

Equivalence Relations and Reductions

1.1 Generalities on Equivalence Relations

Let E be an equivalence relation on a set X . If $A \subseteq X$, we let $E \upharpoonright A = E \cap A^2$ be its **restriction** to A . We also let $[A]_E = \{x \in X : \exists y \in A(xEy)\}$ be its **E -saturation**. The set A is **E -invariant** if $A = [A]_E$. In particular, for each $x \in X$, $[x]_E$ is the **equivalence class**, or **E -class**, of x . A function $f: X \rightarrow Y$ is **E -invariant** if $xEy \implies f(x) = f(y)$. Finally, $X/E = \{[x]_E : x \in X\}$ is the **quotient space** of X modulo E .

Suppose that E, F are equivalence relations on sets X, Y , respectively, and $f: (X/E)^n \rightarrow Y/F, n \geq 1$, is a function. A **lifting** of f is a function $\tilde{f}: X^n \rightarrow Y$ such that $f([x_i]_E) = [\tilde{f}((x_i))]_F, \forall x \in X$. Similarly if $R \subseteq (X/E)^n$, its lifting is $\tilde{R} \subseteq X^n$, where $(x_i) \in \tilde{R} \iff ([x_i]_E) \in R$.

If $E_i, i \in I$, is a family of equivalence relations, with E_i living on X_i , we define the **direct sum** $\bigoplus_i E_i$ to be the equivalence relation on $\bigoplus_i X_i = \{(x, i) : x \in X_i\}$ defined by

$$(x, j) \bigoplus_i E_i (y, k) \iff j = k \ \& \ xE_j y.$$

In particular, we let for $n \geq 1, nE = \bigoplus_{i < n} E$. Also let $\mathbb{N}E = \bigoplus_{i \in \mathbb{N}} E$.

We define the **direct product** $\prod_i E_i$ to be the equivalence relation on the space $\prod_i X_i$ defined by

$$(x_j) \prod_i E_i (y_j) \iff \forall j(x_j E_j y_j).$$

In particular, we let for $n \geq 1, E^n = \prod_{i < n} E$. Also let $E^{\mathbb{N}} = \prod_{i \in \mathbb{N}} E$.

If E, F are equivalence relations on X , then $E \subseteq F$ means that E is a subset of F , when these are viewed as subsets of X^2 , i.e., E is finer than F or equivalently F is coarser than E . The **index** of F over E , in symbols $[F : E]$, is the supremum of the cardinalities of the sets of E -classes contained in an

F -class. Thus $[F : E] \leq \aleph_0$ means that every F -class contains only countably many E -classes.

We denote by $\Delta_X = \{(x, y) : x = y\}$ the equality relation on a set X , and we also let $I_X = X^2$. Note that if $E_Y = E$, $y \in Y$, where E is an equivalence relation on a set X , then $\bigoplus_y E_y = E \times \Delta_Y$.

If $E_i, i \in I$, are equivalence relations on X , we denote by $\bigwedge_i E_i = \bigcap_i E_i$ the largest (under inclusion) equivalence relation contained in all E_i and by $\bigvee_i E_i$ the smallest (under inclusion) equivalence relation containing each E_i . We call $\bigwedge_i E_i$ the **meet** and $\bigvee_i E_i$ the **join** of (E_i) .

If E is an equivalence relation on X , a set $S \subseteq X$ is a **complete section** of E if S intersects every E -class. Moreover, if S intersects every E -class in exactly one point, then S is a **transversal** of E .

Consider now an action $a : G \times X \rightarrow X$ of a group G on a set X . We often write $g \cdot x = a(g, x)$, if there is no danger of confusion. Let $G \cdot x = \{g \cdot x : g \in G\}$ be the **orbit** of $x \in X$. The action a induces an equivalence relation E_a on X whose classes are the orbits, i.e., $x E_a y \iff \exists g (g \cdot x = y)$. When a is understood, sometimes the equivalence relation E_a is also denoted by E_G^X . The action a is **free** if $g \cdot x \neq x$ for every $x \in X, g \in G, g \neq 1_G$.

1.2 Morphisms

Let E, F be equivalence relations on spaces X, Y , resp. A map $f : X \rightarrow Y$ is a **homomorphism** from E to F if $x E y \implies f(x) F f(y)$. In this case we write $f : (X, E) \rightarrow (Y, F)$ or just $f : E \rightarrow F$, if there is no danger of confusion. A homomorphism f is a **reduction** if moreover $x E y \iff f(x) F f(y)$. We denote this by $f : (X, E) \leq (Y, F)$ or just $f : E \leq F$. Note that a homomorphism as above induces a map from X/E to Y/F , which is an injection if f is a reduction. In other words, a homomorphism is a lifting of a map from X/E to Y/F , and a reduction is a lifting of an injection of X/E into Y/F . An **embedding** is an injective reduction. This is denoted by $f : (X, E) \sqsubseteq (Y, F)$ or just $f : E \sqsubseteq F$. An **invariant embedding** is an injective reduction whose range is an F -invariant subset of Y . This is denoted by $f : (X, E) \sqsubseteq^i (Y, F)$ or just $f : E \sqsubseteq^i F$. Finally, an **isomorphism** is a surjective embedding. This is denoted by $f : (X, E) \cong (Y, F)$ or just $f : E \cong F$.

If a, b are actions of a group G on spaces X, Y , resp., a **homomorphism** from a to b is a map $f : X \rightarrow Y$ such that $f(g \cdot x) = g \cdot f(x), \forall g \in G, x \in X$. If f is injective, we call it an **embedding** of a to b .

1.3 The Borel Category

We are interested here in studying (classes of) Borel equivalence relations on **standard Borel spaces** (i.e., Polish spaces with the associated Borel structure). If X is a standard Borel space and E is an equivalence relation on X , then E is Borel if E is a Borel subset of X^2 .

Given a class of functions Φ between standard Borel spaces, we can restrict the above notions of morphism to functions in Φ , in which case we use the subscript Φ in the above notation (e.g., $f: E \rightarrow_{\Phi} F$, $f: E \leq_{\Phi} F$, etc.). In particular if Φ is the class of Borel functions, we write $f: E \rightarrow_B F$, $f: E \leq_B F$, $f: E \sqsubseteq_B F$, $f: E \sqsubseteq_B^i F$, $f: E \cong_B F$ to denote that f is a Borel morphism of the appropriate type. Similarly when we consider the underlying topology, we use the subscript c in the case where Φ is the class of continuous functions between Polish spaces and write $f: E \rightarrow_c F$, $f: E \leq_c F$, $f: E \sqsubseteq_c F$, $f: E \sqsubseteq_c^i F$, $f: E \cong_c F$.

We say that E is **Borel reducible** to F if there is a Borel reduction from E to F . In this case we write $E \leq_B F$. If $E \leq_B F$ and $F \leq_B E$, then E, F are **Borel bireducible**, in symbols $E \sim_B F$. Finally we let $E <_B F$ if $E \leq_B F$ but $F \not\leq_B E$. Similarly we define the notions of E being **Borel embeddable** to F and E being **Borel invariantly embeddable** to F , for which we use the notations $E \sqsubseteq_B F$ and $E \sqsubseteq_B^i F$, respectively. Also we use $E \simeq_B F$, $E \simeq_B^i F$ for the corresponding notions of being **Borel biembeddable** and **Borel invariantly biembeddable** and $E \sqsubset_B F$ and $E \sqsubset_B^i F$ for the corresponding strict notions. More generally, if Φ is as above, we analogously define $E \leq_{\Phi} F$, $E \sqsubseteq_{\Phi} F$, etc.

Finally E, F are **Borel isomorphic**, in symbols $E \cong_B F$, if there is a Borel isomorphism from E to F . Note that by the usual (Borel) Schröder–Bernstein argument, E, F are Borel isomorphic if and only if they are Borel invariantly biembeddable, i.e., $\simeq_B^i = \cong_B$.