Nagoya Math. J., **235** (2019), 115–126
DOI 10.1017/nmj.2018.4

A NOTE ON DORMANT OPERS OF RANK $p-1$ IN CHARACTERISTIC p

YUICHIRO HOSHI

Abstract. In this paper, we prove that the set of equivalence classes of dormant opers of rank $p-1$ over a projective smooth curve of genus $\geqslant 2$ over an algebraically closed field of characteristic $p>0$ is of cardinality one.

Introduction

In this paper, we study *dormant opers of rank* $p-1$ over projective smooth curves of characteristic $p>0$. In particular, we prove that the set of equivalence classes of dormant opers of rank $p-1$ over a curve of characteristic $p>0$ is *of cardinality one*.

The notion of *oper* was introduced in [1] (cf. also [2]). Let k be an algebraically closed field, and let X be a projective smooth curve of genus $\geqslant 2$ over k. Let us recall that an *oper* over X/k is a suitable triple consisting of a locally free coherent $\mathcal{O}_X$-module, a connection on the module relative to X/k, and a filtration of the module. The study of opers in positive characteristic was initiated in, for instance, [4, 5, 9, 10]. Suppose that we are in the situation in which k is of characteristic $p>0$. Then, we shall say that a given oper is *dormant* if the p-curvature of the connection of the oper is zero. We refer to Definition 1.1 (cf. also Definition 1.2) concerning the precise definition of the notion of "dormant oper" discussed in the present paper. It should be noted that a dormant oper of rank two is essentially the same as a *dormant indigenous bundle* studied in *p-adic Teichmüller theory* (cf. [7]).

The main result of the present paper, which is a generalization of the first portion of [3], Theorem A (cf. Remark 2.1.1), is as follows (cf. Proposition 1.4, Theorem 2.1).

Received April 17, 2015. Revised May 15, 2015. Accepted August 1, 2017.
2010 Mathematics subject classification. 14H60.
This research was supported by JSPS KAKENHI Grant Number 15K04780.

THEOREM A. *Let k be an algebraically closed field of characteristic $p > 0$, and let X be a projective smooth curve of genus $\geqslant 2$ over k. Then, the set of equivalence classes of dormant opers of rank $p-1$ over X/k is of cardinality one.*

It should be noted that the fact that the set discussed in Theorem A is *nonempty* was already known (cf. Remark 1.4.2).

In [4], *Joshi* posed a conjecture concerning the number of equivalence classes of dormant opers of rank r over a projective smooth curve of genus $g \geqslant 2$ over an algebraically closed field of characteristic $p > 0$ for $p > C(r, g) \stackrel{\text{def}}{=} r(r-1)(r-2)(g-1)$ (cf. [4, Conjecture 8.1]). Moreover, *Wakabayashi* proved this conjecture for a sufficiently general curve (cf. [10, Theorem 8.7.1]). It should be noted that the triple $(r, g, p) \stackrel{\text{def}}{=} (p-1, g, p)$ (i.e., the triple in the case discussed in Theorem A) does not satisfy the condition $p > C(r, g)$ unless $p \in \{2, 3\}$.

§1. Construction of dormant opers of rank $p-1$

In this section, let p be a prime number, let $g \geqslant 2$ be an integer, let k be an algebraically closed field of characteristic p, and let X be a projective smooth curve over k (i.e., a scheme that is projective, smooth, geometrically connected, and of relative dimension one over k) of genus g. Write $X^{(1)}$ for the projective smooth curve over k obtained by base-changing X via the absolute Frobenius morphism of k, $F\colon X \to X^{(1)}$ for the relative Frobenius morphism over k, $\mathcal{I} \subseteq \mathcal{O}_{X \times_k X}$ for the ideal of $\mathcal{O}_{X \times_k X}$ which defines the diagonal morphism with respect to X/k, and $X_{(n)} \subseteq X \times_k X$ for the closed subscheme of $X \times_k X$ defined by the ideal $\mathcal{I}^{n+1} \subseteq \mathcal{O}_{X \times_k X}$ (where n is a nonnegative integer). In particular, it follows that $\mathcal{I}/\mathcal{I}^2 = \omega_{X/k}$ (resp., $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \tau_{X/k}$), where we use the notation ω (resp., τ) to denote the cotangent (resp., tangent) sheaf. Finally, write $d\colon \mathcal{O}_X \to \omega_{X/k}$ for the exterior differentiation operator.

Let us define the notion of *dormant oper*, as well as the notion of *equivalence of dormant opers*, discussed in the present paper as follows (cf., e.g., [4, §3]; also [5, Definitions 3.1.1, 3.1.2 and (1)–(5) of pp. 51–52]).

DEFINITION 1.1. Let r be a positive integer. Then, we shall say that a collection of data

$$(\mathcal{E},\ \nabla_{\mathcal{E}},\ \{0\} = \mathcal{E}_r \subseteq \mathcal{E}_{r-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E})$$

consisting of a locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$, a connection $\nabla_{\mathcal{E}}$ on $\mathcal{E}$ relative to X/k, and a filtration $\{0\} = \mathcal{E}_r \subseteq \mathcal{E}_{r-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E}$ by $\mathcal{O}_X$-submodules of $\mathcal{E}$ is a *dormant oper* of rank r over X/k if the following five conditions are satisfied.

(1) For every $i \in \{1, \ldots, r\}$, the subquotient $\mathcal{E}_{i-1}/\mathcal{E}_i$ is an invertible sheaf on X.
(2) For every $i \in \{1, \ldots, r\}$, it holds that $\nabla_{\mathcal{E}}(\mathcal{E}_i) \subseteq \mathcal{E}_{i-1} \otimes_{\mathcal{O}_X} \omega_{X/k}$.
(3) For every $i \in \{1, \ldots, r-1\}$, the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{E}_i \stackrel{\nabla_{\mathcal{E}}}{\to} \mathcal{E}_{i-1} \otimes_{\mathcal{O}_X} \omega_{X/k} \twoheadrightarrow (\mathcal{E}_{i-1}/\mathcal{E}_i) \otimes_{\mathcal{O}_X} \omega_{X/k}$$

(cf. (2)) determines an isomorphism $\mathcal{E}_i/\mathcal{E}_{i+1} \stackrel{\sim}{\to} (\mathcal{E}_{i-1}/\mathcal{E}_i) \otimes_{\mathcal{O}_X} \omega_{X/k}$ of invertible sheaves on X (cf. (1)).
(4) The p-curvature of $\nabla_{\mathcal{E}}$ is zero.
(5) There exists an isomorphism $(\det \mathcal{E}, \det \nabla_{\mathcal{E}}) \stackrel{\sim}{\to} (\mathcal{O}_X, d)$.

DEFINITION 1.2. Let r be a positive integer, and let

$$\mathfrak{E} \stackrel{\text{def}}{=} (\mathcal{E},\ \nabla_{\mathcal{E}},\ \{0\} = \mathcal{E}_r \subseteq \mathcal{E}_{r-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E}),$$
$$\mathfrak{F} \stackrel{\text{def}}{=} (\mathcal{F},\ \nabla_{\mathcal{F}},\ \{0\} = \mathcal{F}_r \subseteq \mathcal{F}_{r-1} \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0 = \mathcal{F})$$

be dormant opers of rank r over X/k. Then, we shall say that $\mathfrak{E}$ is *equivalent* to $\mathfrak{F}$ if the following condition is satisfied. There exists a triple $(\mathcal{L}, \nabla_{\mathcal{L}}, \phi)$ consisting of an invertible sheaf $\mathcal{L}$ on X, a connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$ relative to X/k, and a horizontal isomorphism ϕ of $(\mathcal{E}, \nabla_{\mathcal{E}})$ with $(\mathcal{F}, \nabla_{\mathcal{F}}) \otimes_{\mathcal{O}_X} (\mathcal{L}, \nabla_{\mathcal{L}})$ such that ϕ maps, for each $i \in \{0, \ldots, r\}$, the subsheaf $\mathcal{E}_i \subseteq \mathcal{E}$ isomorphically onto the subsheaf $\mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L} \subseteq \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$.

REMARK 1.2.1. It should be noted that the notion of the "equivalence class of a dormant oper of rank r" of the present paper coincides with the notion of the "isomorphism class of a dormant $\mathrm{PGL}(r)$-oper" in the terminology given in [4, §3].

REMARK 1.2.2. It follows immediately from the various definitions involved that the notion of *dormant oper of rank two* is essentially the same as the notion of *dormant indigenous bundle* studied in [7].

In the remainder of the present section, let us construct a dormant oper of rank $p-1$ over X/k. We shall write

$$B_1 \stackrel{\text{def}}{=} \mathrm{Coker}(\mathcal{O}_{X^{(1)}} \to F_*\mathcal{O}_X)$$

for the $\mathcal{O}_{X^{(1)}}$-module obtained by forming the cokernel of the natural homomorphism $\mathcal{O}_{X^{(1)}} \to F_*\mathcal{O}_X$. It should be noted that since F is finite flat of degree p, the $\mathcal{O}_{X^{(1)}}$-module B_1, and hence also the $\mathcal{O}_X$-module F^*B_1, is locally free of rank $p-1$.

LEMMA 1.3. *The following hold.*

(i) *The natural homomorphism $\mathcal{O}_{X^{(1)}} \to F_*\mathcal{O}_X$ and the homomorphism $F^*F_*\mathcal{O}_X \to \mathcal{O}_X$ determined by multiplication of the ring $\mathcal{O}_X$ determine an isomorphism of $\mathcal{O}_X$-modules*

$$F^*F_*\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X \oplus F^*B_1.$$

(ii) *The natural morphism $X \times_{X^{(1)}} X \to X \times_k X$ over k determines an isomorphism over k*

$$X \times_{X^{(1)}} X \xrightarrow{\sim} X_{(p-1)}.$$

(iii) *The closed immersion $X_{(1)} \hookrightarrow X \times_k X$ factors through the closed subscheme $X \times_{X^{(1)}} X \subseteq X \times_k X$:*

$$X_{(1)} \hookrightarrow X \times_{X^{(1)}} X \hookrightarrow X \times_k X.$$

(iv) *The isomorphism $X \times_{X^{(1)}} X \xrightarrow{\sim} X_{(p-1)}$ of* (ii)*, together with the Cartesian diagram*

$$\begin{array}{ccc} X \times_{X^{(1)}} X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle F} \\ X & \xrightarrow[F]{} & X^{(1)}, \end{array}$$

determines isomorphisms of $\mathcal{O}_X$-modules

$$F^*F_*\mathcal{O}_X \xrightarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X \times_{X^{(1)}} X} \xleftarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X_{(p-1)}},$$

which are compatible with the respective natural surjections onto $\mathcal{O}_X$ (arising from the diagonal morphism with respect to $X/X^{(1)}$) from each of these three modules.

(v) *The isomorphisms $F^*F_*\mathcal{O}_X \xrightarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X \times_{X^{(1)}} X} \xleftarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X_{(p-1)}}$ of* (iv) *restrict, relative to the isomorphism $F^*F_*\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X \oplus F^*B_1$ of* (i)*, to isomorphisms of $\mathcal{O}_X$-submodules*

$$F^*B_1 \xrightarrow{\sim} \mathrm{Ker}(\mathrm{pr}_{1*}\mathcal{O}_{X \times_{X^{(1)}} X} \twoheadrightarrow \mathcal{O}_X) \xleftarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p).$$

https://doi.org/10.1017/nmj.2018.4 Published online by Cambridge University Press

(vi) *The* $\mathcal{O}_X$*-module* $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p)$ *admits a filtration*

$$\{0\} = \mathrm{pr}_{1*}(\mathcal{I}^p/\mathcal{I}^p) \subseteq \mathrm{pr}_{1*}(\mathcal{I}^{p-1}/\mathcal{I}^p) \subseteq \cdots \subseteq \mathrm{pr}_{1*}(\mathcal{I}^2/\mathcal{I}^p) \subseteq \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p).$$

(vii) *The* $\mathcal{O}_X$*-module* $F^* \det B_1 = \det(F^* B_1)$ *is isomorphic to* $\omega_{X/k}^{\otimes p(p-1)/2}$.

(viii) *It holds that* $\deg B_1 = (p-1)(g-1)$.

(ix) *There exists an invertible sheaf* $\mathcal{L}^\circ$ *on* $X^{(1)}$ *such that* $\det(\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} B_1)$ $(= (\mathcal{L}^\circ)^{\otimes p-1} \otimes_{\mathcal{O}_{X^{(1)}}} \det B_1)$ *is isomorphic to* $\mathcal{O}_{X^{(1)}}$.

Proof. Assertions (i) and (ii) follow immediately from local explicit calculations. Assertions (iii) and (iv) follow from assertion (ii). Assertion (v) follows from assertion (iv). Assertion (vi) is immediate. Assertion (vii) follows from assertion (vi). Assertion (viii) follows from assertion (vii). Assertion (ix) follows from assertion (viii). This completes the proof of Lemma 1.3. ∎

Let $\mathcal{L}^\circ$ be as in Lemma 1.3(ix) (cf. Remark 1.4.1 below). We shall write

$$\mathcal{E}^\circ \stackrel{\text{def}}{=} F^*(\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} B_1).$$

Then, it follows immediately from the definition of a connection that the factorization of Lemma 1.3(iii) determines a connection on the $\mathcal{O}_X$-module $\mathcal{E}^\circ$ relative to X/k. We shall write

$$\nabla^{\mathrm{can}}_{\mathcal{E}^\circ}$$

for this connection on $\mathcal{E}^\circ$. Moreover, the isomorphism $F^* B_1 \xrightarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p)$ of Lemma 1.3(v) and the filtration of $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p)$ of Lemma 1.3(vi) determine a filtration of $\mathcal{E}^\circ$

$$\{0\} = \mathcal{E}^\circ_{p-1} \subseteq \mathcal{E}^\circ_{p-2} \subseteq \cdots \subseteq \mathcal{E}^\circ_1 \subseteq \mathcal{E}^\circ_0 = \mathcal{E}^\circ.$$

(Thus, it follows immediately from the definition of the filtration that $\mathcal{E}^\circ/\mathcal{E}^\circ_1 \cong F^*\mathcal{L}^\circ \otimes_{\mathcal{O}_X} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^2) = F^*\mathcal{L}^\circ \otimes_{\mathcal{O}_X} \omega_{X/k}$.)

PROPOSITION 1.4. *The collection of data*

$$\mathfrak{E}^\circ \stackrel{\text{def}}{=} (\mathcal{E}^\circ, \nabla^{\mathrm{can}}_{\mathcal{E}^\circ}, \{0\} = \mathcal{E}^\circ_{p-1} \subseteq \mathcal{E}^\circ_{p-2} \subseteq \cdots \subseteq \mathcal{E}^\circ_1 \subseteq \mathcal{E}^\circ_0 = \mathcal{E}^\circ)$$

forms a dormant oper of rank $p-1$ *over* X/k.

https://doi.org/10.1017/nmj.2018.4 Published online by Cambridge University Press

Proof. The assertion that $\mathfrak{E}^\circ$ satisfies condition (1) of Definition 1.1 follows from the definition of the $\mathcal{O}_X$-submodules $\mathcal{E}_i^\circ \subseteq \mathcal{E}^\circ$. The assertion that $\mathfrak{E}^\circ$ satisfies conditions (2) and (3) of Definition 1.1 follows immediately from a local explicit calculation (cf. also [5, Theorem 3.1.6]). The assertion that $\mathfrak{E}^\circ$ satisfies condition (4) of Definition 1.1 follows from the definition of the connection $\nabla_{\mathcal{E}^\circ}^{\mathrm{can}}$. The assertion that $\mathfrak{E}^\circ$ satisfies condition (5) of Definition 1.1 follows from the fact that $\det(\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} B_1)$ is isomorphic to $\mathcal{O}_{X^{(1)}}$ (cf. Lemma 1.3(ix)), together with the definition of the connection $\nabla_{\mathcal{E}^\circ}^{\mathrm{can}}$. This completes the proof of Proposition 1.4. ∎

REMARK 1.4.1. Let us observe that the choice of $\mathcal{L}^\circ$ as in Lemma 1.3(ix) is not unique. More precisely, if we write $(\mathrm{Pic}\, X)[p-1] \subseteq \mathrm{Pic}\, X$ for the subgroup of $\mathrm{Pic}\, X$ obtained by forming the kernel of the endomorphism of $\mathrm{Pic}\, X$ given by multiplication by $p-1$, then the set consisting of isomorphism classes of possible $\mathcal{L}^\circ$s forms a $(\mathrm{Pic}\, X)[p-1]$-torsor. On the other hand, it is immediate from the various definitions involved that the adoption of another possible $\mathcal{L}^\circ$ does not affect the equivalence class of the dormant oper of Proposition 1.4.

REMARK 1.4.2. It should be noted that the assertion that every X/k admits a dormant oper of rank $p-1$ was already essentially proved. For instance, it follows immediately, in light of Remark 1.2.2, from [7, Chapter II, Theorem 2.8] (cf. also the final Remark of [7, Chapter IV, §2.1]), that every X/k admits a dormant oper of rank two; thus, by considering the $(p-2)$-nd symmetric power of a dormant oper of rank two (cf. also the discussion concerning $\mathcal{G}_r(\theta)$ in [5, §3.2]), we conclude that every X/k admits a dormant oper of rank $p-1$.

§2. Uniqueness of dormant opers of rank $p-1$

In this section, we maintain the notation of §1. In particular, we have a projective smooth curve X over k. In this section, we prove the following theorem.

THEOREM 2.1. *Every dormant oper of rank $p-1$ over X/k is equivalent (cf. Definition 1.2) to the dormant oper of Proposition 1.4 (cf. also Remark 1.4.1).*

REMARK 2.1.1. Theorem 2.1 is a generalization of the first portion of [3], Theorem A.

REMARK 2.1.2. As discussed in Remark 1.2.1, the notion of "dormant oper of rank r" of the present paper is essentially the same as the notion of "dormant $\mathrm{PGL}(r)$-oper" in the terminology given in [4, §3]. On the other hand, one may find the definition of the notion of G-oper in, for instance, [1, §3], [2, §1], for a more general algebraic group G. Thus, by imposing a condition similar to condition (4) of Definition 1.1 on such a G-oper, one may define the notion of dormant G-oper (as well as the notion of the isomorphism class of a dormant G-oper).

It should be noted that the proof of Theorem 2.1, that is, the proof of the main result of the present paper, may give a proof of the assertion that the set of isomorphism classes of dormant G-opers over X/k is of cardinality one for a suitable algebraic group G. We leave the routine details of such generalizations to the interested reader.

To complete the verification of Theorem 2.1, let

$$\mathfrak{E} \stackrel{\mathrm{def}}{=} (\mathcal{E},\ \nabla_{\mathcal{E}},\ \{0\} = \mathcal{E}_{p-1} \subseteq \mathcal{E}_{p-2} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E})$$

be a dormant oper of rank $p-1$ over X/k. We shall write

$$\Xi^{\circ} \stackrel{\mathrm{def}}{=} \mathcal{E}^{\circ}/\mathcal{E}_1^{\circ}, \quad \Xi \stackrel{\mathrm{def}}{=} \mathcal{E}/\mathcal{E}_1.$$

Then, it follows from the definition of a dormant oper that both $(\Xi^{\circ})^{\otimes p-1}$ and $\Xi^{\otimes p-1}$ are isomorphic to $\tau_{X/k}{}^{\otimes(p-1)(p-2)/2}$, which thus implies that $\delta \stackrel{\mathrm{def}}{=} \mathcal{H}om_{\mathcal{O}_X}(\Xi, \Xi^{\circ})$ is of order $p-1$. In particular, it is immediate that if we write δ^F for the invertible sheaf on $X^{(1)}$ obtained by pulling back δ via the natural morphism $X^{(1)} \to X$, then it holds that $F^*\delta^F \cong \delta$. Moreover, by considering the connection on $F^*\delta^F$ determined by the factorization of Lemma 1.3(iii), we conclude that $\mathcal{E} \otimes_{\mathcal{O}_X} F^*\delta^F$ determines a dormant oper that is equivalent to $\mathfrak{E}$. Thus, to complete the verification of Theorem 2.1, we may assume without loss of generality, by replacing $\mathcal{E}$ by $\mathcal{E} \otimes_{\mathcal{O}_X} F^*\delta^F$, that

$$\Xi^{\circ} \cong \Xi.$$

By means of such an isomorphism, let us identify Ξ° with Ξ.

Next, let us observe that since the p-curvature of $\nabla_{\mathcal{E}}$ is zero, and $(\det \mathcal{E}, \det \nabla_{\mathcal{E}})$ is isomorphic to $(\mathcal{O}_X, d)$, it follows from a theorem of Cartier (cf., e.g., [6, Theorem 5.1]) that if we write $\mathcal{F} \stackrel{\mathrm{def}}{=} \mathcal{E}^{\nabla_{\mathcal{E}}}$ for the $\mathcal{O}_{X^{(1)}}$-module of horizontal sections of $\mathcal{E}$ with respect to $\nabla_{\mathcal{E}}$, then

(a) $\mathcal{E}$ is naturally isomorphic to $F^*\mathcal{F}$;
(b) the connection $\nabla_{\mathcal{E}}$ arises from the isomorphism of (a) (together with the factorization of Lemma 1.3(iii)); and
(c) $\det \mathcal{F}$ is isomorphic to $\mathcal{O}_{X^{(1)}}$.

Thus, the isomorphism $F^*\mathcal{F} \xrightarrow{\sim} \mathcal{E}$ of (a) and the surjection $\mathcal{E} \twoheadrightarrow \Xi = \Xi^\circ$ determine a surjection of $\mathcal{O}_X$-modules

$$F^*\mathcal{F} \twoheadrightarrow \Xi^\circ,$$

and hence also a homomorphism of $\mathcal{O}_{X^{(1)}}$-modules

$$\mathcal{F} \longrightarrow F_*\Xi^\circ.$$

LEMMA 2.2. *The above homomorphism $\mathcal{F} \to F_*\Xi^\circ$ is a locally split injection.*

Proof. Let us first observe that it is immediate that, to verify Lemma 2.2, it suffices to verify that the homomorphism $\gamma\colon \mathcal{E} \xleftarrow{\sim} F^*\mathcal{F} \to \mathcal{G} \stackrel{\text{def}}{=} F^*F_*\Xi^\circ$ of $\mathcal{O}_X$-modules obtained by pulling back, via F, the homomorphism under consideration is a split injection. Next, let us observe that it follows immediately from the various definitions involved that the composite of γ and the natural homomorphism $\mathcal{G} = F^*F_*\Xi^\circ \to \Xi^\circ$ coincides with the natural surjection $\mathcal{E} \twoheadrightarrow \Xi = \Xi^\circ$.

We write $\nabla_{\mathcal{G}}$ for the connection on $\mathcal{G}$ determined by the factorization of Lemma 1.3(iii). Thus, it follows immediately from the definitions of $\nabla_{\mathcal{E}}$ and $\nabla_{\mathcal{G}}$ that γ is horizontal with respect to $\nabla_{\mathcal{E}}$ and $\nabla_{\mathcal{G}}$. Moreover, for $i \in \{1, \ldots, p\}$, let us define submodules $\mathcal{G}_i \subseteq \mathcal{G}$ inductively as follows. We shall write $\mathcal{G}_1 \stackrel{\text{def}}{=} \operatorname{Ker}(\mathcal{G} = F^*F_*\Xi^\circ \twoheadrightarrow \Xi^\circ)$. If $i \geqslant 2$, then we shall write $\mathcal{G}_i \stackrel{\text{def}}{=} \operatorname{Ker}\big(\mathcal{G}_{i-1} \hookrightarrow \mathcal{G} \xrightarrow{\nabla_{\mathcal{G}}} \mathcal{G} \otimes_{\mathcal{O}_X} \omega_{X/k} \twoheadrightarrow (\mathcal{G}/\mathcal{G}_{i-1}) \otimes_{\mathcal{O}_X} \omega_{X/k}\big)$. Then, it follows that the submodule $\mathcal{G}_i \subseteq \mathcal{G}$ is an $\mathcal{O}_X$-submodule, and $\mathcal{G}_p = \{0\}$. Moreover, it follows immediately from a local explicit calculation (cf. also [5, Theorem 3.1.6]) that the collection of data

$$(\mathcal{G},\ \nabla_{\mathcal{G}},\ \{0\} = \mathcal{G}_p \subseteq \mathcal{G}_{p-1} \subseteq \cdots \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_0 \stackrel{\text{def}}{=} \mathcal{G})$$

satisfies conditions (1), (2), (3), and (4) of Definition 1.1.

Now, I claim that the following assertion holds.

For each $i \in \{1, \ldots, p-1\}$, the composite $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E} \xrightarrow{\gamma} \mathcal{G}$ determines an isomorphism $\mathcal{E}_{i-1}/\mathcal{E}_i \xrightarrow{\sim} \mathcal{G}_{i-1}/\mathcal{G}_i$ of $\mathcal{O}_X$-modules.

Let us verify this claim by induction on i. If $i=1$, then the desired assertion has already been verified (in the first paragraph of this proof). Let $i \in \{2, \ldots, p-1\}$. Suppose that γ determines an isomorphism $\mathcal{E}_{i-2}/\mathcal{E}_{i-1} \xrightarrow{\sim} \mathcal{G}_{i-2}/\mathcal{G}_{i-1}$ of $\mathcal{O}_X$-modules, which thus implies that $\gamma(\mathcal{E}_{i-1}) \subseteq \mathcal{G}_{i-1}$. Thus, since γ is horizontal, the diagram

$$\begin{array}{ccccccc} \mathcal{E}_{i-1} & \xrightarrow{\subseteq} & \mathcal{E} & \xrightarrow{\nabla_{\mathcal{E}}} & \mathcal{E} \otimes_{\mathcal{O}_X} \omega_{X/k} & \longrightarrow & (\mathcal{E}/\mathcal{E}_{i-1}) \otimes_{\mathcal{O}_X} \omega_{X/k} \\ \downarrow & & \gamma \downarrow & & \gamma\otimes\mathrm{id} \downarrow & & \downarrow \\ \mathcal{G}_{i-1} & \xrightarrow{\subseteq} & \mathcal{G} & \xrightarrow{\nabla_{\mathcal{G}}} & \mathcal{G} \otimes_{\mathcal{O}_X} \omega_{X/k} & \longrightarrow & (\mathcal{G}/\mathcal{G}_{i-1}) \otimes_{\mathcal{O}_X} \omega_{X/k} \end{array}$$

commutes. In particular, it follows immediately from condition (3) of Definition 1.1, together with the induction hypothesis, that γ determines an isomorphism $\mathcal{E}_{i-1}/\mathcal{E}_i \xrightarrow{\sim} \mathcal{G}_{i-1}/\mathcal{G}_i$ of $\mathcal{O}_X$-modules, as desired. This completes the proof of the above claim.

It follows immediately from the above claim that the composite $\mathcal{E} \xrightarrow{\gamma} \mathcal{G} \twoheadrightarrow \mathcal{G}/\mathcal{G}_{p-1}$ is an isomorphism of $\mathcal{O}_X$-modules. In particular, γ is a split injection. This completes the proof of Lemma 2.2. □

By Lemma 2.2, together with the fact that Ξ° is isomorphic to $F^*\mathcal{L}^\circ \otimes_{\mathcal{O}_X} \omega_{X/k}$ (cf. the discussion preceding Proposition 1.4), we have an *exact sequence of locally free* $\mathcal{O}_{X^{(1)}}$*-modules*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} F_*\omega_{X/k} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

LEMMA 2.3. *The above* $\mathcal{O}_{X^{(1)}}$*-module* $\mathcal{Q}$ *is isomorphic to* $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} \omega_{X^{(1)}/k}$.

Proof. Let us first observe that $\mathcal{F}$ is of rank $p-1$, and $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} F_*\omega_{X/k}$ is of rank p, which thus implies that $\mathcal{Q}$ is an invertible sheaf on $X^{(1)}$. Thus, by the exact sequence of the discussion preceding Lemma 2.3, together with (c) of the discussion preceding Lemma 2.2, it holds that

$$\mathcal{Q} \cong (\mathcal{L}^\circ)^{\otimes p} \otimes_{\mathcal{O}_{X^{(1)}}} \det F_*\omega_{X/k}.$$

In particular, since (it follows from our choice of $\mathcal{L}^\circ$ – cf. Lemma 1.3(ix) – that) $(\mathcal{L}^\circ)^{\otimes p-1}$ is isomorphic to $\mathcal{H}om_{\mathcal{O}_{X^{(1)}}}(\det B_1, \mathcal{O}_{X^{(1)}})$, we obtain that

$$\mathcal{Q} \cong \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{H}om_{\mathcal{O}_{X^{(1)}}}(\det B_1, \det F_*\omega_{X/k}).$$

Next, let us recall (cf., e.g., [6, Theorem 7.2]) the well-known exact sequence of $\mathcal{O}_{X^{(1)}}$-modules

$$0 \longrightarrow \mathcal{O}_{X^{(1)}} \longrightarrow F_*\mathcal{O}_X \stackrel{F_*d}{\longrightarrow} F_*\omega_{X/k} \stackrel{c}{\longrightarrow} \omega_{X^{(1)}/k} \longrightarrow 0,$$

where we write c for the *Cartier operator*. Thus, it follows from the definition of B_1 that

$$\det F_*\omega_{X/k} \cong \omega_{X^{(1)}/k} \otimes_{\mathcal{O}_{X^{(1)}}} \det B_1.$$

This completes the proof of Lemma 2.3. ∎

By Lemma 2.3, we have an exact sequence of locally free $\mathcal{O}_{X^{(1)}}$-modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} F_*\omega_{X/k} \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} \omega_{X^{(1)}/k} \longrightarrow 0.$$

On the other hand, we have an exact sequence of locally free $\mathcal{O}_{X^{(1)}}$-modules

$$0 \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} B_1 \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} F_*\omega_{X/k} \stackrel{c}{\longrightarrow} \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} \omega_{X^{(1)}/k} \longrightarrow 0$$

(cf. the well-known exact sequence that appears in the second paragraph of the proof of Lemma 2.3).

Let us complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Now, I claim that the following assertion holds.

> Every homomorphism $F_*\omega_{X/k} \to \omega_{X^{(1)}/k}$ of $\mathcal{O}_{X^{(1)}}$-modules is a k-multiple of the Cartier operator c (cf. the well-known exact sequence that appears in the second paragraph of the proof of Lemma 2.3).

Indeed, it follows immediately from a local explicit calculation that the map $F_*\mathcal{O}_X \times F_*\omega_{X/k} \to \omega_{X^{(1)}/k}$ given by mapping (f, θ) to $c(f \cdot \theta)$ determines an isomorphism $F_*\mathcal{O}_X \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X^{(1)}}}(F_*\omega_{X/k}, \omega_{X^{(1)}/k})$ of $\mathcal{O}_{X^{(1)}}$-modules (cf. also the discussion preceding [8, Théorème 4.1.1]). Thus, the above claim follows from the fact that $\Gamma(X^{(1)}, \mathcal{H}om_{\mathcal{O}_{X^{(1)}}}(F_*\omega_{X/k}, \omega_{X^{(1)}/k})) \cong \Gamma(X^{(1)}, F_*\mathcal{O}_{X^{(1)}}) \cong \Gamma(X, \mathcal{O}_X)$ is of dimension one. This completes the proof of the above claim.

Let us recall the two exact sequences of $\mathcal{O}_{X^{(1)}}$-modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} F_*\omega_{X/k} \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} \omega_{X^{(1)}/k} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} B_1 \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} F_*\omega_{X/k} \stackrel{c}{\longrightarrow} \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} \omega_{X^{(1)}/k} \longrightarrow 0$$

appearing in the discussion following Lemma 2.3. It follows immediately from the above claim that we have an equality $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} B_1 = \mathcal{F}$

(in $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^{(1)}}} F_*\omega_{X/k}$). In particular, we have an isomorphism $\phi\colon \mathcal{E}^\circ \xrightarrow{\sim} \mathcal{E}$ (i.e., $F^*\mathrm{id}$). Moreover, it follows immediately from the definition of $\nabla^{\mathrm{can}}_{\mathcal{E}^\circ}$, together with (b) of the discussion preceding Lemma 2.2, that the isomorphism ϕ is horizontal with respect to $\nabla^{\mathrm{can}}_{\mathcal{E}^\circ}$, $\nabla_{\mathcal{E}}$.

Finally, we verify that $\phi(\mathcal{E}^\circ_{p-i}) = \mathcal{E}_{p-i}$ for each $i \in \{1, \ldots, p-1\}$ by induction on i. The equality $(\{0\} =)\ \phi(\mathcal{E}^\circ_{p-1}) = \mathcal{E}_{p-1}\ (= \{0\})$ is immediate. Let $i \in \{2, \ldots, p-1\}$. Suppose that $\phi(\mathcal{E}^\circ_{p-i+1}) = \mathcal{E}_{p-i+1}$ holds. Then, to verify that $\phi(\mathcal{E}^\circ_{p-i}) = \mathcal{E}_{p-i}$, it suffices to verify that the induced isomorphism $\mathcal{E}^\circ/\mathcal{E}^\circ_{p-i+1} \xrightarrow{\sim} \mathcal{E}/\mathcal{E}_{p-i+1}$ maps $\mathcal{E}^\circ_{p-i}/\mathcal{E}^\circ_{p-i+1}$ isomorphically onto $\mathcal{E}_{p-i}/\mathcal{E}_{p-i+1}$. On the other hand, since (it follows immediately from the definition of a dormant oper that)

$$\mathcal{E}^\circ_{j-1}/\mathcal{E}^\circ_j \;\cong\; \mathcal{E}_{j-1}/\mathcal{E}_j \;\cong\; \Xi^\circ \otimes_{\mathcal{O}_X} \omega_{X/k}^{\otimes j-1}$$

for each $j \in \{1, \ldots, p-1\}$, the desired assertion follows immediately from the ampleness of $\omega_{X/k}$. This completes the proof of Theorem 2.1. □

Acknowledgments. The author would like to thank Akio Tamagawa for discussions concerning removal of the hypothesis that the Jacobian variety of the curve under consideration is ordinary from the main result in an earlier version of the manuscript. The author would like to thank Yasuhiro Wakabayashi for explaining basic facts concerning dormant opers. The author also would like to thank the referee for comments concerning Remark 1.4.2 and Remark 2.1.2.

References

[1] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, Available at http://math.uchicago.edu/mitya/langlands/hitchin/BDhitchin.pdf.

[2] A. Beilinsion and V. Drinfeld, *Opers*, Preprint, 2005, arXiv:math/0501398v1 [math.AG].

[3] Y. Hoshi, *Nilpotent admissible indigenous bundles via Cartier operators in characteristic three*, Kodai Math. J. **38**(3) (2015), 690–731.

[4] K. Joshi, *The degree of the dormant operatic locus*, Int. Math. Res. Not. IMRN **9** (2017), 2599–2613.

[5] K. Joshi and C. Pauly, *Hitchin–Mochizuki morphism, opers and Frobenius-destabilized vector bundles over curves*, Adv. Math. **274** (2015), 39–75.

[6] N. M. Katz, *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin*, Publ. Math. Inst. Hautes Études Sci. **39** (1970), 175–232.

[7] S. Mochizuki, *Foundations of p-adic Teichmüller theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society, Providence, RI, International Press, Cambridge, MA, 1999.

[8] M. Raynaud, *Sections des fibrés vectoriels sur une courbe*, Bull. Soc. Math. France **110**(1) (1982), 103–125.
[9] Y. Wakabayashi, *An explicit formula for the generic number of dormant indigenous bundles*, Publ. Res. Inst. Math. Sci. **50**(3) (2014), 383–409.
[10] Y. Wakabayashi, *A theory of dormant opers on pointed stable curves – a proof of Joshi's conjecture*, Preprint, 2014, arXiv:1411.1208v1 [math.AG].

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan
yuichiro@kurims.kyoto-u.ac.jp