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Weak Type Estimates of the Maximal Quasiradial Bochner-Riesz Operator On Certain Hardy Spaces

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Abstract. Let $\{A_t\}_{t>0}$ be the dilation group in $\mathbb{R}^n$ generated by the infinitesimal generator M where $A_t = \exp(M \log t)$, and let $\varrho \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a A_t-homogeneous distance function defined on $\mathbb{R}^n$. For $f \in \mathfrak{S}(\mathbb{R}^n)$, we define the maximal quasiradial Bochner-Riesz operator $\mathfrak{M}_\varrho^\delta$ of index $\delta > 0$ by

$$\mathfrak{M}_\varrho^\delta f(x) = \sup_{t>0} \left| \mathcal{F}^{-1}[(1 - \varrho/t)_+^\delta \hat{f}](x) \right|.$$

If $A_t = tI$ and $\{\xi \in \mathbb{R}^n \mid \varrho(\xi) = 1\}$ is a smooth convex hypersurface of finite type, then we prove in an extremely easy way that $\mathfrak{M}_\varrho^\delta$ is well defined on $H^p(\mathbb{R}^n)$ when $\delta = n(1/p - 1/2) - 1/2$ and $0 < p < 1$; moreover, it is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$.

If $A_t = tI$ and $\varrho \in C^\infty(\mathbb{R}^n \setminus \{0\})$, we also prove that $\mathfrak{M}_\varrho^\delta$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ when $\delta > n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

1 Introduction

Let $\mathfrak{S}(\mathbb{R}^n)$ be the Schwartz space on $\mathbb{R}^n$. For $f \in \mathfrak{S}(\mathbb{R}^n)$, we denote the Fourier transform of f by

$$\mathcal{F}[f](x) = \hat{f}(x) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(\xi)\, d\xi.$$

Then the inverse Fourier transform of f is given by

$$\mathcal{F}^{-1}[f](x) = \check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi)\, d\xi.$$

Let M be a real-valued $n \times n$ matrix whose eigenvalues have positive real parts. Then we consider the dilation group $\{A_t\}_{t>0}$ in $\mathbb{R}^n$ generated by the infinitesimal generator M, where $A_t = \exp(M \log t)$ for $t > 0$. We introduce A_t-homogeneous distance functions ϱ defined on $\mathbb{R}^n$; that is, $\varrho\colon \mathbb{R}^n \to [0,\infty)$ is a continuous function satisfying $\varrho(A_t\xi) = t\varrho(\xi)$ for all $\xi \in \mathbb{R}^n$. One can refer to [3] and [11] for its fundamental properties.

In what follows we shall denote by $\Sigma_\varrho \fallingdotseq \{\xi \in \mathbb{R}^n \mid \varrho(\xi) = 1\}$ the unit sphere of ϱ and denote by $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$. We use polar coordinates; given $x \in \mathbb{R}^n$, we write

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$x = r\theta$ where $r = |x|$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in S^{n-1}$. Given two quantities A and B, we write $A \lesssim B$ or $B \gtrsim A$ if there is a positive constant c (possibly depending on the dimension n and the index p to be given) such that $A \leq cB$. We also write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

For $f \in \mathfrak{S}(\mathbb{R}^n)$, we consider quasiradial Bochner-Riesz means of index $\delta > 0$ defined by

$$\mathfrak{R}^{\delta}_{\varrho,t} f(x) = \mathcal{F}^{-1}[(1-\varrho/t)^{\delta}_{+} \hat{f}](x),$$

and the corresponding maximal operator

$$\mathfrak{M}^{\delta}_{\varrho} f(x) = \sup_{t>0} |\mathfrak{R}^{\delta}_{\varrho,t} f(x)|.$$

In the special case that $\varrho(\xi) = |\xi|^2$ and $A_t = tI$, Stein, Taibleson, and Weiss [10] proved that if $0 < p < 1$, then $\mathfrak{M}^{\delta}_{\varrho}$ is bounded from $H^p(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ at the critical index $\delta = \delta(p) \risingdotseq n(1/p - 1/2) - 1/2$, where $H^p(\mathbb{R}^n)$ is the standard real Hardy space defined in Stein [9] and $L^{p,\infty}(\mathbb{R}^n)$ is one of the Lorentz spaces (which is called *weak-L^p space*) defined in Stein and Weiss [12]. Furthermore Stein obtained the exceptional result that there is $f \in H^1(\mathbb{R}^n)$ such that *a.e.* convergence of the Bochner-Riesz means fails for $p = 1$ and $\delta(1) = (n-1)/2$.

In our first result we shall assume that $\varrho \in C^{\infty}(\mathbb{R}^n_0)$, $A_t = tI$ and Σ_ϱ is a smooth convex hypersurface of $\mathbb{R}^n$ which is of finite type, *i.e.*, every tangent line makes finite order of contact with Σ_ϱ. We say that Σ_ϱ is of finite type $k \geq 2$ if k is the maximal order of contact on Σ_ϱ.

Theorem 1.1 *Suppose that $A_t = tI$, $\varrho \in C^{\infty}(\mathbb{R}^n_0)$ is a A_t-homogeneous distance function defined on $\mathbb{R}^n$, and Σ_ϱ is a smooth convex hypersurface of finite type. Then $\mathfrak{M}^{\delta(p)}_{\varrho}$ is well defined on $H^p(\mathbb{R}^n)$ when $0 < p < 1$; moreover, $\mathfrak{M}^{\delta(p)}_{\varrho}$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$. That is, there is a constant $C = C(n, p, \Sigma_\varrho) > 0$ such that for any $f \in H^p(\mathbb{R}^n)$,*

$$\big|\{x \in \mathbb{R}^n \mid \mathfrak{M}^{\delta(p)}_{\varrho} f(x) > \lambda\}\big| \leq \frac{C}{\lambda^p} \|f\|^p_{H^p(\mathbb{R}^n)}, \quad \lambda > 0,$$

where $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}^n$.

Remark As a matter of fact, we prove this result under more general surface condition than the finite type condition on Σ_ϱ, which is to be called a *spherically integrable condition* of order < 1 in Section 3.

Our second result is to obtain that if $\delta > n(1/p - 1/2) - 1/2$ and $0 < p < 1$ then $\mathfrak{M}^{\delta}_{\varrho}$ admits (H^p, L^p)-estimate under no surface condition on Σ_ϱ.

Theorem 1.2 *Suppose that $A_t = tI$ and $\varrho \in C^{\infty}(\mathbb{R}^n_0)$ is a A_t-homogeneous distance function defined on $\mathbb{R}^n$. If $\delta > \delta(p)$ for $0 < p < 1$, then $\mathfrak{M}^{\delta}_{\varrho}$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$; that is, there is a constant $C = C(n, p) > 0$ such that for any $f \in H^p(\mathbb{R}^n)$,*

$$\|\mathfrak{M}^{\delta}_{\varrho} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{H^p(\mathbb{R}^n)},$$

provided that $\delta > n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

Remark This problem is still left open at the critical index $\delta = n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

2 (H^p, L^p)-Estimate For the Case that $\varrho \in C^\infty(\mathbb{R}^n_0)$ and $\delta > \delta(p)$

We shall employ a decomposition of the Bochner-Riesz multiplier $(1 - \varrho)^\delta_+$ as in A. Córdoba [2]. Let $\phi \in C_0^\infty(1/2, 2)$ satisfy $\sum_{k\in\mathbb{Z}} \phi(2^k t) = 1$ for all $t > 0$. For $k \in \mathbb{N}$, let $\Phi^\delta_k = \phi\big(2^{k+1}(1-\varrho)\big)(1-\varrho)^\delta_+$ and $\Phi^\delta_0 = (1-\varrho)^\delta_+ - \sum_{k\in\mathbb{N}} \Phi^\delta_k$. For each $k \in \mathbb{Z}$, we now introduce a partition of unity $\Xi_{k\ell}$, $\ell = 1, 2, \ldots, N_k$, on the unit sphere Σ_ϱ which we extend to $\mathbb{R}^n$ by way of $\Pi_{k\ell}(A_t\zeta) = \Xi_{k\ell}(\zeta)$, $t > 0$, $\zeta \in \Sigma_\varrho$, and which satisfies the following properties; there are a finite number of points $\zeta_{k1}, \zeta_{k2}, \ldots, \zeta_{kN_k} \in \Sigma_\varrho$ such that for $\ell = 1, 2, \ldots, N_k$,

(i) $\sum_{\ell=1}^{N_k} \Pi_{k\ell}(\zeta) \equiv 1$ for all $\zeta \in \Sigma_\varrho$,
(ii) $\Xi_{k\ell}(\zeta) = 1$ for all $\zeta \in \Sigma_\varrho \cap B(\zeta_{k\ell}; 2^{-k/2})$,
(iii) $\Xi_{k\ell}$ is supported in $\Sigma_\varrho \cap B(\zeta_{k\ell}; c_1 2^{-k/2})$,
(iv) $|\mathcal{D}^\alpha \Pi_{k\ell}(\xi)| \le c_2 2^{|\alpha|k/2}$ for any multiindex α, if $1/2 \le \varrho(\xi) \le 2$,
(v) $N_k \le c_3 2^{(n-1)k/2}$ for fixed k,

where $B(\zeta_0; s)$ denotes the ball in $\mathbb{R}^n$ with center $\zeta_0 \in \Sigma_\varrho$ and radius $s > 0$ and the positive constants c_1, c_2, c_3 do not depend upon k. For each $k \in \mathbb{Z}$, let $\mathcal{H}^\delta_{\varrho k\ell} = \mathcal{F}^{-1}[\Phi^\delta_k \Pi_{k\ell}]$ and $\mathcal{H}_0 = \mathcal{F}^{-1}[\Phi^\delta_0]$.

Next we invoke a simple observation used in [8] to obtain decay estimate for kernels $\mathcal{H}_{k\ell}$, $\mathcal{H}_0$ corresponding to the decomposition of the Bochner-Riesz multiplier defined in the above. Without loss of generality, we can assume that $\varrho \in C^\infty(\mathbb{R}^n)$ because we can replace ϱ by ϱ^N for sufficiently large $N > 0$ by a subordination argument in [3]. Then we easily see that the kernel $\mathcal{H}_0$ has a nice decay, and so its corresponding maximal operator admits $(H^p, L^{p,\infty})$-estimate for the critical index $\delta(p) = n(1/p - 1/2) - 1/2$ and $0 < p < 1$ as in that of Stein, Taibleson, and Weiss [10]. Thus we concentrate upon obtaining the decay estimate for the kernels $\mathcal{H}^\delta_{\varrho k\ell}$.

Lemma 2.1 *For fixed $k \in \mathbb{N}$ and for $\ell = 1, 2, \ldots, N_k$, let $T_{\zeta_{k\ell}}(\Sigma_\varrho)$ be the tangent space of Σ_ϱ at $\zeta_{k\ell} \in \Sigma_\varrho$, $\{e^j_{k\ell}\}_{j=1}^{n-1}$ be an orthonormal basis of $T_{\zeta_{k\ell}}(\Sigma_\varrho)$, and $e^0_{k\ell}$ be the outer unit normal vector to Σ_ϱ at $\zeta_{k\ell} \in \Sigma_\varrho$. Then we have the following estimate*

$$|\mathcal{H}^\delta_{\varrho k\ell}(x)| \le \frac{C_N 2^{-k(\delta+1+(n-1)/2)}}{\big(1 + 2^{-k}|\langle x, e^0_{k\ell}\rangle|\big)^N \prod_{j=1}^{n-1}\big(1 + 2^{-k/2}|\langle x, e^j_{k\ell}\rangle|\big)^N}$$

for any $N \in \mathbb{N}$.

Proof We need the following simple observation:

Let $\varrho \in C^N(\mathbb{R}^n)$ and $F \in C^N(\mathbb{R}_+)$. For $e \in S^{n-1}$, let $\mathcal{D}_e f$ be the directional derivative $\langle e, \nabla f\rangle$. Then one can have the formula (see [8])

$$\mathcal{D}^N_e(F \circ \varrho) = \sum_{\nu=1}^N F^{(\nu)} \circ \varrho \sum_{\beta \in \mathcal{Y}^N_\nu} \sum_{m=1}^{\nu} c_{N,\beta_m} \mathcal{D}^\beta_e \varrho \tag{2.1}$$

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where

$$\mathcal{Y}_\nu^N = \left\{ \beta \;\Big|\; \sum_{m=1}^{\nu} \beta_m = N,\ \text{at least } \nu - \frac{N}{2} \text{ of the numbers } \beta_m \text{ are equal to } 1 \right\},$$

$\beta = (\beta_1, \dots, \beta_\nu)$ is a multiindex, and c_{N,β_m}s are some constants. For $k \in \mathbb{N}$, let $F_k(t) = \phi\big(2^{k+1}(1-t)\big)(1-t)_+^\delta$. Then it follows from simple computation that

$$F_k^{(\nu)}(t) = (-1)^\nu \sum_{i=0}^{\nu} C(\nu, i) C(\delta, \nu - i) 2^{i(k+1)} \phi^{(i)}\big(2^{k+1}(1-t)\big)(1-t)^{\delta-\nu+i} \tag{2.2}$$

where $C(\nu, i) = \nu(\nu-1)(\nu-2)\cdots(\nu-i+1)$ for positive integers ν, i, and $C(\nu, 0) = 1$. For fixed k, ℓ, we have the estimate

$$\|\mathcal{D}_{e_{k\ell}^0}^N [\Phi_k^\delta \Pi_{k\ell}]\|_{L^1} \le c 2^{-k(\frac{n+1}{2})} 2^{-k\delta} 2^{kN} \tag{2.3}$$

for any $N \in \mathbb{N}$. Since we have the better estimate $|\mathcal{D}_{e_{k\ell}^j} \varrho| \le c 2^{-k/2}$ on the support of $\mathcal{F}[\mathcal{H}_{\varrho k\ell}^\delta]$ for fixed j, k, ℓ, it follows from (2.1) and Taylor's theorem that

$$\|\mathcal{D}_{e_{k\ell}^j}^N [\Phi_k^\delta \Pi_{k\ell}]\|_{L^1} \le c 2^{-k(\frac{n+1}{2})} 2^{-k\delta} 2^{kN/2} \tag{2.4}$$

for any $N \in \mathbb{N}$. Using integration by parts, it follows from (2.3) and (2.4) that

$$|\mathcal{H}_{\varrho k\ell}^\delta(x)| \le \frac{C_N 2^{-(\delta+1+(n-1)/2)k}}{\big(1 + 2^{-k}|\langle x, e_{k\ell}^0\rangle|\big)^N \prod_{j=1}^{n-1}\big(1 + 2^{-k/2}|\langle x, e_{k\ell}^j\rangle|\big)^N} \tag{2.5}$$

for any $N \in \mathbb{N}$. ■

We now introduce the real Hardy space $H^p(\mathbb{R}^n)$ defined in terms of atomic decompositions along the pattern of Stein [9]. For $0 < p \le 1$, a function $\mathfrak{a} \in L^\infty(\mathbb{R}^n)$ is called a (p, μ)-*atom centered* at $x_0 \in \mathbb{R}^n$ if it satisfies

(i) there is a ball $B(x_0; s)$ with $\operatorname{supp}(\mathfrak{a}) \subset B(x_0; s)$,
(ii) $\|\mathfrak{a}\|_{L^\infty} \le |B(x_0; s)|^{-1/p}$, and
(iii) $\int_{\mathbb{R}^n} \mathfrak{a}(x) x^\alpha \, dx = 0$ for $|\alpha| \le \mu$,

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is an n-tuple of nonnegative integers and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. If $f = \sum_{k=1}^\infty c_k \mathfrak{a}_k$ where the $\mathfrak{a}_k$s are (p, μ)-atoms and $\{c_k\} \in \ell^p$, then $f \in H^p(\mathbb{R}^n)$ and $\|f\|_{H^p}^p \lesssim \sum_k |c_k|^p$, and the converse inequality also holds. Here we note that if $\delta > n(1/p - 1/2) - 1/2$ then $\mu = n(1/p' - 1)$ is enough for our oncoming estimates where $p' < p$ is a positive number satisfying $\delta = n(1/p' - 1/2) - 1/2$.

For $f \in \mathfrak{S}(\mathbb{R}^n)$, $\delta > 0$, $k \in \mathbb{N}$, and $\ell = 1, 2, \dots, N_k$, let

$$\mathfrak{M}_{\varrho k\ell}^\delta f(x) = \sup_{t>0} |\mathcal{H}_{\varrho k\ell}^{\delta, t} * f(x)|$$

where $\mathcal{H}^{\delta,t}_{\varrho k\ell}(x) = t^n \mathcal{H}^{\delta}_{\varrho k\ell}(A^*_t x)$, and let $\mathfrak{M}^{\delta}_{\varrho k} f(x) = \sum_{\ell=1}^{N_k} \mathfrak{M}^{\delta}_{\varrho k\ell} f(x)$.

Lemma 2.2 *If $\delta > n(1/p - 1/2) - 1/2$ for $0 < p < 1$, let a positive number $p' < p$ be chosen so that $\delta = n(1/p' - 1/2) - 1/2$. For fixed $k \in \mathbb{N}$ and for $\ell = 1, 2, \ldots, N_k$, let $T_{\zeta_{k\ell}}(\Sigma_\varrho)$ be the tangent space of Σ_ϱ at $\zeta_{k\ell} \in \Sigma_\varrho$, $\{e^j_{k\ell}\}_{j=1}^{n-1}$ be an orthonormal basis of $T_{\zeta_{k\ell}}(\Sigma_\varrho)$, and $e^0_{k\ell}$ be the outer unit normal vector to Σ_ϱ at $\zeta_{k\ell} \in \Sigma_\varrho$. Then we have the following estimate*

$$|\mathcal{H}^{\delta}_{\varrho k\ell}(x)| + |\nabla \mathcal{H}^{\delta}_{\varrho k\ell}(x)| \le \frac{C_p 2^{-k(\frac{n-1}{2p'})}}{\prod_{j=0}^{n-1}(1 + |\langle x, e^j_{k\ell}\rangle|)^{1/p'}} \eqqcolon C_p 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x).$$

Proof This can easily be obtained by choosing $\delta = n(1/p' - 1/2) - 1/2$ and $N = 1/p'$ in Lemma 2.1. We also observe that $\nabla \mathcal{H}^{\delta}_{\varrho k\ell} = \varphi * \mathcal{H}^{\delta}_{\varrho k\ell}$ for some $\varphi \in \mathfrak{S}(\mathbb{R}^n)$. ■

Lemma 2.3 *If $\delta > n(1/p - 1/2) - 1/2$ for $0 < p < 1$, let a positive number $p' < p$ be chosen so that $\delta = n(1/p' - 1/2) - 1/2$. Suppose that $\mathfrak{a}$ is a $\big(p, n(1/p' - 1)\big)$-atom on $\mathbb{R}^n$ which is supported in the ball $B(x_0; s)$ with center $x_0 \in \mathbb{R}^n$ and radius $s > 0$. Then there is a constant $C = C(n, p) > 0$ such that*

(a) $|\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a}(x)| \le C s^{-n/p} 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(\frac{x - x_0}{s})$ *for any $x \in B(x_0; 2s)^c$,*

(b) $\|(\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a})\chi_{B(x_0;2s)^c}\|_{L^p} \le C 2^{-k(\frac{n-1}{2p'})}$,

where $P_{k\ell}(x)$ is the function given in Lemma 2.2.

Proof (a) We first assume that $\mathfrak{a}$ is a $\big(p, n(1/p' - 1)\big)$-atom which is supported in the unit ball $B(0; 1)$ centered at the origin and let $N \in \mathbb{N}$ be an integer satisfying $N < n(1/p' - 1) \le N + 1$, *i.e.*, $n/(n + N + 1) \le p' < n/(n + N)$. If $x \in B(0; 2)^c$ and $t > 1$, then it easily follows from Lemma 2.2 that

$$|\mathcal{H}^{\delta,t}_{\varrho k\ell} * \mathfrak{a}(x)| \le C t^{n(1-1/p')} 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x).$$

Since $n(1 - 1/p') < 0$, we have that

$$\sup_{t>1} |\mathcal{H}^{\delta,t}_{\varrho k\ell} * \mathfrak{a}(x)| \le C 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x). \tag{2.6}$$

If $x \in B(0; 2)^c$ and $0 < t \le 1$, let $\mathfrak{Q}_{t,x}(y)$ be the N-th order Taylor polynomial of the function $y \mapsto \mathcal{H}^{\delta(p)}_{\varrho k\ell}\big(A^*_t(x - y)\big)$ expanded near the origin. Using the moment conditions on the atom $\mathfrak{a}$ and Taylor's theorem, we obtain the estimate

$$\begin{aligned}
|\mathfrak{M}^{\delta,t}_{\varrho k\ell} * \mathfrak{a}(x)| &= t^n \Big| \int_{\mathbb{R}^n} [\mathcal{H}^{\delta}_{\varrho k\ell}\big(A^*_t(x - y)\big) - \mathfrak{Q}_{t,x}(y)] \mathfrak{a}(y)\, dy \Big| \\
&\lesssim t^{n+(N+1)} \int_0^1 \int_{B(0;1)} \big| \nabla^{N+1} \mathcal{H}^{\delta}_{\varrho k\ell}\big(A^*_t(x - \tau y)\big) \big|\, dy\, d\tau \\
&\lesssim t^{n+(N+1)-n/p'} 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x)
\end{aligned}$$

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because $n + (N+1) - n/p' \geq 0$. Thus we have that

$$\sup_{0<t\leq 1} |\mathcal{H}^{\delta,t}_{\varrho k\ell} * \mathfrak{a}(x)| \lesssim 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x). \tag{2.7}$$

By (2.6) and (2.7) we have that $\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a}(x) \lesssim 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x)$.

Finally, let $\mathfrak{a}$ be a $\big(p, n(1/p'-1)\big)$-atom which is supported in that ball $B(x_0; s)$. Without loss of generality, we assume that $x_0 = 0$. Let $\mathfrak{b}(x) = s^{n/p}\mathfrak{a}(A_s x)$. Then $\mathfrak{b}$ is clearly a $\big(p, n(1/p'-1)\big)$-atom supported in the unit ball $B(0;1)$. We also observe that

$$\begin{aligned} \mathcal{H}^{\delta,1/t}_{\varrho k\ell} * \mathfrak{a}(x) &= \int_{\mathbb{R}^n} \mathcal{H}^{\delta}_{\varrho k\ell}(A_{1/t}x - y)\mathfrak{a}(A_t y)\,dy \\ &= s^{-n/p}\int_{\mathbb{R}^n} \mathcal{H}^{\delta}_{\varrho k\ell}(A_{s/t}A_{1/s}x - y)\mathfrak{b}(A_{t/s}y)\,dy \\ &= s^{-n/p}(t/s)^{-n}\int_{\mathbb{R}^n} \mathcal{H}^{\delta}_{\varrho k\ell}\big(A_{s/t}(A_{1/s}x - y)\big)\,\mathfrak{b}(y)\,dy \\ &= s^{-n/p}\mathcal{H}^{\delta,s/t}_{\varrho k\ell} * \mathfrak{b}(A_{1/s}x). \end{aligned} \tag{2.8}$$

Therefore, combining this with the above estimate, we complete the part (a).

(b) We observe that there is a constant $C = C(n,p) > 0$ such that for any $x_0 \in \mathbb{R}^n$ and for any $k \in \mathbb{N}$, $\ell = 1, 2, \dots, N_k$,

$$\|P_{k\ell}(\cdot - x_0)\|_{L^p} \leq C. \tag{2.9}$$

Then it easily follows from the change of variable and (2.9) that

$$\|(\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a})\chi_{B(x_0;2s)^c}\|_{L^p} \leq C2^{-k(\frac{n-1}{2p'})}\|P_{k\ell}(\cdot - x_0/s)\|_{L^p} \leq C2^{-k(\frac{n-1}{2p'})}. \qquad \blacksquare$$

Proof of Theorem 1.2 First of all, we prove that if $\delta > n(1/p - 1/2) - 1/2$ for $0 < p < 1$, then $\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a} \in L^p(\mathbb{R}^n)$ for any $\big(p, n(1/p'-1)\big)$-atom on $\mathbb{R}^n$ where $p' < p$ is a positive number satisfying $\delta = n(1/p' - 1/2) - 1/2$, and moreover there is a constant $C > 0$ independent of such atoms such that $\|\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a}\|_{L^p} \leq C$. For $t > 0$ and $\delta > 0$, let $\mathcal{H}^{\delta}_{\varrho,t}(x) = \mathcal{F}^{-1}[(1-\varrho/t)^{\delta}_{+}](x)$ and let $\mathcal{H}^{\delta}_{\varrho,1}(x) = \mathcal{H}^{\delta}_{\varrho}(x)$. Let $\mathfrak{a}$ be a $\big(p, n(1/p'-1)\big)$-atom supported in the ball $B(x_0; s)$ with center $x_0 \in \mathbb{R}^n$ and radius $s > 0$. Then we see that $\mathfrak{R}^{\delta}_{\varrho,t}\mathfrak{a}(x) = \mathcal{H}^{\delta}_{\varrho,t} * \mathfrak{a}(x)$. Since $\mathcal{H}^{\delta}_{\varrho} \in L^1(\mathbb{R}^n)$ by Lemma 2.2, if $x \in B(0; 2s)$ is given then we have that

$$|\mathfrak{R}^{\delta}_{\varrho,t}\mathfrak{a}(x)| \leq \|\mathcal{H}^{\delta}_{\varrho,t}\|_{L^1}\|\mathfrak{a}\|_{L^\infty} \leq \|\mathcal{H}^{\delta}_{\varrho}\|_{L^1}|B(x_0;s)|^{-1/p},$$

and so

$$\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a}(x) \lesssim |B(x_0;s)|^{-1/p}.$$

Since $0 < p < 1$, it easily follows from (b) of Lemma 2.3 that

$$
\begin{aligned}
(2.10) \qquad \|\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a}\|^{p}_{L^p} &= \|(\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a})\chi_{B(x_0;2s)}\|^{p}_{L^p} + \|(\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a})\chi_{B(x_0;2s)^c}\|^{p}_{L^p} \\
&\le 2^n + \sum_{k=1}^{\infty}\sum_{\ell=1}^{N_k} \|(\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a})\chi_{B(x_0;2s)^c}\|^{p}_{L^p} \\
&\lesssim 2^n + C\sum_{k=1}^{\infty} 2^{-k(\frac{p}{p'}-1)(\frac{n-1}{2})} \le C.
\end{aligned}
$$

Finally, if $f = \sum_{j=1}^{\infty} c_j\mathfrak{a}_j$ where the $\mathfrak{a}_j$s are $\big(p, n(1/p'-1)\big)$-atoms and $\{c_j\} \in \ell^p$, then by (2.10) we have the estimate

$$
\|\mathfrak{M}^{\delta}_{\varrho} f\|^{p}_{L^p} \le \sum_{j} |c_j|^p \|\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a}_j\|^{p}_{L^p} \lesssim \sum_{j} |c_j|^p.
$$

Hence this completes the proof. ■

3 $(H^p, L^{p,\infty})$-Estimate For the Case that Σ_ϱ is a Smooth Convex Hypersurface of Finite Type

In this section we shall focus upon obtaining $(H^p, L^{p,\infty})$-mapping properties of the maximal operator $\mathfrak{M}^{\delta(p)}_{\varrho}$, $p < 1$, under the condition that Σ_ϱ is a smooth convex hypersurface of finite type.

Let Σ be a smooth convex hypersurface of $\mathbb{R}^n$ and let $d\sigma$ be the induced surface area measure on Σ. Let $\mathcal{E}(\Sigma)$ be the set of points of Σ at which the Gaussian curvature κ vanishes, and let $\mathcal{N}(\Sigma) = \{n(\xi) \mid \xi \in \mathcal{E}(\Sigma)\}$ where $n(\xi)$ denotes the outer unit normal to Σ at $\xi \in \Sigma$. For $x \in \mathbb{R}^n$, denote by $d\big(x/|x|, \mathcal{N}(\Sigma)\big)$ the geodesic distance on S^{n-1} between $x/|x|$ and $\mathcal{N}(\Sigma)$, and by $\mathcal{B}\big(\xi(x), s\big)$ the spherical cap near $\xi(x) \in \Sigma$ cut off from Σ by a plane parallel to $T_{\xi(x)}(\Sigma)$ (the affine tangent plane to Σ at $\xi(x)$) at distance $s > 0$ from it; that is,

$$
\mathcal{B}\big(\xi(x), s\big) = \big\{\xi \in \Sigma \mid d\big(\xi, T_{\xi(x)}(\Sigma)\big) < s\big\},
$$

where $\xi(x)$ is the point of Σ whose outer unit normal is in the direction x. These spherical caps play an important role in furnishing the decay of the Fourier transform of the measure $d\sigma$. It is well known [7,9] that the function

$$
(3.1) \qquad \Omega(\theta) \fallingdotseq \sup_{r>0} \sigma\big[\mathcal{B}\big(\xi(r\theta), 1/r\big)\big](1+r)^{\frac{n-1}{2}}
$$

is bounded on S^{n-1} provided that Σ has nonvanishing Gaussian curvature.

Definition 3.1 Σ be a smooth convex hypersurface of $\mathbb{R}^n$. Then we say that Σ satisfies a spherically integrable condition of order < 1 if $\Omega \in L^p(S^{n-1})$ for any $p < 1$.

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Remark (i) B. Randol [7] proved that if Σ is a real analytic convex hypersurface of $\mathbb{R}^n$ then $\Omega \in L^p(S^{n-1})$ for some $p > 2$. Thus any real analytic convex hypersurface satisfies a spherically integrable condition of order < 1.

(ii) Let Σ be a smooth convex hypersurface of finite type $k \geq 2$ and suppose that $\mathcal{N}(\Sigma)$ is a m-dimensional submanifold of $\mathbb{R}^n$ which is on S^{n-1}, where $m < [k(n-1)]/[2(k-1)]$. Then we see (refer to [4]) that Σ satisfies a spherically integrable condition of order < 1. Moreover, it is not hard to see that Σ satisfies a spherically integrable condition even for $m \leq n-2$. We mention for reader that it can be shown by [4, Lemma 2.8] and the fact Σ is of finite type $P(k)$; *i.e.*, there is some constant $C = C(\Sigma) > 0$ such that for any $\theta \in S^{n-1}$,

$$\Omega(\theta) \leq \frac{C}{d\big(\theta, \mathcal{N}(\Sigma)\big)^{\frac{k-2}{2(k-1)}(n-1)}}.$$

Since Σ is smooth and of finite type, it is absolutely impossible that $\mathcal{N}(\Sigma)$ is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ which is on S^{n-1}.

(iii) More generally, it was shown by I. Svensson [13] that if Σ is a smooth convex hypersurface of finite type $k \geq 2$ then $\Omega \in L^p(S^{n-1})$ for some $p > 2$.

Thus, by the above remark (iii), it is natural for us to obtain the following lemma.

Lemma 3.2 *Any smooth convex hypersurface of finite type always satisfies a spherically integrable condition of order* < 1.

Sharp decay estimates for the Fourier transform of surface measure on a smooth convex hypersurface Σ of finite type $k \geq 2$ have been obtained by Bruna, Nagel, and Wainger [1]; precisely speaking, $|\mathcal{F}[d\sigma](x)|$ is equivalent to $\sigma\big[\mathcal{B}\big(\xi(x), 1/|x|\big)\big]$. They define a family of anisotropic balls on Σ by letting

$$\mathcal{B}(\xi_0, s) = \big\{\, \xi \in \Sigma \mid d\big(\xi, T_{\xi_0}(\Sigma)\big) < s \big\}$$

where $\xi_0 \in \Sigma$. We now recall some properties of the anisotropic balls $\mathcal{B}(\xi_0, s)$ associated with Σ. The proof of the doubling property in [1] makes it possible to obtain the following stronger estimate for the surface measure of these balls:

$$\sigma[\mathcal{B}(\xi_0, \gamma s)] \lesssim \begin{cases} \gamma^{\frac{n-1}{2}} \sigma[\mathcal{B}(\xi_0, s)], & \gamma \geq 1, \\ \gamma^{\frac{n-1}{k}} \sigma[\mathcal{B}(\xi_0, s)], & \gamma < 1. \end{cases} \tag{3.2}$$

It also follows from the triangle inequality and the doubling property [1] that there is a positive constant $C > 0$ independent of $s > 0$ such that

$$\frac{1}{C}\sigma[\mathcal{B}(\xi_0, s)] \leq \sigma[\mathcal{B}(\xi, s)] \leq C\sigma[\mathcal{B}(\xi_0, s)] \quad \text{for any } \xi \in \mathcal{B}(\xi_0, s). \tag{3.3}$$

Next we recall a useful lemma [10] due to E. M. Stein, M. H. Taibleson, and G. Weiss on summing up weak type functions.

Lemma 3.3 *Let* $0 < p < 1$. *Suppose that* $\{\mathfrak{h}_k\}$ *is a sequence of measurable functions such that for all* $k \in \mathbb{N}$,

$$\|\mathfrak{h}_k\|_{L^{p,\infty}} \leq 1.$$

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If $\{c_k\} \in \ell^p$, then we have the following estimate

$$\Big\| \sum_{k=1}^{\infty} c_k \mathfrak{h}_k \Big\|_{L^{p,\infty}} \leq \Big(\frac{2-p}{1-p} \Big)^{1/p} \|\{c_k\}\|_{\ell^p}.$$

We now state an elementary lemma without proof which will be useful to measure the distance from a point of $\mathcal{B}(\xi_0, s)$ to the affine tangent plane to Σ at $\xi_0 \in \Sigma$ in higher dimensions.

Lemma 3.4 *Let Σ be a smooth simple closed convex curve in $\mathbb{R}^2$ whose graph near $(0,0)$ is given as $\big(t, g(t)\big)$ where $g(t) = bt^m + c$ is a convex function defined on $[-d, d]$ for some sufficiently small constant b, c, $d > 0$ and an integer $m \geq 2$. For $|t| \leq d$, we denote by $\Theta(t)$ the angle between $n\big(0, g(0)\big)$ and $n\big(t, g(t)\big)$. For some small angle $\Theta_0 > 0$ with $\Theta_0 \leq \max\{\Theta(-d), \Theta(d)\}$, let t_0 be chosen so that $\Theta(t_0) = \Theta_0$ and $|t_0| \leq d$. Then we have the following estimate*

$$|g(t_0) - c| \sim |b|^{-\frac{1}{m-1}} m^{-\frac{m}{m-1}} \Theta_0^{\frac{m}{m-1}}.$$

Lemma 3.5 *Let Σ be a smooth convex hypersurface of $\mathbb{R}^n$ which is of finite type $k \geq 2$. Then there is a constant $C = C(\Sigma) > 0$ such that for any $y \in B(0; s)$ and $x \in B(0; 2s)^c$, $0 < s \leq 1$,*

$$\xi(x-y) \in \mathcal{B}\big(\xi(x), C/|x|\big)$$

where $\xi(x)$ is the point of Σ whose outer unit normal is in the direction x.

Proof We observe that the following inequality always holds for any $x, y \in \mathbb{R}^n$ with $|x| > 2|y|$;

$$\Big| \frac{x-y}{|x-y|} - \frac{x}{|x|} \Big| \leq 2\frac{|y|}{|x|}. \tag{3.4}$$

Near $\xi(x/|x|) \in \Sigma$, the hypersurface Σ can be given as the graph of a smooth convex function defined on $D \fallingdotseq T_{\xi(x/|x|)}(\Sigma) \cap B\big(\xi(x/|x|); 1/2\big)$. To be precise, let Ψ be a smooth convex function defined on D such that $\big(\xi_0', \Psi(\xi_0')\big) = \xi(x/|x|)$, and for $|t| < 1/2$ and $\eta \in T^{n-1} \fallingdotseq [T_{\xi(x/|x|)}(\Sigma) - \xi(x/|x|)] \cap S^{n-1}$,

$$\Psi(\xi_0' + t\eta) = \sum_{i=0}^{k} \frac{1}{i!} \mathcal{D}_\eta^i \Psi(\xi_0') t^i + \mathcal{O}(t^{k+1}). \tag{3.5}$$

Using (3.5), we now estimate the distance from $\xi = \big(\xi', \Psi(\xi')\big) \in \Sigma$ to the tangent space $T_{\xi(x/|x|)}(\Sigma)$ as follows: since Σ is of finite type $k \geq 2$, for each $\eta \in T^{n-1}$ there is an integer m with $2 \leq m \leq k$ such that for $-1/2 < t < 1/2$,

$$\Psi(\xi_0' + t\eta) - \Psi(\xi_0') - \mathcal{D}_\eta \Psi(\xi_0') t = \frac{1}{m!} \mathcal{D}_\eta^m \Psi(\xi_0') t^m + \mathcal{O}(t^{m+1}).$$

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Thus by (3.4) and Lemma 3.4 we have that

$$\begin{aligned}\left\langle \xi\left(\frac{x}{|x|}\right)-\xi\left(\frac{x-y}{|x-y|}\right),\frac{x}{|x|}\right\rangle &= \Psi(\xi_0'+t_1\eta)-\Psi(\xi_0')-\mathcal{D}_\eta\Psi(\xi_0')t_1\\ &\lesssim \Big[\frac{m^m}{m!}|\mathcal{D}_\eta^m\Psi(\xi_0')|\Big]^{-\frac{1}{m-1}}\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right|^{\frac{m}{m-1}}\\ &\le M_0\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right|\le\frac{2M_0}{|x|}\end{aligned}$$

where t_1, $|t_1|<1/2$, is some number so that $\big(\xi_0'+t_1\eta,\Psi(\xi_0'+t_1\eta)\big)=\xi(\frac{x-y}{|x-y|})$ and $M_0=\sup_{2\le m\le k}\sup_{\eta\in T^{n-1}}[\frac{m^m}{m!}|\mathcal{D}_\eta^m\Psi(\xi_0')|]^{-\frac{1}{m-1}}$. Hence we complete the proof. ■

Lemma 3.6 *Let Σ be a smooth convex hypersurface of $\mathbb{R}^n$ which is of finite type $k\ge 2$. Then there is a constant $C=C(\Sigma)>0$ such that for any $y\in B(0;s)$ and $x\in B(0;2s)^c$, $0<s\le 1$,*

$$\Omega\left(\frac{x-y}{|x-y|}\right)\le C\Omega\left(\frac{x}{|x|}\right)$$

where Ω is the radial function defined as in (3.1).

Proof It easily follows from (3.2), (3.3), the definition of Ω, and Lemma 3.5 that for any $y\in B(0;s)$ and $x\in B(0;2s)^c$, $0<s\le 1$,

$$\begin{aligned}\Omega\left(\frac{x-y}{|x-y|}\right)&=\sup_{r>0}\sigma\big[\mathcal{B}\big(\xi(x-y),1/r\big)\big](1+r)^{\frac{n-1}{2}}\\ &\lesssim\sup_{r>0}\sigma\big[\mathcal{B}\big(\xi(x),1/r\big)\big](1+r)^{\frac{n-1}{2}}=\Omega\left(\frac{x}{|x|}\right).\end{aligned}$$ ■

Proof of Theorem 1.1 Fix $0<p<1$. Let $\mathfrak{a}$ be a $\big(p,n(1/p-1)\big)$-atom supported in the ball $B(x_0;s)$ with center $x_0\in\mathbb{R}^n$ and radius $s>0$. Then we see that $\mathfrak{R}^\delta_{\varrho,t}\mathfrak{a}(x)=\mathcal{H}^\delta_{\varrho,t}*\mathfrak{a}(x)$. Recalling the lemma [6] about asymptotics of quasiradial Bochner-Riesz kernel and the result of Bruna, Nagel, and Wainger [1], we get that

$$(3.6)\qquad |\mathcal{H}^{\delta(p)}_\varrho(x)|\sim|\nabla\mathcal{H}^{\delta(p)}_\varrho(x)|\sim\frac{1}{(1+|x|)^{\frac{n}{p}-\frac{n-1}{2}}}\sigma\big[\mathcal{B}\big(\xi(x),1/|x|\big)\big]$$

where we consider Σ_ϱ as Σ given in the above. Since $\mathcal{H}^{\delta(p)}_\varrho\in L^1(\mathbb{R}^n)$ by (3.6) and Lemma 3.2, if $x\in B(0;2s)$ is given then we have that

$$|\mathfrak{R}^{\delta(p)}_{\varrho,t}\mathfrak{a}(x)|\le\|\mathcal{H}^{\delta(p)}_{\varrho,t}\|_{L^1}\|\mathfrak{a}\|_{L^\infty}\le\|\mathcal{H}^{\delta(p)}_\varrho\|_{L^1}|B(x_0;s)|^{-1/p},$$

and so

$$\mathfrak{M}^{\delta(p)}_\varrho\mathfrak{a}(x)\lesssim|B(x_0;s)|^{-1/p}.$$

Thus we have that for all $\lambda > 0$,

$$\big|\{x \in B(x_0; 2s) \mid \mathfrak{M}_\varrho^{\delta(p)}\mathfrak{a}(x) > \lambda/2\}\big| \lesssim \lambda^{-p}. \tag{3.7}$$

Next we shall obtain the following inequality

$$\big|\{x \in B(x_0; 2s)^c \mid \mathfrak{M}_\varrho^{\delta(p)}\mathfrak{a}(x) > \lambda/2\}\big| \lesssim \lambda^{-p}, \quad \lambda > 0. \tag{3.8}$$

As in the argument of (2.8), without loss of generality we can assume that a $\big(p, n(1/p-1)\big)$-atom $\mathfrak{a}$ is supported in the unit ball $B(0;1)$ centered at the origin. We now consider the case that $x \in B(0;2)^c$ and $t > 1$. Then it follows from (3.1), (3.2), (3.6), and Lemma 3.6 that

$$\begin{aligned}
|\mathcal{H}_{\varrho,t}^{\delta(p)} * \mathfrak{a}(x)| &\lesssim t^n \int_{B(0;1)} \big|\mathcal{H}_\varrho^{\delta(p)}\big(A_t(x-y)\big)\big|\, dy \\
&\lesssim \frac{t^{n-n/p}}{(1+|x|)^{\frac{n}{p}}} \int_{B(0;1)} \Omega\Big(\frac{x-y}{|x-y|}\Big)\, dy \\
&\lesssim \frac{t^{n-n/p}}{(1+|x|)^{\frac{n}{p}}} \Omega\Big(\frac{x}{|x|}\Big) \\
&\lesssim \frac{1}{(1+|x|)^{\frac{n}{p}}} \Omega\Big(\frac{x}{|x|}\Big),
\end{aligned}$$

because $n(1-1/p) < 0$. So we have that

$$\sup_{t>1} |\mathcal{H}_{\varrho,t}^{\delta(p)} * \mathfrak{a}(x)| \lesssim \frac{1}{(1+|x|)^{\frac{n}{p}}} \Omega\Big(\frac{x}{|x|}\Big). \tag{3.9}$$

Let $N \in \mathbb{N}$ be an integer satisfying $N < n(1/p-1) \le N+1$, *i.e.*, $n/(n+N+1) \le p < n/(n+N)$. If $x \in B(0;2)^c$ and $0 < t \le 1$, let $\mathcal{Q}_{t,x}(y)$ be the N-th order Taylor polynomial of the function $y \mapsto \mathcal{H}_\varrho^{\delta(p)}\big(A_t^*(x-y)\big)$ expanded near the origin, where $\mathcal{H}_\varrho^{\delta(p)}(x) = \mathcal{F}^{-1}[(1-\varrho)_+^{\delta(p)}](x)$. Then it follows from the moment condition on the atom $\mathfrak{a}$, Taylor's theorem, (3.1), (3.2), (3.6), and Lemma 3.6 that

$$\begin{aligned}
|\mathcal{H}_{\varrho,t}^{\delta(p)} * \mathfrak{a}(x)| &= t^n \Big| \int_{\mathbb{R}^n} \big[\mathcal{H}_\varrho^{\delta(p)}\big(A_t(x-y)\big) - \mathcal{Q}_{t,x}(y)\big]\, \mathfrak{a}(y)\, dy \Big| \\
&\lesssim t^{n+(N+1)} \int_0^1 \int_{B(0;1)} \big|\nabla^{N+1}\mathcal{H}_\varrho^{\delta(p)}\big(A_t(x-\tau y)\big)\big|\, dy\, d\tau \\
&\lesssim \frac{t^{n+(N+1)-n/p}}{(1+|x|)^{\frac{n}{p}}} \int_0^1 \int_{B(0;1)} \Omega\Big(\frac{x-\tau y}{|x-\tau y|}\Big)\, dy\, d\tau \\
&\lesssim \frac{t^{n+(N+1)-n/p}}{(1+|x|)^{\frac{n}{p}}} \Omega\Big(\frac{x}{|x|}\Big) \\
&\lesssim \frac{1}{(1+|x|)^{\frac{n}{p}}} \Omega\Big(\frac{x}{|x|}\Big)
\end{aligned}$$

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because $n + (N+1) - n/p \geq 0$. Thus we have that

$$\sup_{0<t\leq 1} |\mathcal{H}_{\varrho,t}^{\delta(p)} * \mathfrak{a}(x)| \lesssim \frac{1}{(1+|x|)^{\frac{n}{p}}} \Omega\left(\frac{x}{|x|}\right). \tag{3.10}$$

Thus by (3.9) and (3.10) we conclude that

$$\mathfrak{M}_{\varrho}^{\delta(p)} \mathfrak{a}(r\theta) \lesssim \frac{1}{(1+r)^{\frac{n}{p}}} \Omega(\theta).$$

Hence we have the following estimate

$$\int_{\{x \in B(0;2)^c | \mathfrak{M}_{\varrho}^{\delta(p)} \mathfrak{a}(x) > \lambda\}} dx \lesssim \int_{S^{n-1}} \int_{\{r>0 | 2<r<\lambda^{-p/n}\Omega(\theta)^{p/n}\}} r^{n-1}\, dr\, d\theta \lesssim \lambda^{-p}$$

because $\Omega \in L^p(S^{n-1})$ for any $p < 1$ by Lemma 3.2. Therefore, by (3.7), (3.8), and Lemma 3.3, we complete the proof. ■

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