two vectors a and b will be denoted by $a \otimes b$, while $a \odot b = (a \otimes b + b \otimes a)/2$.

2. Some motivations

In this section, we provide two examples of models in which the incompatibility plays a crucial role and must be expressed in a curvilinear system.

2.1. The incompatibility operator in linearized elasticity with dislocations

The strain energy density in small-strain elasticity and for an isotropic material reads

$$W_{\mathrm{e}}(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\mathbb{A}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon},$$

where $\boldsymbol{\varepsilon}$ is the linearized elastic strain tensor. The stress tensor is classically defined as $\boldsymbol{\sigma} = \partial W_{\mathrm{e}}/\partial \boldsymbol{\varepsilon} = \mathbb{A}\boldsymbol{\varepsilon}$. Furthermore, by the symmetry property of $\boldsymbol{\varepsilon}$, Beltrami decomposition (1.2) holds. The potential energy is defined as

$$\mathcal{W}(\boldsymbol{\varepsilon}) = \int_{\Omega}(W_{\mathrm{e}}(\boldsymbol{\varepsilon}) - \boldsymbol{f} \cdot \boldsymbol{u} - \mathbb{G} \cdot \boldsymbol{F})\, dx,$$

which, in the absence of dislocations, that is, as $\boldsymbol{F} = 0$, yields, by minimization, the standard equilibrium equation

$$-\mathrm{div}(\mathbb{A}\boldsymbol{\varepsilon}) = -\mathrm{div}(\mathbb{A}\boldsymbol{\nabla}^{S}\boldsymbol{u}) = \boldsymbol{f}, \tag{2.1}$$

where $\boldsymbol{f}$ is the body force, $\boldsymbol{u}$ is the displacement field and $\boldsymbol{\nabla}^{S}\boldsymbol{u} = (\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^{t}\boldsymbol{u})/2$.

Now, in the general case where dislocation lines are present, the minimization problem is written as

$$\min_{\boldsymbol{\varepsilon}}\mathcal{W}(\boldsymbol{\varepsilon}) = \min_{\substack{\boldsymbol{u},\boldsymbol{F}:\\ \boldsymbol{\varepsilon}=\boldsymbol{\nabla}^{s}u+\mathrm{inc}\,\boldsymbol{F}}}\mathcal{W}(\boldsymbol{\varepsilon}).$$

Letting $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{F}}$ be variations in appropriate function spaces with vanishing boundary conditions, Euler–Lagrange equations yield

$$\left\langle \frac{\delta\mathcal{W}(\boldsymbol{\varepsilon})}{\delta\boldsymbol{u}}, \tilde{\boldsymbol{u}} \right\rangle = \int_{\Omega}(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}^{S}\tilde{\boldsymbol{u}} - \boldsymbol{f} \cdot \tilde{\boldsymbol{u}})\, dx = 0,$$

$$\left\langle \frac{\delta\mathcal{W}(\boldsymbol{\varepsilon})}{\delta\boldsymbol{F}}, \tilde{\boldsymbol{F}} \right\rangle = \int_{\Omega}(\boldsymbol{\sigma} \cdot \mathrm{inc}\,\tilde{\boldsymbol{F}} - \mathbb{G} \cdot \tilde{\boldsymbol{F}})\, dx = 0.$$

After some easy integration by parts, these provide the strong forms

$$\begin{cases} -\mathrm{div}\,\boldsymbol{\sigma} = \boldsymbol{f}, \\ \mathrm{inc}\,\boldsymbol{\sigma} = \mathbb{G}, \end{cases}$$

which appear clearly as a generalization of (2.1). Recalling (1.2), the complete problem consists of solving the coupled problem with unknowns $\boldsymbol{u}$ and $\boldsymbol{F}$

$$\begin{cases} -\mathrm{div}(\mathbb{A}\boldsymbol{\nabla}^{S}\boldsymbol{u}) = \boldsymbol{f} + \mathrm{div}(\mathbb{A}\,\mathrm{inc}\,\boldsymbol{F}), \\ \mathrm{inc}(\mathbb{A}\,\mathrm{inc}\,\boldsymbol{F}) = \mathbb{G} - \mathrm{inc}(\mathbb{A}\boldsymbol{\nabla}^{S}\boldsymbol{u}). \end{cases} \tag{2.2}$$

Material isotropy yields $\mathbb{A} = 2\mu\mathbb{I}_{4} + \lambda\mathbb{I}_{2} \otimes \mathbb{I}_{2}$, and hence (2.2) can be rewritten as

$$\begin{cases} -\mathrm{div}(\mathbb{A}\boldsymbol{\nabla}^{S}\boldsymbol{u}) = \boldsymbol{f} + \mathrm{div}(\lambda\,\mathrm{tr}\,\boldsymbol{\varepsilon}^{0}\mathbb{I}_{2}), \\ \mathrm{inc}(\mathbb{A}\,\mathrm{inc}\,\boldsymbol{F}) = \mathbb{G} - \mathrm{inc}(\lambda\,\mathrm{div}\,\boldsymbol{u}\mathbb{I}_{2}). \end{cases}$$

Note that the decoupled problem is found as soon as either $\lambda = 0$ or incompressibility is assumed, that is, $\mathrm{tr}\,\boldsymbol{\varepsilon} = \mathrm{tr}\,\boldsymbol{\varepsilon}^{0} = \mathrm{div}\,\boldsymbol{u} = 0$, and it reads

$$\begin{cases} -\mathrm{div}(\tilde{\mathbb{A}}\boldsymbol{\nabla}^{S}\boldsymbol{u}) = \boldsymbol{f} \\ \mathrm{inc}(\tilde{\mathbb{A}}\,\mathrm{inc}\,\boldsymbol{F}) = \mathbb{G} \end{cases} \quad \text{in } \Omega,$$

where the special form of $\tilde{\mathbb{A}}$ due to incompressibility can be found in the work of Hoger and Johnson [9].

It is useful to have an expression of incompatibility in curvilinear coordinates (components) systems in accordance with the geometry of Ω.

2.2. Dislocation-induced dissipation Let us define the system specific Helmholtz free energy density [8] as

$$\Psi = W_e(\boldsymbol{\varepsilon}) + W_{\text{dislo}}(\text{curl}\,\boldsymbol{\kappa}), \tag{2.3}$$

where the elastic part is simply the strain energy of the previous section and the defect part is assumed to depend on a unique internal variable, namely, the curl of the contortion tensor $\boldsymbol{\kappa}$. Therefore, the free energy Ψ is partially of second order in the sense that the defect internal variable appears in the form of its derivatives (here its curl). For simplicity, let us assume a quadratic law in the higher-order terms, namely, $W_{\text{dislo}}(\text{curl}\,\boldsymbol{\kappa}) = (\mathbb{M}/2)\,\text{curl}\,\boldsymbol{\kappa} \cdot \text{curl}\,\boldsymbol{\kappa}$, with $\mathbb{M}$ a positive definite fourth-rank tensor. By Kröner's relation (1.1), the energy of an inclusion $\omega \subset \Omega$ reads

$$\mathcal{W}_{\text{dislo}} = \int_\omega \frac{1}{2}\mathbb{M}\,\text{inc}\,\boldsymbol{\varepsilon}^0 \cdot \text{inc}\,\boldsymbol{\varepsilon}^0\,dx.$$

Therefore, minimizing this energy will again lead one to evaluate or express the incompatibility in a local basis appropriate to the geometry of ω.

Note that a full second-order energy density would be, for instance,

$$\Psi = W_e(\boldsymbol{\varepsilon}) + \hat{W}_e(\text{curl}^t\,\boldsymbol{\varepsilon}, \text{div}\,\boldsymbol{\varepsilon}) + \hat{W}_{\text{dislo}}(\boldsymbol{\kappa}, \text{curl}\,\boldsymbol{\kappa}, \text{div}\,\boldsymbol{\kappa}) + \bar{W}_{\text{dislo}}(\boldsymbol{\varepsilon}^0),$$

where $\text{curl}^t\,\boldsymbol{\varepsilon}$ is recognized as the Frank tensor, that is, the gradient of the rotation field, as introduced by Van Goethem and Dupret [18].

3. Extension and differentiation of the normal and tangent vectors to a surface

The aim here is to construct a curvilinear basis on the boundary, which should be smooth and also orthonormal, starting from the vector $N_{\partial\Omega}$ normal to the boundary and defining two tangent vectors perpendicular to $N_{\partial\Omega}$. This basis is then extended to the whole body. The natural moving frame sought is close in spirit to the Darboux frame of surfaces [4, 7, 11], although, in principle, the latter may only be defined at nonumbilical points. As a matter of fact, in order to achieve a certain level of generality, we will not consider principal lines of curvature with their associated principal curvatures, and hence the gradient of the normal vector will be given by a symmetrical matrix with possibly nonzero extradiagonal components. Details on this section and, in particular, the proofs can be found in [1].

3.1. Signed distance function and extended unit normal We denote the outward unit normal to $\partial\Omega$ by $N_{\partial\Omega}$ and the signed distance to $\partial\Omega$ by b: that is,

$$b(x) = \begin{cases} \text{dist}(x, \partial\Omega) & \text{if } x \notin \Omega \\ -\text{dist}(x, \partial\Omega) & \text{if } x \in \Omega. \end{cases}$$

We recall the following results [5, Chapter 5, Theorems 3.1 and 4.3].

THEOREM 3.1. *There exists an open neighbourhood W of $\partial\Omega$ such that the following conditions hold:*

(1) *the function $b(x)$ is smooth in W;*
(2) *every $x \in W$ admits a unique projection $p_{\partial\Omega}(x)$ onto $\partial\Omega$;*
(3) *this projection satisfies $p_{\partial\Omega}(x) = x - (1/2)\boldsymbol{\nabla}b^2(x), x \in W$; and*
(4) *$\boldsymbol{\nabla}b(x) = N_{\partial\Omega}(p_{\partial\Omega}(x)), x \in W$.*

In particular, the last property shows that $\boldsymbol{\nabla}b(x) = N_{\partial\Omega}(x)$ for all $x \in \partial\Omega$ and that $|\boldsymbol{\nabla}b(x)| = 1$ for all $x \in W$. Therefore, we define the extended unit normal as

$$N(x) = \boldsymbol{\nabla}b(x) = N_{\partial\Omega}(p_{\partial\Omega}(x)) \quad x \in W.$$

3.2. Tangent vectors on $\partial\Omega$ For all $x \in \partial\Omega$, we denote by $T_{\partial\Omega}(x)$ the tangent plane to $\partial\Omega$ at x, that is, the orthogonal complement of $N_{\partial\Omega}(x)$. As $\partial\Omega$ is smooth, there exists a covering of $\partial\Omega$ by open balls $B_1, \ldots, B_M$ of $\mathbb{R}^3$ such that, for each index k, two smooth vector fields $\boldsymbol{\tau}_{\partial\Omega}^A, \boldsymbol{\tau}_{\partial\Omega}^B$ can be constructed on $\partial\Omega \cap B_k$, where, for all $x \in \partial\Omega \cap B_k$, $(\boldsymbol{\tau}_{\partial\Omega}^A(x), \boldsymbol{\tau}_{\partial\Omega}^B(x))$ is an orthonormal basis of $T_{\partial\Omega}(x)$. In all the subsequent work, the index k will be implicitly considered as fixed and the restriction to B_k will be omitted. In fact, for our needs, global properties and constructions will be easily obtained from local ones through a partition of unity, subordinate to the covering.

Using the fact that the Jacobian matrix $DN(x) = D^2b(x)$ of $N(x)$ is symmetrical, differentiating the equality $|N(x)|^2 = 1$ entails

$$\partial_N N(x) = DN(x)N(x) = 0 \quad x \in W.$$

In other words, $N(x)$ is an eigenvector of $DN(x)$ for the eigenvalue 0. For all $x \in \partial\Omega$, the system $(\tau_{\partial\Omega}^A(x), \tau_{\partial\Omega}^B(x), N_{\partial\Omega}(x))$ is an orthonormal basis of $\mathbb{R}^3$. In this basis, $DN(x)$ takes the form

$$DN(x) = \begin{pmatrix} \kappa_{\partial\Omega}^A(x) & \xi_{\partial\Omega}(x) & 0 \\ \xi_{\partial\Omega}(x) & \kappa_{\partial\Omega}^B(x) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x \in \partial\Omega,$$

where $\kappa_{\partial\Omega}^A, \kappa_{\partial\Omega}^B$ and ξ are smooth scalar fields defined on $\partial\Omega$.

If $R \in \{A, B\}$, we denote by R^* the complementary index of R: that is, $R^* = B$ if $R = A$ and $R^* = A$ if $R = B$.

3.3. Extended tangent vectors in Ω and their curvilinear differentials   Let d be defined in W by $d = (1 + b\kappa_{\partial\Omega}^A \circ p_{\partial\Omega})(1 + b\kappa_{\partial\Omega}^B \circ p_{\partial\Omega}) - (b\xi_{\partial\Omega} \circ p_{\partial\Omega})^2$. Possibly adjusting W so that $d(x) > 0$ for all $x \in W$ and for $R = A, B$, we define, in W,

$$\tau^R = \tau_{\partial\Omega}^R \circ p_{\partial\Omega}, \quad \kappa^R = d^{-1}\{(1 + b\kappa_{\partial\Omega}^{R^*} \circ p_{\partial\Omega})(\kappa_{\partial\Omega}^R \circ p_{\partial\Omega}) - b(\xi_{\partial\Omega} \circ p_{\partial\Omega})^2\},$$
$$\xi = d^{-1}\xi_{\partial\Omega} \circ p_{\partial\Omega}, \quad \kappa = \kappa^A + \kappa^B, \quad \gamma^R = \operatorname{div}\tau^R.$$

Obviously, for each $x \in W$, the triple $(\tau^A(x), \tau^B(x), N(x))$ forms an orthonormal basis of $\mathbb{R}^3$. Next, we compute the normal and tangential derivatives of these vectors. We denote the tangential derivative ∂_{τ^R} by ∂_R for simplicity: that is, $\partial_R u = Du\tau^R$, where Du stands for the differential of u and $\partial_R u$ is its value in the direction τ^R.

THEOREM 3.2 [1]. *In W*,

$$\partial_N \boldsymbol{\tau}^R = 0, \quad \partial_R \boldsymbol{N} = \kappa^R \boldsymbol{\tau}^R + \xi \boldsymbol{\tau}^{R^*}, \quad \partial_R \boldsymbol{\tau}^R = -\kappa^R \boldsymbol{N} - \gamma^{R^*} \boldsymbol{\tau}^{R^*}, \quad \partial_{R^*} \boldsymbol{\tau}^R = \gamma^R \boldsymbol{\tau}^{R^*} - \xi \boldsymbol{N},$$
$$\operatorname{div} \boldsymbol{N} = \operatorname{tr} D\boldsymbol{N} = \Delta b = \kappa.$$

COROLLARY 3.3 [1]. *If f is twice differentiable in* Ω, *then*

$$\partial_R \partial_N f = \partial_N \partial_R f + \kappa^R \partial_R f + \xi \partial_{R^*} f.$$

4. Differential geometry on the boundary with curvinormal basis

At each point $x \in \partial\Omega$, the curvilinear basis $(\boldsymbol{g}^i(x))_{i=A,B,N} = (\boldsymbol{\tau}^A(x), \boldsymbol{\tau}^B(x), \boldsymbol{N}_{\partial\Omega}(x))$ is orthonormal and differentiable, by Theorem 3.2. Therefore, it will be called *curvinormal*. Note that indices P, Q, R stand for A or B and denote one of the two orthogonal tangent vectors on the boundary, whereas index N will always be associated to the normal $\boldsymbol{N}_{\partial\Omega}$. Let $\boldsymbol{N}$ be the extension of $\boldsymbol{N}_{\partial\Omega}$ in a neighbourhood of $\partial\Omega$. In some sense, the chosen curvilinear basis is a generalization to general surfaces of the spherical or cylindrical bases. We recall that ∂_i means the differential in the direction $\boldsymbol{g}^i$. Let u be a scalar. Then $\partial_i u = \partial_R u = \boldsymbol{\tau}^R \cdot \boldsymbol{\nabla} u = Du\boldsymbol{\tau}^R$ for $R = A, B$ or $\partial_N u = \boldsymbol{N} \cdot \boldsymbol{\nabla} u = Du\boldsymbol{N}$ for $i = N$, with $\boldsymbol{\nabla} = \boldsymbol{e}^i \partial_{x_i}$ being the Cartesian gradient operator, where $\boldsymbol{e}^i$ stands for the ith Cartesian basis vector. For instance, in spherical coordinates, the gradient $\boldsymbol{\nabla} u = \partial_r u \boldsymbol{g}^r + (\partial_\phi/r) u \boldsymbol{g}^\phi + (\partial_\theta/r \sin \phi) u \boldsymbol{g}^\theta$ and hence $\partial_A = \partial_\phi/r$ and $\partial_B = \partial_\theta/r \sin \phi$. Recall that partial curvilinear derivatives do not commute, as shown in Corollary 3.3.

4.1. Christoffel symbols and Riemannian curvature A general vector field will be written as $\boldsymbol{v} = v_i \boldsymbol{g}^i$, with v_i being its covariant components. Moreover, the extrinsic metric is Euclidean since $g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j = \delta^{ij}$. Let $\boldsymbol{g}_i = \delta_{ij} \boldsymbol{g}^j$ be the dual of the basis vector, where $g_{ij} = \delta_{ij}$ is the inverse of $g^{ij} = \delta^{ij}$. The second Christoffel symbol Γ_{ij}^p is defined as the linear operator [3]

$$\partial_j \boldsymbol{g}^p = -\Gamma_{ij}^p \boldsymbol{g}^i. \tag{4.1}$$

In other words, $\Gamma_{ij}^p = -\boldsymbol{g}_i \cdot \partial_j \boldsymbol{g}^p = \boldsymbol{g}^p \cdot \partial_j \boldsymbol{g}_i$. Note also that, since Ω is embedded in a Euclidean space, $\partial_j \boldsymbol{g}_i = \delta_{ik} \partial_j \boldsymbol{g}^k$.

Connection. As a consequence, for vector $\boldsymbol{v}$,

$$\partial_j \boldsymbol{v} = \partial_j (v_i \boldsymbol{g}^i) = (\partial_j v_i - \Gamma_{ij}^p v_p) \boldsymbol{g}^i = v_{i\|j} \boldsymbol{g}^i,$$

where the covariant derivative of the covariant component of $\boldsymbol{v}$ is given as

$$v_{i\|j} = \partial_j v_i - \Gamma_{ij}^p v_p. \tag{4.2}$$

Thus, for vector $\boldsymbol{v} = v_i \boldsymbol{g}^i = \hat{v}_j \boldsymbol{e}^j$, $(\boldsymbol{\nabla} \boldsymbol{v})_{mn} = \partial_{x_m} v_n$, and hence

$$\operatorname{grad} \boldsymbol{v} = (\boldsymbol{\nabla} \boldsymbol{v})_{mn} \boldsymbol{e}^m \otimes \boldsymbol{e}^n = v_{i\|j} \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \partial_j \boldsymbol{v} \otimes \boldsymbol{g}^j.$$

Accordingly, the curl of a vector $\boldsymbol{v}$ in the curvinormal basis is written as

$$\operatorname{curl} \boldsymbol{v} = (\operatorname{curl} \boldsymbol{v})_k \boldsymbol{g}^k = \epsilon_{kij} v_{i\|j} \boldsymbol{g}^k. \tag{4.3}$$

Curvilinear coordinates.   Let $q_R \in \omega^R$ be the curvilinear coordinate associated to $\boldsymbol{g}^R$ in the sense that $\boldsymbol{g}^R = \partial_{q_R}\boldsymbol{x}/G_R$ with $G_R = \|\partial_{q_R}\boldsymbol{x}\|$, where $\boldsymbol{x}$ is the position vector of a point. In other words, q_R is the curvilinear abscissa of the curve with tangent vector $\boldsymbol{\tau}^R$. In general,

$$\partial_{q_R}u = \frac{\partial x_i}{\partial q_R}\partial_{x_i}u = g_i^R G^R \partial_{x_i}u = G_R \partial_R u.$$

Hence the gradient of scalar u is given as (with no summation on the underlined index)

$$\text{grad}u = \partial_i u \boldsymbol{g}^i = \frac{1}{G_{\underline{i}}}\partial_{q_i}u \boldsymbol{g}^i \tag{4.4}$$

and of vector $\boldsymbol{v} = v_i\boldsymbol{g}^i$ as

$$\text{grad}\boldsymbol{v} = \partial_j \boldsymbol{v} \otimes \boldsymbol{g}^j = \frac{1}{G_{\underline{j}}}\partial_{q_j}\boldsymbol{v} \otimes \boldsymbol{g}^j.$$

We call the curvilinear expression of the gradient the operator $\boldsymbol{\nabla}^{\text{CURV}}(\cdot) = h_j \partial_{q_j}(\cdot)\boldsymbol{g}^j$ with $h_j = 1/G_j$, the jth metric factor. Notice that the ∂_R derivatives do not commute, in contrast to ∂_{q_R}, because of the factors G_R. As an example, consider the spherical base system, where $G_N = G_r = 1, G_A = G_\phi = 1/r, G_B = G_\theta = 1/(r\sin\phi)$ and $q_A = \phi$ (polar angle), $q_B = \theta$ (azimuthal angle). Then

$$(\partial_A \partial_r - \partial_r \partial_A) = \frac{1}{r^2}\partial_\phi, \quad (\partial_B \partial_r - \partial_r \partial_B) = \frac{1}{r^2\sin\phi}\partial_\theta, \tag{4.5}$$

$$(\partial_B \partial_A - \partial_A \partial_B) = \frac{1}{r^2\sin\phi\tan\phi}\partial_\theta. \tag{4.6}$$

Christoffel symbols in the curvinormal basis.   By Theorem 3.2, it is easily deduced, by identification with (4.1), that the only nonvanishing components of Γ_{ij}^p (with no sum on repeated indices) are

$$\Gamma_{RR^*}^N = -\xi, \quad \Gamma_{RR}^N = -\kappa^R, \quad \Gamma_{NR}^{R^*} = \xi, \quad \Gamma_{R^*R}^R = \gamma^{R^*}, \quad \Gamma_{NR}^R = \kappa^R, \quad \Gamma_{R^*R^*}^R = -\gamma^R.$$

Moreover, it is observed that Γ_{ij}^p is not symmetrical: that is, $\Gamma_{ij}^p \neq \Gamma_{ij}^p$. Therefore, the torsion is nonvanishing (this is due to the fact that the moving curvinormal frame is nonholonomic, whereas the connection is obviously symmetric) and

$$T_{ij}^p = \Gamma_{ij}^p - \Gamma_{ji}^p.$$

In the curvinormal basis, the only nonvanishing components of T_{ij}^p are

$$T_{ij}^R = \kappa^R \delta_{iN}\delta_{jR} + \xi\delta_{iN}\delta_{jR^*} + (\gamma^{R^*} - \gamma^R)\delta_{iR^*}\delta_{jR}.$$

Note that the Riemann curvature tensor is defined as [6]

$$\text{Riem}_{ijk}^q = \partial_k\Gamma_{ij}^q - \partial_j\Gamma_{ik}^q + \Gamma_{ij}^p\Gamma_{pk}^q - \Gamma_{ik}^p\Gamma_{pj}^q.$$

Spherical system.   As an example, in a spherical coordinate (component) system, $i, j \in \{\phi, \theta\}$, $\kappa^R = 1/r$, $\gamma^\phi = 1/\tan\phi$, $\gamma^\theta = 0$, where ϕ and θ denote the polar and azimuthal coordinates, respectively, and hence

$$
\Gamma_{ij}^r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\dfrac{1}{r} & 0 \\ 0 & 0 & -\dfrac{1}{r} \end{pmatrix}, \quad
\Gamma_{ij}^\phi = \begin{pmatrix} 0 & \dfrac{1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{r\tan\phi} \end{pmatrix}, \quad
\Gamma_{ij}^\theta = \begin{pmatrix} 0 & 0 & \dfrac{1}{r} \\ 0 & 0 & \dfrac{1}{r\tan\phi} \\ 0 & 0 & 0 \end{pmatrix}.
$$

Hence we have the torsion

$$
T_{ij}^r = 0, \quad
T_{ij}^\phi = \begin{pmatrix} 0 & \dfrac{1}{r} & 0 \\ -\dfrac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
T_{ij}^\theta = \begin{pmatrix} 0 & 0 & \dfrac{1}{r} \\ 0 & 0 & \dfrac{1}{r\tan\phi} \\ -\dfrac{1}{r} & -\dfrac{1}{r\tan\phi} & 0 \end{pmatrix}. \tag{4.7}
$$

Accordingly, the covariant derivative expression reads

$$
(v_{i\|j})_{ij} = \begin{pmatrix}
\partial_r v_r & \dfrac{1}{r}\partial_\phi v_r - \dfrac{v_\phi}{r} & \dfrac{1}{r\sin\phi}\partial_\theta v_r - \dfrac{v_\theta}{r} \\[2mm]
\partial_r v_\phi & \dfrac{1}{r}\partial_\phi v_\phi + \dfrac{v_r}{r} & \dfrac{1}{r\sin\phi}\partial_\theta v_\phi - \dfrac{v_\theta}{r\tan\phi} \\[2mm]
\partial_r v_\theta & \dfrac{1}{r}\partial_\phi v_\theta & \dfrac{1}{r\sin\phi}\partial_\theta v_\theta + \dfrac{v_r}{r} + \dfrac{v_\phi}{r\tan\phi}
\end{pmatrix}_{ij}. \tag{4.8}
$$

Hence, by using (4.3) and (4.8), the curl of a vector v, $(\operatorname{curl} v)_i = \epsilon_{ikj} v_{k\|j}$ can be written as

$$
(\operatorname{curl} v)_i = \begin{pmatrix}
\dfrac{1}{r\tan\phi} v_\theta + \dfrac{1}{r}\partial_\phi v_\theta - \dfrac{1}{r\sin\phi}\partial_\theta v_\phi \\[3mm]
\dfrac{1}{r\sin\phi}\partial_\theta v_r - \dfrac{1}{r} v_\theta - \partial_r v_\theta \\[3mm]
\dfrac{1}{r} v_\phi + \partial_r v_\phi - \dfrac{1}{r}\partial_\phi v_r
\end{pmatrix}_i. \tag{4.9}
$$

4.2. Commutation operator in the curvinormal basis   The covariant components of a second-rank tensor T reads [6]

$$
T_{ij\|k} = \partial_k T_{ij} - \Gamma_{ik}^l T_{lj} - \Gamma_{jk}^l T_{il}. \tag{4.10}
$$

Let $T_{ij} = v_{i\|j}$. Then, by (4.10), $v_{i\|jk} = (u_{i\|j})_{\|k}$, and hence,

$$
\begin{aligned}
v_{i\|jk}\boldsymbol{g}^i &= \partial_k v_{i\|j}\boldsymbol{g}^i - (\Gamma_{ik}^l v_{l\|j} + \Gamma_{jk}^l v_{i\|l})\boldsymbol{g}^i = \partial_k(v_{i\|j}\boldsymbol{g}^i) + v_{i\|j}\Gamma_{lk}^i\boldsymbol{g}^l - (\Gamma_{ik}^l v_{l\|j} + \Gamma_{jk}^l v_{i\|l})\boldsymbol{g}^i \\
&= \partial_k(v_{i\|j}\boldsymbol{g}^i) - \Gamma_{jk}^l v_{i\|l}\boldsymbol{g}^i,
\end{aligned}
$$

where (4.2) and a change of dummy indices have been used. Therefore,

$$\partial_k(\partial_j \boldsymbol{v}) = (v_{i\|jk} + \Gamma^l_{jk} v_{i\|l}) \boldsymbol{g}^i.$$

In particular,

$$v_{i\|jk} - v_{i\|kj} = (\partial_k \partial_j - \partial_l \partial_k) \boldsymbol{v} \cdot \boldsymbol{g}^i - T^l_{jk} v_{i\|l}, \qquad (4.11)$$

which can be rewritten [6] by means of the Riemann curvature as

$$v_{i\|jk} - v_{i\|kj} = \text{Riem}_{iqkj} v_q - T^l_{jk} v_{i\|l}. \qquad (4.12)$$

Notice that, in spherical coordinates and by (4.7), (4.5) and (4.6), equation (4.11) yields

$$v_{i\|r\phi} - v_{i\|\phi r} = v_{i\|r\theta} - v_{i\|\theta r} = v_{i\|\theta\phi} - v_{i\|\phi\theta} = 0, \qquad (4.13)$$

(with a slight abuse of notation, since q_R is used as covariant differentiation index in (4.13) instead of R, as it should be according to (4.5) and (4.6)). Therefore, the second covariant derivatives commute in spherical coordinates/components; in particular, $\epsilon_{ljk} v_{i\|jk} = \epsilon_{ljk}(v_{i\|j})_{\|k} = 0$. Note that the identity curl $\boldsymbol{\nabla} u = 0$ is immediate if u is a smooth scalar-valued function, whereas its vector counterpart

$$\text{curl}\, \boldsymbol{\nabla}\boldsymbol{v} = \text{curl}((\boldsymbol{\nabla}v)^{\text{CART}}_{ij} \boldsymbol{e}^i \otimes \boldsymbol{e}^j) = \text{curl}((\boldsymbol{\nabla}v)^{\text{CURV}}_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j).$$

That is, in the Cartesian system,

$$\text{curl}((\boldsymbol{\nabla}v)^{\text{CART}}_{ij} \boldsymbol{e}^i \otimes \boldsymbol{e}^j) = -\epsilon_{ljk}(v_{i,j})_{,k} \boldsymbol{e}^i \otimes \boldsymbol{e}^l = 0,$$

by Schwarz lemma. However, in a general curvilinear system,

$$\text{curl}((\boldsymbol{\nabla}v)^{\text{CURV}}_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j) = \boldsymbol{\nabla}^{\text{CURV}}(\boldsymbol{\nabla}v)^{\text{CURV}}_{ij} \times \boldsymbol{g}^i \otimes \boldsymbol{g}^j + (\boldsymbol{\nabla}v)^{\text{CURV}}_{ij} \text{curl}^{\text{CURV}}(\boldsymbol{g}^i \otimes \boldsymbol{g}^j) = 0,$$

and hence

$$\boldsymbol{\nabla}^{\text{CURV}}(\boldsymbol{\nabla}v)^{\text{CURV}}_{ij} \times (\boldsymbol{g}^i \otimes \boldsymbol{g}^j) = (v_{i\|j})_{\|k} \boldsymbol{g}^i \otimes \epsilon_{lkj} \boldsymbol{g}^l = -(\boldsymbol{\nabla}v)^{\text{CURV}}_{ij} \text{curl}^{\text{CURV}}(\boldsymbol{g}^i \otimes \boldsymbol{g}^j). \quad (4.14)$$

Note that, in the spherical system, the first term on the right-hand side (RHS) of (4.14) vanishes by (4.13).

Summarizing, equation (4.14) shows that the noncommutation operator in the RHS of (4.12) is related in the curvilinear system to the curl of the basis diads. This fact will appear crucial in the calculations of the following sections.

4.3. Expression of the incompatibility in the curvinormal basis Now the incompatibility operator on a second-rank tensor $\boldsymbol{T}$ is defined as

$$\text{inc}\, \boldsymbol{T} = \text{curl}\, \text{curl}^t \boldsymbol{T}, \qquad (4.15)$$

(for some authors, such as Malvern [13], the incompatibility is defined with a negative sign) which, in a Cartesian system, is equivalent to writing componentwise as

$$\text{inc}\, \boldsymbol{T} = \epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l T_{mn} \boldsymbol{e}^i \otimes \boldsymbol{e}^j.$$

In a general curvilinear system, equation (4.15) shows that it suffices to express the curl of a tensor and apply the curl operator twice. In fact, (4.15) can be rewritten as

$$\text{inc } \boldsymbol{T} = \text{curl}(\text{curl}^t(T_{ij}^{\text{CART}} \boldsymbol{e}^i \otimes \boldsymbol{e}^j)) = \text{curl}(\text{curl}^t(T_{ij}^{\text{CURV}} \boldsymbol{g}^i \otimes \boldsymbol{g}^j)),$$

with

$$\text{curl}(T_{ij}^{\text{CURV}} \boldsymbol{g}^i \otimes \boldsymbol{g}^j) = \boldsymbol{\nabla}^{\text{CURV}} T_{ij}^{\text{CURV}} \times \boldsymbol{g}^i \otimes \boldsymbol{g}^j + T_{ij}^{\text{CURV}} \text{curl}^{\text{CURV}}(\boldsymbol{g}^i \otimes \boldsymbol{g}^j). \quad (4.16)$$

Note that, compared with the Cartesian system case, the second term in the RHS of (4.16) is nonvanishing and requires computing of the curl of the basis diads. Summarizing, we get

$$\text{curl}^t(T_{ij}^{\text{CURV}} \boldsymbol{g}^i \otimes \boldsymbol{g}^j) = S_{ij}^{\text{CURV}} \boldsymbol{g}^i \otimes \boldsymbol{g}^j$$

for some components S_{ij}^{CURV} obtained by rearranging (4.16).   Hence the incompatibility in the curvilinear system yields

$$
\begin{aligned}
\text{inc } \boldsymbol{T} &= \boldsymbol{\nabla}^{\text{CURV}} S_{ij}^{\text{CURV}} \times (\boldsymbol{g}^i \otimes \boldsymbol{g}^j) + S_{ij}^{\text{CURV}} \text{curl}^{\text{CURV}}(\boldsymbol{g}^i \otimes \boldsymbol{g}^j) \\
&= \boldsymbol{\nabla}^{\text{CURV}} S_{ij}^{\text{CURV}} \times (\boldsymbol{g}^i \otimes \boldsymbol{g}^j) + S_{ij}^{\text{CURV}}((\boldsymbol{\nabla}^{\text{CURV}} \boldsymbol{g}^i) \times \boldsymbol{g}^j + \boldsymbol{g}^i \otimes \text{curl}^{\text{CURV}} \boldsymbol{g}^j) \\
&= \eta_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j.
\end{aligned}
$$

Obviously, η_{ij} is symmetrical as soon as $\boldsymbol{T}$ is, by the symmetry property of its Cartesian counterpart $\epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l T_{mn}$. Moreover, its explicit expression only requires us to determine the gradient of the scalar S_{ij}^{CURV} in the curvilinear system, which is expressed by means of the tangent vectors as

$$\boldsymbol{\nabla}^{\text{CURV}} S_{ij}^{\text{CURV}} = DS_{ij}^{\text{CURV}}(x)[\boldsymbol{g}^l] \boldsymbol{g}^l = (\boldsymbol{g}^l \cdot \boldsymbol{\nabla} S_{ij}^{\text{CURV}}) \boldsymbol{g}^l$$

(see also equation (4.4)), together with the curvilinear differentials of the basis tensors in Section 3.3 (by equations (4.1)–(4.3)), and expressed by means of κ^R, γ^R and ξ, which are intrinsic numbers of the boundary as related to the choice of the basis. Note that here we use the identity

$$\text{curl } \boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{\nabla} \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \otimes \text{curl } \boldsymbol{b}. \quad (4.17)$$

5. Incompatibility in the spherical system

Recall that ϕ and θ are, respectively, polar and azimuthal angles. Moreover, $\boldsymbol{x} = r\boldsymbol{g}^r$ with r as the radius. The spherical system consists of the triad $\{\boldsymbol{g}^r, \boldsymbol{g}^\phi, \boldsymbol{g}^\theta\}$ and, according to our conventions, $\boldsymbol{N} = \boldsymbol{g}^r$, $\boldsymbol{\tau}^A = \boldsymbol{g}^\phi$ and $\boldsymbol{\tau}^B = \boldsymbol{g}^\theta$, with the latter two being tangential to the sphere with radius r and normal $\boldsymbol{g}^r$.

We consider a symmetrical tensor

$$
\begin{aligned}
\boldsymbol{T} = {}& T_{rr}\boldsymbol{g}^r \otimes \boldsymbol{g}^r + T_{\phi\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi + T_{\theta\theta}\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta + 2T_{r\phi}\boldsymbol{g}^r \odot \boldsymbol{g}^\phi \\
& + 2T_{r\theta}\boldsymbol{g}^r \odot \boldsymbol{g}^\theta + 2T_{\phi\theta}\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta.
\end{aligned}
$$

5.1.   Curl of the diads   By virtue of (4.8) and (4.9) and recalling (4.17), let us first compute the curls of the base diads.

$$\text{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r) = -\frac{1}{r}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi),$$

$$\text{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi) = \frac{1}{r}(\boldsymbol{g}^r \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^r),$$

$$\text{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^\theta) = -\frac{1}{r}(\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^r) + \frac{1}{r\tan\phi}\boldsymbol{g}^r \otimes \boldsymbol{g}^r,$$

$$\text{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r) = \frac{1}{r}\boldsymbol{g}^r \otimes \boldsymbol{g}^\theta + \frac{1}{r\tan\phi}\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi,$$

$$\text{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi) = -\frac{1}{r\tan\phi}\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r + \frac{1}{r}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta,$$

$$\text{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta) = -\frac{1}{r}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r + \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi) + \frac{1}{r\tan\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r,$$

$$\text{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r) = -\frac{1}{r}\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi - \frac{1}{r\tan\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi,$$

$$\text{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi) = \frac{1}{r}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r + \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta) + \frac{1}{r\tan\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r,$$

$$\text{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta) = \frac{1}{r\tan\phi}\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r - \frac{1}{r}\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi.$$

5.2.   Computation of inc $T_{rr}\boldsymbol{g}^r \otimes \boldsymbol{g}^r$ Let us first compute curl $\boldsymbol{T} = T_{rr}\boldsymbol{g}^r \otimes \boldsymbol{g}^r$, by using

$$\text{curl}(T_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j) = \boldsymbol{\nabla}T_{ij} \times (\boldsymbol{g}^i \otimes \boldsymbol{g}^j) + T_{ij}\,\text{curl}(\boldsymbol{g}^i \otimes \boldsymbol{g}^j),$$

which yields

$$\begin{aligned}
\text{curl}\,\boldsymbol{T} &= \boldsymbol{\nabla}T_{rr} \times (\boldsymbol{g}^r \otimes \boldsymbol{g}^r) + T_{rr}\,\text{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r)\\
&= -\frac{\partial_\phi T_{rr}}{r}\boldsymbol{g}^r \otimes \boldsymbol{g}^\theta + \frac{\partial_\theta T_{rr}}{r\sin\phi}\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi - \frac{T_{rr}}{r}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi).
\end{aligned}$$

Hence

$$\begin{aligned}
\text{curl}\,\text{curl}^t\,\boldsymbol{T} = {}&-\boldsymbol{\nabla}\!\left(\frac{\partial_\phi T_{rr}}{r}\right) \times (\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r) + \boldsymbol{\nabla}\!\left(\frac{\partial_\phi T_{rr}}{r\sin\phi}\right) \times (\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r)\\
&+ \boldsymbol{\nabla}\!\left(\frac{T_{rr}}{r}\right) \times (\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi) - \frac{\partial_\phi T_{rr}}{r}\,\text{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r)\\
&+ \frac{\partial_\phi T_{rr}}{r\sin\phi}\,\text{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r) + \frac{T_{rr}}{r}\,\text{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi)
\end{aligned}$$

and

$$\text{inc}\,\boldsymbol{T} = -\frac{2T_{rr}}{r^2}\boldsymbol{g}^r \otimes \boldsymbol{g}^r - \left(\frac{\partial_r T_{rr}}{r} - \frac{\partial_\phi T_{rr}}{r^2 \tan\phi} - \frac{\partial_\theta^2 T_{rr}}{r^2 \sin^2\phi}\right)\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi$$

$$+ \left(\frac{\partial_\phi^2 T_{rr}}{r^2} - \frac{\partial_r T_{rr}}{r}\right)\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta + \frac{2\partial_\phi T_{rr}}{r^2}\boldsymbol{g}^r \odot \boldsymbol{g}^\phi$$

$$+ \frac{2\partial_\theta T_{rr}}{r^2 \sin\phi}\boldsymbol{g}^r \odot \boldsymbol{g}^\theta + \frac{2}{r^2 \sin\phi}\left(\frac{\partial_\theta T_{rr}}{\tan\phi} - \partial_\theta\partial_\phi T_{rr}\right)\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta.$$

5.3. Complete expression of the incompatibility   Collecting (A.1) and all the computations of the Appendix, we arrive at the general formulae

$$(\text{inc}\,\boldsymbol{T})_{rr} = -\frac{2T_{rr}}{r^2} + \left(\frac{\partial_\theta^2 T_{\phi\phi} + T_{\phi\phi}}{r^2 \sin\phi} - \frac{\partial_\phi T_{\phi\phi}}{r^2 \tan\phi} + \frac{\partial_r T_{\phi\phi}}{r} + \frac{2T_{\phi\phi}}{r^2}\right)$$

$$+ \left(\frac{\partial_\phi^2 T_{\theta\theta}}{r^2} + \frac{2\partial_\phi T_{\theta\theta}}{r^2 \tan\phi} + \frac{\partial_r T_{\theta\theta}}{r}\right) - \frac{2\partial_\phi(T_{r\phi}\sin\phi)}{r^2 \sin\phi}$$

$$- \frac{2\partial_\theta T_{r\theta}}{r^2 \sin\phi} - \frac{2}{\sin\phi}\left(\frac{\partial_\theta\partial_\phi T_{\phi\phi}}{r^2} + \frac{\partial_\theta T_{\phi\phi}}{r^2 \tan\phi}\right), \tag{5.1}$$

$$(\text{inc}\,\boldsymbol{T})_{\phi\phi} = -\left(\frac{\partial_r T_{rr}}{r} - \frac{\partial_\phi T_{rr}}{r^2 \tan\phi} - \frac{\partial_\theta^2 T_{rr}}{r^2 \sin^2\phi}\right) + \left(\partial_r^2 T_{\theta\theta} + \frac{2\partial_r T_{\theta\theta}}{r}\right)$$

$$- \frac{4T_{r\phi}}{r^2 \tan\phi} - 2\left(\frac{\partial_\theta T_{r\phi}}{r^2 \sin\phi} + \frac{\partial_\theta\partial_r T_{r\theta}}{r \sin\phi}\right),$$

$$(\text{inc}\,\boldsymbol{T})_{\theta\theta} = \left(\frac{\partial_\phi^2 T_{rr}}{r^2} - \frac{\partial_r T_{rr}}{r}\right) + \left(\partial_r^2 T_{\phi\phi} + \frac{2\partial_r T_{\phi\phi}}{r}\right) - 2\left(\frac{\partial_r\partial_\phi T_{r\phi}}{r} + \frac{\partial_\phi T_{r\phi}}{r^2}\right), \tag{5.2}$$

$$(\text{inc}\,\boldsymbol{T})_{r\phi} = \frac{\partial_\phi T_{rr}}{r^2} + \frac{\partial_r T_{\phi\phi}}{r \tan\phi} - \left(\frac{\partial_r\partial_\phi T_{\theta\theta}}{r} + \frac{\partial_r T_{\theta\theta}}{r \tan\phi}\right) - \left(\frac{\partial_\theta^2 T_{r\phi}}{r^2 \sin^2\phi} + \frac{2T_{r\phi}}{r^2}\right)$$

$$+ \frac{\partial_\phi(\partial_\theta T_{r\theta}\sin\phi)}{r^2 \sin^2\phi} + \frac{\partial_\phi\partial_r T_{\phi\theta}}{r \sin\phi},$$

$$(\text{inc}\,\boldsymbol{T})_{r\theta} = \frac{\partial_\theta T_{rr}}{r^2 \sin\phi} - \frac{\partial_r\partial_\theta T_{\phi\phi}}{r \sin\phi} + \frac{1}{\sin\phi}\left(\frac{\partial_\phi\partial_\theta T_{r\phi}}{r^2} - \frac{\partial_\theta T_{r\phi}}{r^2 \tan\phi}\right)$$

$$- \left(\frac{\partial_\phi(\partial_\theta T_{r\theta}\sin\phi)}{r^2 \sin\phi} + \frac{T_{r\theta}}{r^2} - \frac{T_{r\theta}}{r^2 \tan^2\phi}\right) + \left(\frac{2\partial_r T_{\phi\theta}}{r \tan\phi} + \frac{\partial_r\partial_\phi T_{\phi\theta}}{r}\right),$$

$$(\text{inc}\,\boldsymbol{T})_{\phi\theta} = \frac{1}{r^2 \sin\phi}\left(\frac{\partial_\theta T_{rr}}{\tan\phi} - \partial_\theta\partial_\phi T_{rr}\right) + \left(\frac{\partial_r\partial_\theta T_{r\phi}}{r \sin\phi} + \frac{\partial_\theta T_{r\phi}}{r^2 \sin\phi}\right)$$

$$+ \left(\frac{\partial_r\partial_\phi T_{r\theta}}{r} - \frac{\partial_r T_{r\theta}}{r \tan\phi} - \frac{T_{r\theta}}{r^2 \tan\phi} + \frac{\partial_\phi T_{r\theta}}{r^2}\right) - \left(\partial_r^2 T_{\theta\phi} + \frac{2\partial_r T_{\theta\phi}}{r}\right).$$

5.4. Application: determining the dislocation-induced force in linearized elasticity A simple application of our full expression is presented here. Recall the general form of the second-order free energy $\Psi = W_e(\boldsymbol{\varepsilon}) + \hat{W}_e(\text{curl}^t\,\boldsymbol{\varepsilon}, \text{div}\,\boldsymbol{\varepsilon}) + \hat{W}_{\text{dislo}}(\boldsymbol{\kappa}, \text{curl}\,\boldsymbol{\kappa}, \text{div}\,\boldsymbol{\kappa}) + \bar{W}_{\text{dislo}}(\boldsymbol{\varepsilon}^0)$. Let $\varphi_e(\boldsymbol{u}) = \int_\Omega W_e(\boldsymbol{\nabla u})\,dx$ and $\varphi_{\text{dislo}}(\boldsymbol{F}) = \int_\Omega \bar{W}_{\text{dislo}}(\text{inc}\,\boldsymbol{F})\,dx$. The Fréchet derivative of φ_e at $\boldsymbol{u}$ in the direction $\boldsymbol{v}$ reads $D\varphi_e(\boldsymbol{u})[\boldsymbol{v}] = \int_\Omega \mathbb{A}\boldsymbol{\nabla u} \cdot \boldsymbol{\nabla v}\,dx = -\int_\Omega \text{div}(\mathbb{A}\boldsymbol{\nabla u}) \cdot \boldsymbol{v}\,dx$, that is, by Riesz theorem [2], the differential $\varphi'_e(\boldsymbol{u}) := D\varphi_e(\boldsymbol{u}) = \boldsymbol{f}$. Also, $D\varphi_{\text{dislo}}(\boldsymbol{F})[\boldsymbol{V}] = \int_\Omega \text{inc}\,\bar{W}'_{\text{dislo}}(\text{inc}\,\boldsymbol{F}) \cdot \boldsymbol{V}\,dx$ and we set $\mathbb{G} = \varphi'_{\text{dislo}}(\boldsymbol{F}) = \text{inc}\,\bar{W}'_{\text{dislo}}(\text{inc}\,\boldsymbol{F})$, which is symmetric and divergence free. Assume also that $\mathbb{G}$ is independent of $\boldsymbol{\varepsilon}^0$. Now we would like to solve (2.3) with Ω, the unit sphere, in the simplified case where $\mathbb{A} = \alpha\mathbb{I}_4$ and where we take $\mathbb{G} = \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta$: that is, we seek F such that $\text{inc}(\alpha\,\text{inc}\,\boldsymbol{F}) = \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta$.

The solution of $\text{inc}\,\boldsymbol{T} = \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta$ is found by (5.2) as $T(r)\boldsymbol{g}^r \otimes \boldsymbol{g}^r$ with $T(r) = -r^2/2 + c$. Moreover, by (5.1), $\boldsymbol{F} = F(r)\boldsymbol{g}^r \otimes \boldsymbol{g}^r$ is a solution with $F(r) = r^4/4\alpha - r^2/2\alpha$. Uniqueness is obtained by imposing the natural homogeneous Dirichlet conditions $\boldsymbol{F} = \text{curl}^t\,\boldsymbol{F} \times N = 0$ on $\partial\Omega$ at $r = 1$: that is, $c = 1/2$. Thus $\boldsymbol{\varepsilon}^0 = -\{(r^2 - 1)/2\alpha\}\boldsymbol{g}^r \otimes \boldsymbol{g}^r$. Now the displacement $\boldsymbol{u}$ is obtained by solving $-\text{div}(\mathbb{A}\boldsymbol{\nabla u}) = f + \text{div}(\mathbb{A}\boldsymbol{\varepsilon}^0)$, where $\text{div}(\mathbb{A}\boldsymbol{\varepsilon}^0) = -(2r - 1/r)\boldsymbol{g}^r$ is a radial force due to the presence of dislocations.

6. Concluding remarks

In this paper, a method to compute the incompatibility operator in a system of curvilinear components (coordinates) is proposed. Moreover, an in-extenso expression of the incompatibility is given in a spherical system. It is shown that the incompatibility of the elastic strain is directly linked to the dislocation density of a solid. Therefore, in the first step, our method allows us to compute the energy related to dislocations in spherical inclusions. In a second step, as carried out in two dimensions, our computations will allow us to optimize the location of these spherical inclusions in the elastic solid with a view to minimizing or maximizing certain cost functionals and to predict the creation of plastic regions, that is, regions with high dislocation mobility and strain incompatibility. This will be the purpose of our future work in three dimensions.

Appendix A. Other terms of the incompatibility

A.1. Computation of inc $T_{\phi\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi$ Let us compute the curl of $\boldsymbol{T} = T_{\phi\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi$.

$$\text{curl}\,\boldsymbol{T} = \boldsymbol{\nabla}T_{\phi\phi} \times (\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi) + T_{\phi\phi}\,\text{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi)$$
$$= \partial_r T_{\phi\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta - \frac{1}{r\sin\phi}\partial_\theta T_{\phi\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r + \frac{T_{\phi\phi}}{r}\left(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta - \frac{1}{\tan\phi}\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r\right).$$

Hence

$$
\text{curl curl}^t \, \boldsymbol{T} = \boldsymbol{\nabla}(\partial_r T_{\phi\phi}) \times (\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi) - \boldsymbol{\nabla}\!\left(\frac{\partial_\theta T_{\phi\phi}}{r \sin \phi}\right) \times (\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi)
$$

$$
+ \boldsymbol{\nabla}\!\left(\frac{T_{\phi\phi}}{r}\right) \times \left(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi - \frac{1}{\tan \phi} \times \boldsymbol{g}^r \otimes \boldsymbol{g}^\theta\right)
$$

$$
+ \partial_r T_{\phi\phi} \, \text{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi) - \frac{\partial_\theta T_{\phi\phi}}{r \sin \phi} \, \text{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi)
$$

$$
+ \frac{T_{\phi\phi}}{r} \, \text{curl}\!\left(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi - \frac{1}{\tan \phi} \boldsymbol{g}^r \otimes \boldsymbol{g}^\theta\right)
$$

and

$$
\text{inc} \, \boldsymbol{T} = \left(\frac{\partial_\theta^2 T_{\phi\phi} + T_{\phi\phi}}{r^2 \sin \phi} - \frac{\partial_\phi T_{\phi\phi}}{r^2 \tan \phi} + \frac{\partial_r T_{\phi\phi}}{r} + \frac{2 T_{\phi\phi}}{r^2}\right) \boldsymbol{g}^r \otimes \boldsymbol{g}^r
$$

$$
+ \left(\partial_r^2 T_{\phi\phi} + \frac{2 \partial_r T_{\phi\phi}}{r}\right) \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta
$$

$$
+ \frac{2 \partial_r T_{\phi\phi}}{r \tan \phi} \boldsymbol{g}^r \odot \boldsymbol{g}^\phi - \frac{2 \partial_r \partial_\theta T_{\phi\phi}}{r \sin \phi} \boldsymbol{g}^r \odot \boldsymbol{g}^\theta. \tag{A.1}
$$

A.2.   Computation of inc $T_{\theta\theta} \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta$ Let us compute the curl of $\boldsymbol{T} = T_{\theta\theta} \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta$.

$$
\text{curl} \, \boldsymbol{T} = \boldsymbol{\nabla} T_{\theta\theta} \times (\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta) + T_{\theta\theta} \, \text{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta)
$$

$$
= -\partial_r T_{\theta\theta} \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi + \frac{\partial_\phi T_{\theta\theta}}{r} \boldsymbol{g}^\theta \otimes \boldsymbol{g}^r + \frac{T_{\theta\theta}}{r}\left(\frac{1}{\tan \phi} \boldsymbol{g}^\theta \otimes \boldsymbol{g}^r - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi\right).
$$

Hence

$$
\text{curl curl}^t \, \boldsymbol{T} = -\boldsymbol{\nabla}(\partial_r T_{\theta\theta}) \times (\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta) + \boldsymbol{\nabla}\!\left(\frac{\partial_\phi T_{\theta\theta}}{r}\right) \times (\boldsymbol{g}^r \otimes \boldsymbol{g}^\theta)
$$

$$
+ \boldsymbol{\nabla}\!\left(\frac{T_{\theta\theta}}{r}\right) \times \left(\frac{1}{\tan \phi} \boldsymbol{g}^r \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta\right) - \partial_r T_{\theta\theta} \, \text{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta)
$$

$$
+ \frac{\partial_\phi T_{\theta\theta}}{r} \, \text{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^\theta) + \frac{T_{\theta\theta}}{r} \, \text{curl}\!\left(\frac{1}{\tan \phi} \boldsymbol{g}^r \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta\right)
$$

and

$$
\text{inc} \, \boldsymbol{T} = \left(\frac{\partial_\phi^2 T_{\theta\theta}}{r^2} + \frac{2 \partial_\phi T_{\theta\theta}}{r^2 \tan \phi} + \frac{\partial_r T_{\theta\theta}}{r}\right) \boldsymbol{g}^r \otimes \boldsymbol{g}^r + \left(\partial_r^2 T_{\theta\theta} + \frac{2 \partial_r T_{\theta\theta}}{r}\right) \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi
$$

$$
- 2\left(\frac{\partial_r \partial_\phi T_{\theta\theta}}{r} + \frac{\partial_r T_{\theta\theta}}{r \tan \phi}\right) \boldsymbol{g}^r \odot \boldsymbol{g}^\phi.
$$

A.3.   Computation of inc $2 T_{r\phi} \boldsymbol{g}^r \odot \boldsymbol{g}^\phi$ Let us compute the curl of $\boldsymbol{T} = 2 T_{r\phi} \boldsymbol{g}^r \odot \boldsymbol{g}^\phi$.

$$
\text{curl} \, \boldsymbol{T} = 2 \boldsymbol{\nabla} T_{r\phi} \times (\boldsymbol{g}^r \odot \boldsymbol{g}^\phi) + 2 T_{r\phi} \, \text{curl}(\boldsymbol{g}^r \odot \boldsymbol{g}^\phi)
$$

$$
= \partial_r T_{r\phi} \boldsymbol{g}^r \otimes \boldsymbol{g}^\theta + \frac{\partial_\phi T_{r\phi}}{r} \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta - \frac{\partial_\theta T_{r\phi}}{r \sin \phi} \boldsymbol{g}^r \otimes \boldsymbol{g}^r + \frac{\partial_\theta T_{r\phi}}{r \sin \phi} \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi
$$

$$
+ \frac{T_{r\phi}}{r}\left(2 \boldsymbol{g}^r \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^r + \frac{1}{\tan \phi} \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi\right).
$$

Hence

$$\operatorname{curl}\operatorname{curl}^t \boldsymbol{T} = \boldsymbol{\nabla}(\partial_r T_{r\phi}) \times (\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r) - \boldsymbol{\nabla}\Big(\frac{\partial_\phi T_{r\phi}}{r}\Big) \times (\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi) - \boldsymbol{\nabla}\Big(\frac{\partial_\theta T_{r\phi}}{r\sin\phi}\Big) \times (\boldsymbol{g}^r \otimes \boldsymbol{g}^r)$$

$$+ \boldsymbol{\nabla}\Big(\frac{\partial_\theta T_{r\phi}}{r\sin\phi}\Big) \times (\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi) + \boldsymbol{\nabla}\Big(\frac{T_{r\phi}}{r}\Big) \times (2\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r - \boldsymbol{g}^r \otimes \boldsymbol{g}^\theta)$$

$$+ \boldsymbol{\nabla}\Big(\frac{T_{r\phi}}{r\tan\phi}\Big) \times (\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta) + \partial_r T_{r\phi} \operatorname{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r) - \frac{\partial_\phi T_{r\phi}}{r} \operatorname{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi)$$

$$- \frac{\partial_\theta T_{r\phi}}{r\sin\phi} \operatorname{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r) + \frac{\partial_\theta T_{r\phi}}{r\sin\phi} \operatorname{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi)$$

$$+ \frac{T_{r\phi}}{r} \operatorname{curl}(2\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r - \boldsymbol{g}^r \otimes \boldsymbol{g}^\theta) + \frac{T_{r\phi}}{r\tan\phi} \operatorname{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta)$$

and

$$\operatorname{inc}\boldsymbol{T} = -2\Big(\frac{\partial_\phi T_{r\phi}}{r^2} + \frac{T_{r\phi}}{r^2\tan\phi}\Big)\boldsymbol{g}^r \otimes \boldsymbol{g}^r - \frac{4T_{r\phi}}{r^2\tan\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi$$

$$- 2\Big(\frac{\partial_r\partial_\phi T_{r\phi}}{r} + \frac{\partial_\phi T_{r\phi}}{r^2}\Big)\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta - 2\Big(\frac{\partial_\theta^2 T_{r\phi}}{r^2\sin^2\phi} + \frac{2T_{r\phi}}{r^2}\Big)\boldsymbol{g}^r \odot \boldsymbol{g}^\phi$$

$$+ 2\Big(\frac{\partial_\phi\partial_\theta T_{r\phi}}{r^2\sin\phi} - \frac{\partial_\theta T_{r\phi}}{r^2\sin\phi\tan\phi}\Big)\boldsymbol{g}^r \odot \boldsymbol{g}^\theta + 2\Big(\frac{\partial_r\partial_\theta T_{r\phi}}{r\sin\phi} + \frac{\partial_\theta T_{r\phi}}{r^2\sin\phi}\Big)\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta.$$

A.4.   Computation of inc $2T_{r\theta}\boldsymbol{g}^r \odot \boldsymbol{g}^\theta$ Let us compute the curl of $\boldsymbol{T} = 2T_{r\theta}\boldsymbol{g}^r \odot \boldsymbol{g}^\theta$.

$$\operatorname{curl}\boldsymbol{T} = 2\boldsymbol{\nabla}T_{r\theta} \times (\boldsymbol{g}^r \odot \boldsymbol{g}^\theta) + 2T_{r\theta}\operatorname{curl}(\boldsymbol{g}^r \odot \boldsymbol{g}^\theta)$$

$$= -\partial_r T_{r\theta}\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi + \frac{\partial_\phi T_{r\theta}}{r}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta) + \frac{\partial_\theta T_{r\theta}}{r\sin\phi}\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\phi$$

$$+ \frac{T_{r\theta}}{r}\Big(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r - 2\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi + \frac{1}{\tan\phi}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi)\Big).$$

Hence

$$\operatorname{curl}\operatorname{curl}^t \boldsymbol{T} = -\boldsymbol{\nabla}(\partial_r T_{r\theta}) \times (\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r) + \boldsymbol{\nabla}\Big(\frac{\partial_\phi T_{r\theta}}{r}\Big) \times (\boldsymbol{g}^r \otimes \boldsymbol{g}^r - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta)$$

$$+ \boldsymbol{\nabla}\Big(\frac{\partial_\theta T_{r\theta}}{r\sin\phi}\Big) \times (\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta)$$

$$+ \boldsymbol{\nabla}\Big(\frac{T_{r\theta}}{r}\Big) \times \Big(\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi - 2\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r + \frac{1}{\tan\phi}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi)\Big)$$

$$- \partial_r T_{r\theta} \operatorname{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r) + \frac{\partial_\phi T_{r\theta}}{r} \operatorname{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r - \boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta)$$

$$+ \frac{\partial_\theta T_{r\theta}}{r\sin\phi} \operatorname{curl}(\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\theta)$$

$$+ \frac{T_{r\theta}}{r} \operatorname{curl}\Big(\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi - 2\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r + \frac{1}{\tan\phi}(\boldsymbol{g}^r \otimes \boldsymbol{g}^r - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi)\Big)$$

and

$$
\begin{aligned}
\text{inc } \boldsymbol{T} = {} & -\frac{2\partial_\theta T_{r\theta}}{r^2 \sin\phi}\boldsymbol{g}^r \otimes \boldsymbol{g}^r - 2\Big(\frac{\partial_\theta T_{r\phi}}{r^2 \sin\phi} + \frac{\partial_\theta \partial_r T_{r\theta}}{r \sin\phi}\Big)\boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi \\
& + \frac{2}{\sin\phi}\Big(\frac{\partial_\phi \partial_\theta T_{r\theta}}{r^2} + \frac{\partial_\theta T_{r\theta}}{r^2 \tan\phi}\Big)\boldsymbol{g}^r \odot \boldsymbol{g}^\phi \\
& - 2\Big(\frac{\partial_\phi^2 T_{r\theta}}{r^2} + \frac{T_{r\theta}}{r^2} - \frac{T_{r\theta}}{r^2 \tan^2\phi} + \frac{\partial_\phi T_{r\theta}}{r^2 \tan\phi}\Big)\boldsymbol{g}^r \odot \boldsymbol{g}^\theta \\
& + 2\Big(\frac{\partial_r \partial_\phi T_{r\theta}}{r} - \frac{\partial_r T_{r\theta}}{r \tan\phi} - \frac{T_{r\theta}}{r^2 \tan\phi} + \frac{\partial_\phi T_{r\theta}}{r^2}\Big)\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta.
\end{aligned}
$$

A.5. Computation of inc $2T_{\phi\theta}\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta$ Let us compute the curl of $\boldsymbol{T} = 2T_{\phi\theta}\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta$.

$$
\begin{aligned}
\text{curl } \boldsymbol{T} = {} & 2\boldsymbol{\nabla} T_{\phi\theta} \times (\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta) + 2T_{\phi\theta}\,\text{curl}(\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta) \\
= {} & \partial_r T_{\phi\theta}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi) + \frac{\partial_\phi T_{\phi\theta}}{r}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r - \frac{\partial_\theta T_{\theta\phi}}{r \sin\phi}\boldsymbol{g}^\theta \otimes \boldsymbol{g}^r \\
& + \frac{T_{\phi\theta}}{r}\Big(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi + \frac{2}{\tan\phi}\boldsymbol{g}^\phi \otimes \boldsymbol{g}^r\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\text{curl curl}^t \boldsymbol{T} = {} & \boldsymbol{\nabla}(\partial_r T_{\phi\theta}) \times (\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi) + \boldsymbol{\nabla}\Big(\frac{\partial_\phi T_{\phi\theta}}{r}\Big) \times (\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi) \\
& - \boldsymbol{\nabla}\Big(\frac{\partial_\theta T_{\phi\theta}}{r \sin\phi}\Big) \times (\boldsymbol{g}^r \otimes \boldsymbol{g}^\theta) + \boldsymbol{\nabla}\Big(\frac{T_{\phi\theta}}{r}\Big) \times \Big(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi + \frac{2}{\tan\phi}\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi\Big) \\
& + \partial_r T_{\phi\theta}\,\text{curl}(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi) + \frac{\partial_\phi T_{\phi\theta}}{r}\,\text{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi) \\
& - \frac{\partial_\theta T_{\phi\theta}}{r \sin\phi}\,\text{curl}(\boldsymbol{g}^r \otimes \boldsymbol{g}^\theta) + \frac{T_{\phi\theta}}{r}\,\text{curl}\Big(\boldsymbol{g}^\theta \otimes \boldsymbol{g}^\theta - \boldsymbol{g}^\phi \otimes \boldsymbol{g}^\phi + \frac{2}{\tan\phi}\boldsymbol{g}^r \otimes \boldsymbol{g}^\phi\Big), \\
\text{inc } \boldsymbol{T} = {} & -\frac{2}{\sin\phi}\Big(\frac{\partial_\theta \partial_\phi T_{\phi\theta}}{r^2} + \frac{\partial_\theta T_{\phi\theta}}{r^2 \tan\phi}\Big)\boldsymbol{g}^r \otimes \boldsymbol{g}^r + 2\frac{\partial_\phi \partial_r T_{\phi\theta}}{r \sin\phi}\boldsymbol{g}^r \odot \boldsymbol{g}^\phi \\
& + 2\Big(\frac{2\partial_r T_{\phi\theta}}{r \tan\phi} + \frac{\partial_r \partial_\phi T_{\phi\theta}}{r}\Big)\boldsymbol{g}^r \odot \boldsymbol{g}^\theta - 2\Big(\partial_r^2 T_{\theta\phi} + \frac{2\partial_r T_{\phi\theta}}{r}\Big)\boldsymbol{g}^\phi \odot \boldsymbol{g}^\theta.
\end{aligned}
$$

Acknowledgements

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