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A FEW MORE BALANCED ROOM SQUARES

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Abstract

The existence problem for balanced Room squares is, in general, unsolved. Recently, B. A. Anderson gave a construction for a class of these designs with side $2^n - 1$, where n is odd and $n \geqslant 3$. For n even, the existence has not yet been settled. In this paper, we use the affine geometry of dimension $2k$ and order 2, and a hill-climbing algorithm, to construct a number of new balanced Room squares directly. Recursive techniques based on finite geometries then give new squares of side $2^{2k} - 1$ for infinitely many values of k.

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1. Introduction

A *Room square* of side r defined on an $(r + 1)$-set V is an $r \times r$ array A satisfying the following conditions:

(1) each cell of A is either empty or contains an unordered pair of distinct elements from V;

(2) each element of V is contained in precisely one cell of each row and each column of A;

(3) every pair of distinct elements from V is contained in exactly one cell of A.

We denote a Room square of side r by $RS(r)$.

The spectrum for Room squares was determined in 1973.

THEOREM 1.1 (Mullin and Wallis [8]). *There exists an $RS(r)$ if and only if r is an odd positive integer other than* 3 *or* 5.

A Room square $RS(r)$ defined on set V is said to be *ordered* if every pair $\{x, y\}$ in the array is replaced by one of the ordered pairs (x, y) or (y, x). Let F_i denote the set of all first components taken from the ordered pairs in row i and let S_i denote the set of all second components in row i. Clearly, $F_i \cup S_i = V$, $1 \leqslant i \leqslant r$. Let $\beta = \{F_i, S_i: 1 \leqslant i \leqslant r\}$.

If (V, β) is a balanced incomplete block design, then the ordered Room square is called a *balanced* Room square of side r and is denoted $BRS(r)$.

EXAMPLE

0∞	26	45		13		
	1∞	30	56		24	
		2∞	41	60		35
46			3∞	52	01	
	50			4∞	63	12
23		61			5∞	04
15	34		02			6∞

A $BRS(7)$

$F_1 = \{0,2,4,1\}$, $S_1 = \{\infty,5,5,3\}$.
$F_2 = \{1,3,5,2\}$, $S_2 = \{\infty,0,6,4\}$.
$F_3 = \{2,4,6,3\}$, $S_3 = \{\infty,1,0,5\}$.
$F_4 = \{4,3,5,0\}$, $S_4 = \{6,\infty,2,1\}$.
$F_5 = \{5,4,6,1\}$, $S_5 = \{0,\infty,3,2\}$.
$F_6 = \{2,6,5,0\}$, $S_6 = \{3,1,\infty,4\}$.
$F_7 = \{1,3,0,6\}$, $S_7 = \{5,4,2,\infty\}$.

A balanced Room square $BRS(r)$ is sometimes called a *complete balanced Howell rotation* on $(r + 1)$ teams. This terminology arises from some of the early history of the subject in connection with designing duplicate bridge tournaments with various properties. The interested reader is referred to [9].

Unlike Room squares, the existence question for balanced Room squares is unsolved. It is well known [3, 9, 10] that a necessary condition for a $BRS(r)$ to exist is $r \equiv 3 \pmod 4$. In this paper, we are interested in the existence of $BRS(2^n - 1)$. In a recent paper by B. Anderson [1], it is shown that for all odd $n \geqslant 3$, such a design exists. We will briefly consider this construction in the next section. In Section 3 we construct by direct methods a number of new $BRS(2^n - 1)$ for various even values of n and in Section 4 recursive techniques are applied to produce infinitely many new squares. The constructions considered throughout this paper are based on finite affine geometries.

We identify the points of $AG(n,2)$, the affine geometry of order 2 and dimension n, with the elements of the field $GF(2^n)$. Let α be a generator for $GF(2^n)$ and define $\alpha^\infty = 0$. Lines in this geometry contain two points. If $L = \{\alpha^a, \alpha^b\}$ is a line, then we will usually write this as $L = \{a, b\}$. Two lines $L_1 = \{\alpha^a, \alpha^b\}$ and $L_2 = \{\alpha^c, \alpha^d\}$ are parallel if and only if $\alpha^a + \alpha^b = \alpha^c + \alpha^d$. A set of lines $S = \{L_i: 1 \leqslant i \leqslant 2^{n-1}\}$ is said to be a *skew resolution class* of the geometry if S is a partition of the point set and no two distinct lines of S are parallel. A *skew resolution* is a set of skew resolution classes which partition the lines of $AG(n,2)$. We will make extensive use of the following result.

THEOREM 1.2 (Fuji-Hara and Vanstone [7]). *If there exists a skew resolution in $AG(n,2)$, then there exists a $BRS(2^n - 1)$.*

PROOF. Let $r = 2^n - 1$ and $P_1, P_2, \ldots, P_r$ be the r parallel classes of lines in the geometry. Let $R_1, R_2, \ldots, R_r$ be r skew resolution classes forming a skew resolution. Index the rows and columns of an $r \times r$ array A by the parallel classes and skew resolution classes respectively. In cell (P_i, R_j) place the set $P_i \cap R_j$. Clearly, every cell is either empty or contains a line of $AG(n,2)$. It is easy to see that A is a Room square. We now order the pairs in A to form a balanced Room square.

Since P_i is a parallel class of lines there is a unique pair of parallel hyperplanes F_i and S_i such that each line of P_i contains one point from each. If $L = \{x, y\} \in P_i$ and $x \in F_i$, $y \in S_i$, then replace L in A by (x, y). Since the set of all hyperplanes forms a $BIBD$ the resulting array is a $BRS(2^n - 1)$.

2. Preliminaries

In [1], B. A. Anderson proved the existence of a $BRS(2^n - 1)$ for all odd $n \geqslant 3$. In this section, we briefly describe this construction in terms of affine geometries and Theorem 1.2 above.

Consider $AG(n,2)$ (n odd, $n \geqslant 3$) and let α be a generator for $GF(2^n)$. Let

$$P = \{\{\alpha^i, 1 + \alpha^i\} : i = \infty, 0 \leqslant i \leqslant 2^n - 2\}.$$

Then P is a parallel class of lines which generates all other parallel classes of lines under a cyclic automorphism of order $2^n - 1$. Let

$$S = \{\{\alpha^\infty, \alpha^0\}\} \cup \{\{1/\alpha^i, 1/(1 + \alpha^i)\} : 0 \leqslant i \leqslant 2^n - 2\}.$$

Then S is a skew resolution class of lines which generates a skew resolution under the action of the automorphism group of order $2^n - 1$. It is not difficult to show that S is a partition of the point set. Suppose two lines of S are parallel. For some i and j then either

(I) $1/\alpha^i + 1/(1 + \alpha^i) = 1/\alpha^j + 1/(1 + \alpha^j)$

or

(II) $\alpha^\infty + \alpha^0 = 1/\alpha^i + 1/(1 + \alpha^i)$.

In (I) we get that $(\alpha^i + \alpha^j) + (\alpha^i + \alpha^j) = 0$ implying $\alpha^i + \alpha^j = 0$ or $\alpha^i + \alpha^j = 1$. In either case the two lines are identical. In (II), $1 = 1/\alpha^i(1 + \alpha^i)$ or $\alpha^{2i} + \alpha^i + 1 = 0$. Hence α^i is a cube root of unity (not 1), which is impossible since n odd implies $2^n - 1$ is not divisible by 3. We summarize this in the following theorem.

THEOREM 2.1. *There exists a cyclically generated skew resolution in $AG(n,2)$ for all odd $n \geqslant 3$.*

COROLLARY 2.1. *There exists a $BRS(2^n - 1)$ for all odd $n \geqslant 3$.*

It is easy to see why the construction given here fails for n even. The lines $\{0, \infty\}$ and $\{(2^n - 1)/3, (2^{n+1} - 2)/3\}$ are both in P and S and, hence, S is not a skew resolution class.

We require several more definitions. The concepts of starter adder and strong starter are fundamental in the study of Room squares.

A *starter* T in a finite abelian group G of odd order is a partition of $G \setminus \{0\}$ into pairs which form a difference set (i.e., $\{\pm(a - b): \{a, b\} \in T\} = G \setminus \{0\}$).

An *adder* A for a starter T in G is an injective mapping from T into $G \setminus \{0\}$ such that

$$\{a + A(\{a, b\}), b + A(\{a, b\}): \{a, b\} \in T\} = G \setminus \{0\}.$$

If T is a starter and $\Sigma T = \{a + b: \{a, b\} \in T\}$ is a set of $|T|$ distinct elements of $G \setminus \{0\}$ then T is called a *strong starter*.

It is well known [8] that a starter and adder in a finite abelian group G of order r imply the existence of an $RS(r)$ and that a strong starter in G implies the existence of a starter and adder in G.

Consider the parallel class P in $AG(n,2)$ as given above. P is a parallel class for all values of n. If $\{\alpha^i, 1 + \alpha^i\}$ is a line of P, then replace this pair by $\{i, z(i)\}$ where $1 + \alpha^i = \alpha^{z(i)}$ to get a set of pairs P'; P' is a starter in $G = Z_r$ where $r = 2^n - 1$. Anderson showed that P' is a strong starter in G. This is, of course, not true when n is even. In the next section we construct adders for the starter P' for various values of n even. It should be clear that P' and an adder will produce a skew resolution class which will generate a skew resolution.

3. Some new balanced room squares

Given a starter in a finite abelian group G, Dinitz and Stinson [4] have devised a hill-climbing algorithm for finding an adder for the starter. Applying this algorithm to P' of the previous section we obtain the following results.

THEOREM 3.1. *There exists a skew resolution in $AG(n,2)$ for $n = 4, 6, 8$ and 14.*

PROOF. For $n = 4, 6, 8$, we list the polynomial used to define $GF(2^n)$, the parallel resolution class P', an adder A for P', and the corresponding skew resolution class S.

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$n = 4 \qquad f(x) = x^4 + x + 1$

P'	A	S
11,12	1	12,13
7, 9	14	6, 8
1, 4	3	4, 7
14, 3	2	1, 5
5,10	4	9,14
2, 8	9	11, 2
6,13	12	3,10

$n = 6 \qquad f(x) = x^6 + x + 1$

P'	A	S	P'	A	S	P'	A	S	P'	A	S
57,58	59	53,54	18,27	25	43,52	16,33	13	29,46	52,14	7	59,21
51,53	26	14,16	2,12	6	8,18	36,54	58	31,49	56,19	5	61,24
31,34	39	7,10	38,40	37	12,23	7,26	43	50, 6	45, 9	38	20,47
39,43	23	62, 3	61,10	41	39,51	4,24	36	40,60	22,50	46	5,33
1, 6	35	36,41	28,41	4	32,45	21,42	34	55,13	3,32	53	56,22
62, 5	12	11,17	11,25	33	44,58	13,35	2	15,37	17,47	17	34, 1
37,44	28	2, 9	40,55	50	27,42	48, 8	40	25,48	29,60	60	26,57
15,23	15	30,38	30,46	52	19,35	59,20	8	4,28			

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$n = 8 \qquad f(x) = x^8 + x^4 + x^3 + x^2 + 1$

P'	A	S	P'	A	S	P'	A	S	P'	A	S
230,231	241	216,217	205,207	192	142,144	32, 35	68	100,103	155,159	129	29, 33
117,122	104	221,226	64, 70	186	250, 1	143,150	47	190,197	55, 63	90	145,153
135,144	105	240,249	234,244	82	61, 71	10, 21	215	225,236	128,140	242	115,127
156,169	153	54, 67	31, 45	143	174,188	222,237	32	254, 14	110,126	224	79, 95
187,204	155	87,104	15, 33	79	94,112	163,182	5	168,187	213,233	49	7, 27
245, 11	6	251, 17	20, 42	30	50, 72	59, 82	99	158,181	1, 25	197	198,222
254, 24	140	139,164	57, 83	217	19, 45	151,178	15	166,193	62, 90	113	175,203
74,103	96	170,199	189,219	122	56, 86	210,241	67	22, 53	220,252	189	154,186
240, 18	25	10, 43	119,153	7	126,160	223, 3	193	161,196	30, 66	146	176,212
76,113	188	9, 46	71,109	37	108,146	149,188	205	99,138	171,211	132	48, 88
98,139	199	42, 83	235, 22	52	32, 74	134,177	75	209,252	40, 84	107	147,191
224, 14	164	133,178	118,164	65	183,229	154,201	127	26, 73	2, 50	80	82,130
58,107	12	70,119	253, 48	167	165,215	17, 68	22	39, 90	114,166	29	143,195
108,161	77	185,238	47,101	212	4, 58	192,247	83	20, 75	124,180	124	248, 49
172,229	111	28, 85	148,206	216	109,167	173,232	179	97,156	123,183	88	211, 16
69,130	100	169,230	165,227	62	227, 34	200, 8	120	65,128	185,249	206	136,200
93,158	198	36,101	225, 36	237	207, 18	39,106	133	172,239	238, 51	81	64,132
125,194	38	163,232	191, 6	213	149,219	146,217	78	224, 40	60,132	247	52,124
19, 92	218	237, 55	152,226	92	224, 63	56,131	229	30,105	142,218	115	2, 78
27,104	175	202, 24	43,121	4	47,125	81,160	254	80,159	87,167	228	60,140
95,176	163	3, 84	196, 23	150	91,173	198, 26	178	121,204	215, 44	222	182, 11
85,170	191	21,106	13, 99	63	76,162	88,175	172	5, 92	80,168	112	192, 25
52,141	234	31,120	193, 28	173	111,201	46,137	187	233, 69	236, 73	170	151,243
97,190	181	23,116	53,147	84	137,231	79,174	71	150,245	4,100	110	114,210
65,162	203	13,110	116,214	235	96,194	242, 86	161	148,247	251, 96	45	41,141
208, 54	169	122,223	34,136	43	77,179	181, 29	33	214, 62	228, 77	184	157, 6
7,112	59	66,171	216, 67	223	184, 35	197, 49	8	205, 57	94,202	40	134,242
184, 38	69	253,107	129,239	134	8,118	9,120	108	117,228	248,105	136	129,241
179, 37	56	235, 93	89,203	204	38,152	12,127	86	98,213	41,157	61	102,218
133,250	73	206, 68	91,209	201	37,155	102,221	165	12,131	246,111	24	15,135
212, 78	156	113,234	138, 5	39	177, 44	72,195	51	123,246	75,199	145	220, 89
61,186	128	189, 59	145, 16	35	180, 51	243,115	93	81,208			

In $AG(14, 2)$ there are 16,384 points and a skew resolution contains 8,192 lines. We omit a listing of the skew resolution found. The hill-climbing algorithm took 11 seconds of CUP time on an Amdahl 580 computer to find an adder for the parallel class P'.

Corollary 3.1. *There exists a* $BRS(2^n - 1)$ *for* $n = 4, 6, 8$ *and* 14.

The case $n = 6$ was recently done independently by B. A. Anderson [2] using a related but somewhat different approach. Anderson begins with a pair of parallel hyperplanes in $AG(6,2)$ which he writes as a pair of supplementary difference sets A and B in Z_{63}.

$$A = \{\infty, 1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 17, 19, 20, 25, 27, 28, 29, 33, 34, 36, 37, 39, 42, 46, 49, 50, 53, 55, 57\}.$$

$$B = \{0, 6, 11, 12, 16, 18, 21, 22, 23, 24, 26, 30, 31, 32, 35, 38, 40, 41, 43, 44, 45, 47, 48, 51, 52, 54, 56, 58, 59, 60, 61, 62\}.$$

Using a modified version of the Dinitz-Stinson hill-climbing algorithm he constructs a strong starter T in Z_{63} where each pair in T contains one element from A and one from B. The strong starter T is listed below.

$\{46,45\}$, $\{33,35\}$, $\{9,6\}$, $\{57,61\}$, $\{53,48\}$, $\{49,43\}$, $\{37,44\}$,
$\{13,21\}$, $\{7,16\}$, $\{36,26\}$, $\{20,31\}$, $\{5,56\}$, $\{1,51\}$, $\{25,11\}$, $\{8,23\}$,
$\{15,62\}$, $\{29,12\}$, $\{42,24\}$, $\{10,54\}$, $\{27,47\}$, $\{39,18\}$, $\{17,58\}$, $\{55,32\}$,
$\{2,41\}$, $\{34,59\}$, $\{14,40\}$, $\{3,30\}$, $\{50,22\}$, $\{4,38\}$, $\{19,52\}$, $\{28,60\}$.

T generates a $BRS(63)$ whose associated $BIBD$ is isomorphic to the design consisting of the points and hyperplanes of $AG(6,2)$. This $BRS(63)$ is not isomorphic to the one displayed in Theorem 3.1. Neither the rows nor the columns form the parallel resolution of lines in $AG(6,2)$.

4. Recursive constructions

Unlike the Room square case, there are very few recursive constructions for balanced Room squares. In terms of finite geometries and skew resolutions, several recursive constructions do exist. We will use these techniques to produce infinitely many new balanced Room squares.

The definition of skew resolution class and skew resolution extend to $AG(n,q)$.

THEOREM 4.1 (Fuji-Hara and Vanstone [6]). *If there exists a skew resolution in $AG(m,q)$ and a skew resolution in $AG(n,q^m)$, then there exists a skew resolution in $AG(mn,q)$.*

THEOREM 4.2 (Fuji-Hara and Vanstone [6]). *If there exists a skew resolution in $AG(m+1,q)$ and a skew resolution in $AG(n,q^m)$, then there exists a skew resolution in $AG(mn+1,q)$.*

In order to apply Theorems 4.1 and 4.2 we need to know something about the existence of skew resolutions in $AG(n,q)$ for $q > 2$.

THEOREM 4.3 (Fuji-Hara and Vanstone [5]). *There exists a skew resolution in* $AG(2^i - 1, q)$ *for all* q *and* $i \geqslant 2$. *In particular, there exists a skew resolution in* $AG(2^i - 1, 2^j)$ *for all* $i \geqslant 2, j \geqslant 2$.

Applying Theorems 4.1 and 4.3 we get the following theorem.

THEOREM 4.4. *There exists a skew resolution in* $AG(n(2^i - 1), 2)$ *for* $n = 4, 6, 8$ *and* 14 *and all* $i \geqslant 2$.

Applying Theorems 4.2 and 4.3 we get

THEOREM 4.5. *There exists a skew resolution in* $AG((n - 1)(2^i - 1) + 1, 2)$ *for* $n = 4, 6, 8$ *and* 14 *and all* $i \geqslant 2$.

In terms of balanced Room squares we summarize the above results.

THEOREM 4.6. *There exists a* $BRS(r)$ *for all*
(1) $r = n(2^i - 1) - 1, i \geqslant 2, n = 4, 6, 8$ *and* 14,
(2) $r = (n - 1)(2^i - 1), i \geqslant 2, n = 4, 6, 8$ *and* 14.

The smallest value of k for which a skew resolution in $AG(2k, 2)$ is not yet known to exist is $k = 10$ and, in terms, of balanced Room squares the smallest $BRS(2^{2k} - 1)$ unknown is $k = 10$.

5. Conclusion

In this paper we have been concerned with balanced Room squares of side $2^n - 1$ where n is even and we have constructed infinitely many new ones in this class. Since $BRS(2^m - 1)$, m odd and $m \geqslant 3$, are all known to exist, what would be ideal is a recursive construction which doubles the point set. Such a construction would prove the existence of $BRS(2^{2k} - 1)$. Unfortunately, no such method is known. In terms of affine geometries the problem could be solved by showing that a skew resolution in a space of dimension $2^k - 1$ implies the existence of a skew resolution in a space of dimension $2k$. Again, such a construction is not known.

We finish by mentioning that it has recently been shown [11] that there are at least 279 non-isomorphic skew resolutions in $AG(5, 2)$, and hence at least 279 non-isomorphic $BRS(31)$.

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