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An Extension of Nikishin's Factorization Theorem

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Abstract. A Nikishin–Maurey characterization is given for bounded subsets of weak-type Lebesgue spaces. New factorizations for linear and multilinear operators are shown to follow.

1 Introduction

Let (Ω, μ) be a non-atomic probability space. The space of scalar-valued μ-measurable functions $L_0 = L_0(d\mu)$ is equipped with the topology of convergence in measure and $f = g$ in L_0 if $f(\omega) = g(\omega)$ for μ-almost every $\omega \in \Omega$.

For $0 < p < \infty$, the Lebesgue space L_p is defined by

$$f \in L_p \Leftrightarrow \|f\|_p = \Big(\int |f|^p \, d\mu \Big)^{1/p} < \infty.$$

Of course, $\|\cdot\|_p$ is a quasi-norm making L_p into a quasi-Banach space and L_p is a Banach space if $1 \le p \le \infty$ where L_∞ is the space of essentially bounded functions equipped with the essential supremum norm.

For $0 < p < \infty$, the weak Lebesgue space $L_{p,\infty}$ is defined by

$$f \in L_{p,\infty} \Leftrightarrow \|f\|_{p,\infty} := \sup_{t>0} t\mu(\{|f| > t\})^{1/p} < \infty.$$

The quantity $\|\cdot\|_{p,\infty}$ is a (complete) quasi-norm [3, Chapter 1.1].

The intent of this article is to extend the existing Nikishin–Maurey theory. Although this theory is often presented in the context of factoring operators, the foundational results of this theory are concerned with bounded subsets of non-negative measurable functions. If $\mathcal{F}$ is such a set of measurable functions and $0 < p < \infty$, Nikishin [8] characterized the existence of a positive measurable function g such that $\sup_{f \in \mathcal{F}} \|f/g\|_{p,\infty} < \infty$. The existence of such a function is equivalent to the existence of a decreasing function $C\colon (0,\infty) \to (0,\infty)$ such that $\lim_{t\to\infty} C(t) = 0$ and

$$\mu(\{\sup_j |c_j f_j| > t\}) \le C(t)$$

for all $t > 0$, finitely supported sequences (f_j) from $\mathcal{F}$ and scalars (c_j) satisfying $\sum_j |c_j|^p \le 1$. In other words, g exists if and only if

$$\{\sup_j |c_j f_j| : n \in \mathbb{N}, f_1, \dots, f_n \in \mathcal{F}, |c_1|^p + \cdots + |c_n|^p \le 1\}$$

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is bounded in L_0. The key step in proving g's existence from the boundedness of this set is purely constructive. A nice presentation of the proof of Nikishin's [8] characterization may be found in [10, Proposition III.H.2]. To summarize, for $0 < \epsilon < 1$, the boundedness of the above set within L_0 implies the existence of a subset E_ϵ of Ω and a constant $C_\epsilon < \infty$ such that $\mu(E_\epsilon) \geq 1 - \epsilon$ and

$$\sup_{f \in \mathcal{F}} \|1_{E_\epsilon} f\|_{p,\infty} \leq C_\epsilon.$$

Of course, the constant C_ϵ is tied to the function C from the above maximal estimate, and thus it may be that $\lim_{\epsilon \to 0} C_\epsilon = \infty$. If this limit were finite, there would be nothing to prove, as the set $\mathcal{F}$ would be bounded in $L_{p,\infty}$. Given any decreasing null sequence (ϵ_n) such that $0 < \epsilon_n < 1$, g is constructed by selecting an unbounded increasing sequence of positive scalars (a_n) such that $g = a_1 1_{E_{\epsilon_1}} + \sum_{n>1} a_n 1_{E_{\epsilon_n} \setminus E_{\epsilon_{n-1}}}$, where the scalars (a_n) are chosen to counteract the growth of constants (C_{ϵ_n}) in order to obtain the factorization $\mathcal{F} = g(g^{-1}\mathcal{F})$, where $g^{-1}\mathcal{F}$ is bounded in $L_{p,\infty}$. Of course, any selection of scalars (a_n) defines g as a positive element of L_0. However, it is of interest to identify a more specific space for g based on the particular function $C(t)$ from the above maximal estimate. Theorem 1.1 is a formulation of Nikishin's theorem for the specific case that $\mathcal{F}$ is a subset of $L_{q,\infty}$ for some $0 < q < p$. In this case, it is natural to consider $C(t) = t^{-q}$. Much of the proof follows from technical adaptations of the arguments used to prove Nikishin's [8] theorem as presented in [10]. However, the aforementioned construction of the function g is insufficient to obtain g from the natural weak-Lebesgue space, and an extra compactness argument will be used to prove (N3) $\Leftrightarrow$ (N1).

Theorem 1.1 *Let $0 < q < p < \infty$, $1/r = 1/q - 1/p$, and let $\mathcal{F}$ be a subset of non-negative elements of $L_{q,\infty}$. Then the following conditions are equivalent.*

(N1) *There exist a constant $C < \infty$ and a positive $g \in L_{r,\infty}$ so that $\|g\|_{r,\infty} = 1$ and*

$$\sup_{f \in \mathcal{F}} \|f/g\|_{p,\infty} \leq C.$$

(N2) *There exists a constant $C < \infty$ so that $\| \sup_j |c_j f_j| \|_{q,\infty} \leq C\big(\sum_j |c_j|^p\big)^{1/p}$ for all finitely supported sequences (f_j) from $\mathcal{F}$ and scalars (c_j).*

(N3) *There exists a constant $C < \infty$ such that for any $0 < \epsilon < 1$ there exists a measurable set E_ϵ such that $\mu(E_\epsilon) \geq 1 - \epsilon$ and $\sup_{f \in \mathcal{F}} \|1_{E_\epsilon} f\|_{p,\infty} \leq C\epsilon^{-1/r}$.*

Theorem 1.1 will be proved in Section 3. Section 2 contains applications related to the factorization of linear and multilinear operators. For more on Nikishin–Maurey theory and its applications to Banach space theory, the reader is referred to [1,4,5,10].

2 Factoring Operators

The topology of any locally bounded topological vector space X is induced by a function $\|\cdot\|: X \to [0, \infty)$ satisfying the following conditions.

- $\|x\| > 0$ for all $x \neq 0$.
- $\|cx\| = |c|\|x\|$ for all scalars c and all $x \in X$.

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- There exists $1 \le C < \infty$ so that $\|x+y\| \le C(\|x\|+\|y\|)$ for all $x, y \in X$.

The function $\|\cdot\|$ is called a quasi-norm and X is a quasi-Banach space if X is complete with respect to $\|\cdot\|$. Furthermore, if $C = 1$, then X is a Banach space.

For $0 < p \le 2$, a quasi-Banach space X has Rademacher type p if there exists a constant $T_p(X) < \infty$ such that $\mathbb{E}\|\sum_j \epsilon_j x_j\|^p \le T_p(X)^p \sum_j \|x_j\|^p$ for all finitely supported sequences (x_j) from X. Here (ϵ_j) is a sequence of independent Bernoulli random variables satisfying $P(\epsilon_j = 1) = 1/2 = P(\epsilon_j = -1)$.

For $k \ge 1$ and quasi-Banach space(s) $X_1, \ldots, X_k$, $T: X_1 \times \cdots \times X_k \to L_0$ is k-sublinear if each coordinate map is sublinear. If each coordinate map is linear, then T is a k-linear operator. In [2] it was shown that for a bounded k-sublinear operator $T: X_1 \times \cdots \times X_k \to L_0$ there exists a decreasing function $C: (0,\infty) \to (0,\infty)$ such that $\lim_{t\to\infty} C(t) = 0$ and $\mu(\{\sup_j |T(x_{1,j}, \cdots, x_{k,j})| > t\}) \le C(t)$ for all finitely supported sequences $(x_{i,j})_j$ from X_i, $1 \le i \le k$, such that $\sum_j(\|x_{1,j}\| \cdots \|x_{k,j}\|)^p \le 1$, where $1/p = 1/p_1 + \cdots + 1/p_k$ and $0 < p_1, \ldots, p_k \le 2$ are the respective Rademacher types of the quasi-Banach spaces $X_1, \ldots, X_k$. Thus, by Nikishin's [8] characterization there exists a positive measurable g so that $\|g^{-1}T(x_1, \cdots, x_k)\|_{p,\infty} \le \|x_1\| \cdots \|x_k\|$ for all $x_i \in X_i$, $1 \le i \le k$. Of course, the case $k = 1$ is due to Nikishin [8]. Analogous results are also shown in [2] for operators mapping into L_q for some $0 < q < p$. These results utilize Pisier's [9] characterization for the factorization of subsets of L_q through $L_{p,\infty}$ if $0 < q < p$.

Maurey [7] characterized when a subset of L_q can be factored through L_p for $0 < q < p < \infty$. Due to the stable laws and the Kahane–Khintchine inequalities, this result only applies to linear operators defined on a space of type 2, *i.e.*, every linear operator $T: X \to L_q$ factors through L_2 if $0 < q < 2$. There are no direct applications of Maurey's results for factoring multilinear operators through L_p using Rademacher type. However, partial analogs of Maurey's result are shown to hold in [2,6]. Suppose X_1 and X_2 have type 2 and $T: X_1 \times X_2 \to L_0$ is a bounded bilinear operator. Then not only does T factor through $L_{1,\infty}$, but the range of T is locally convex in L_0. This is shown in [2] as an application of the characterization of the Rademacher decoupling property for quasi-Banach spaces given in [6]. Thus, the T-induced linear map is continuous from the projective tensor-product $X_1 \widehat{\otimes} X_2$ into L_0. This means that T factors through a Banach space, *i.e.*, there exists a Banach space Z, a bilinear operator $B: X_1 \times X_2 \to Z$, and a linear operator $L: Z \to L_0$ such that $T = LB$.

As with previously existing Nikishin–Maurey theorems, Theorem 1.1 applies to linear and multilinear operators due to their homogeneity [2]. Suppose $k \ge 1$ and $0 < p_1, \ldots, p_k \le 2$ are the respective Rademacher types of the quasi-Banach spaces $X_1, \ldots, X_k$. If $1/p = 1/p_1 + \cdots + 1/p_k$ and $T: X_1 \times \cdots \times X_k \to L_{q,\infty}$ is a bounded k-linear operator and $0 < q < p$, then the Kahane–Khintchine inequalities imply that there exists a constant $C = C(q, p_1, \ldots, p_k, T) < \infty$ such that

$$\Big\| \Big(\sum_{j_1,\ldots,j_k} |T(x_{1,j_1}, \ldots, x_{k,j_k})|^2 \Big)^{1/2} \Big\|_{q,\infty} \le C \prod_{i=1}^{k} \Big(\sum_{j_i} \|x_{i,j_i}\|^{p_i} \Big)^{1/p_i}$$

for all finitely supported sequences $(x_{i,j})_j$ from X_i, $1 \le i \le k$. Of course, this estimate implies that

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$$\|\sup_j |T(x_{1,j},\dots,x_{k,j})|\|_{q,\infty} \le C\prod_{i=1}^k \Big(\sum_j \|x_{i,j}\|^{p_i}\Big)^{1/p_i}$$

for all finitely supported sequences $(x_{i,j})_j$ from X_i, $1 \le i \le k$. As illustrated in [2, 6], the homogeneity of T and of the identity $1/p = 1/p_1 + \cdots + 1/p_k$ implies

$$\|\sup_j |T(x_{1,j},\dots,x_{k,j})|\|_{q,\infty} \le C\Big(\sum_j (\|x_{1,j}\|\cdots\|x_{k,j}\|)^p\Big)^{1/p}$$

for all finitely supported sequences $(x_{i,j})_j$ from X_i, $1 \le i \le k$. An immediate consequence of this estimate and Theorem 1.1 is the following theorem.

Theorem 2.1 *Let $k \ge 1$, $0 < p_1,\dots,p_k \le 2$, $1/p = 1/p_1 + \cdots + 1/p_k < 1/q < \infty$, $1/r = 1/q - 1/p$, and suppose $T\colon X_1 \times \cdots \times X_k \to L_{q,\infty}$ is a continuous k-linear operator where $p_1,\dots,p_k$ are the respective Rademacher types of the quasi-Banach spaces $X_1,\dots,X_k$. Then there exists a positive $g \in L_{r,\infty}$ such that $\|g^{-1}T(x_1,\dots,x_k)\|_{p,\infty} \le \|x_1\|\cdots\|x_k\|$ for all $x_i \in X_i$, $1 \le i \le k$.*

Theorem 2.1 identifies new factorizations not established by the characterizations of Nikishin [8], Maurey [7], or Pisier [9], due the the fact that the operator takes values in $L_{q,\infty}$ and that the factorization $T = g(g^{-1}T)$ is defined by an element g of $L_{r,\infty}$, where $1/r = 1/q - 1/p$.

3 Proof of Theorem 1.1

All Nikishin–Maurey characterizations like Theorem 1.1 may be reduced to any particular value of p by considering $\mathcal{F}^t = \{f^t : f \in \mathcal{F}\}$ for any $t > 0$. Since $1/r = 1/q - 1/p$, it follows that $1/(r/t) = 1/(q/t) + 1/(p/t)$, and all the conditions (N1)–(N3) translate into equivalent conditions about $\mathcal{F}^t$. Picking t so that p/t has a particular value and proving the equivalence of these new conditions for $\mathcal{F}^t$, implies the general equivalence of (N1)–(N3).

The following proof of (N1) $\Rightarrow$ (N2) $\Rightarrow$ (N3) follows from a technical reworking of the arguments used to prove the analogous implications of Nikishin's [8] theorem as presented in [10]. However, the aforementioned constructive arguments used to prove the final implication of Nikshin's theorem are insufficient to construct the function g from (N1) because of the added requirement that $g \in L_{r,\infty}$ where $1/r = 1/q - 1/p$. However, assuming (N3) and using the aforementioned reduction with the assumption that $r > 2$, one can construct a sequence (g_n) such that $\sup_n \sup_{f\in\mathcal{F}} \|f/g_n\|_{p,\infty} < \infty$ and $\sup_n \|g_n\|_{r_n,\infty} < \infty$, where $2 < r_n < r$ and (r_n) is increasing to r. Thus, (g_n) is bounded in L_2, and the Banach–Saks theorem guarantees that (g_n) has a subsequence with convergent Cesàro means. The above two conditions will be used to show that the limit g of these Cesàro means will satisfy (N1), and this will complete the proof by establishing (N3) $\Rightarrow$ (N1).

Proof Assuming (N1), let (f_j) be a finitely supported sequence from $\mathcal{F}$, (c_j) be scalars, and define $F = \sup_j |c_j f_j|$. Then $1/r = 1/q - 1/p$ implies that $\|F\|_{q,\infty} \le \|F/g\|_{p,\infty}\|g\|_{r,\infty}$. Since (f_j) is a finitely supported sequence from $\mathcal{F}$, there is a partition (E_j) of Ω so that $F = |c_k f_k|$ almost everywhere on E_k. The disjointness of the partition implies

$$\mu(\{F/g > t\}) \le \sum_k \mu(\{|c_k f_k|/g > t\}).$$

Hence, (N1) implies

$$\mu(\{\sup_j |c_j f_j|/g > t\}) \le \sum_k \mu(\{|c_k f_k| > t\}) \le C t^{-p} \sum_k |c_k|^p,$$

and (N2) follows with the same constant as in (N1) because $\|g\|_{r,\infty} = 1$.

Now assume (N2) holds. Without loss of generality, assume $C = 1$ by normalizing $\mathcal{F}$. Fix $0 < \epsilon < 1$ and consider a measurable set B to be ϵ-bad if there exists $f \in \mathcal{F}$ so that $\mu(B)|f(\omega)|^p > \epsilon^{-p/q}$ for all $\omega \in B$.

If there are no ϵ-bad sets for some $0 < \epsilon < 1$, then for every $f \in \mathcal{F}$ and every $t > 0$, there exists $\omega \in \{|f| > t\}$ so that

$$\mu(\{|f| > t\}) t^p < \mu(\{|f| > t\}) |f(\omega)|^p \le \epsilon^{-p/q}.$$

In this case $\mathcal{F}$ is bounded in $L_{p,\infty}$, and (N3) follows in a trivial manner.

Suppose there are ϵ-bad sets for every $0 < \epsilon < 1$. Fix ϵ and suppose (B_j) is a maximal family of disjoint ϵ-bad sets. Thus, for each j there exists $f_j \in \mathcal{F}$ so that

$$\mu(B_j)|f_j(\omega)|^p > \epsilon^{-p/q}$$

for all $\omega \in B_j$. For each j let $c_j = \mu(B_j)^{1/p}$. Notice that $\sup_j |c_j f_j|^p > \epsilon^{-p/q}$ everywhere on $B = \bigcup_j B_j$. Thus, for any $n \in \mathbb{N}$, (N2) implies that

$$\mu\Big(\bigcup_{j=1}^{n} B_j\Big) \le \mu(\{\sup_{j \le n} |c_j f_j| > \epsilon^{-1/q}\}) \le \epsilon \Big(\sum_{j=1}^{n} |c_j|^p\Big)^{q/p} \le \epsilon \mu(B)^{q/p}.$$

Letting n tend to infinity implies that $\mu(B)^{1-q/p} \le \epsilon$. Therefore, $\mu(B) \le \epsilon^{p/(p-q)}$. So if $E_{\epsilon^{p/(p-q)}} = \Omega \setminus B$, then $\mu(E_{\epsilon^{p/(p-q)}}) \ge 1 - \epsilon^{p/(p-q)}$ and $E_{\epsilon^{p/(p-q)}}$ is not ϵ-bad. Therefore,

$$\|1_{E_{\epsilon^{p/(p-q)}}} f\|_{p,\infty}^p \le \epsilon^{-p/q}.$$

This holds for all $0 < \epsilon < 1$. By making the substitution $\delta = \epsilon^{p/(p-q)}$, then for every $0 < \delta < 1$, the condition $1/r = 1/q - 1/p$ implies that there exists E_δ such that $\mu(E_\delta) \ge 1 - \delta$ and $\|1_{E_\delta} f\|_{p,\infty}^p \le \delta^{-(p-q)/q} = \delta^{-p/r}$. By undoing the assumed normalization of (N2), it follows that (N3) holds with the same constant as in (N2).

Assume (N3) with $C = 1$. Moreover, the fact that $\mathcal{F}$ is a set of positive measurable functions, the homogeneity of the desired estimates on the indices q, p, and r due to the identity $1/r = 1/q - 1/p$ allows for the assumption that $r > 2$.

Fix n and let $(\epsilon_{n,m})_m$ be the sequence defined by $\epsilon_{n,m} = (1/(2m^n))_m$. For each m, let $E_{n,m}$ be the set $E_{\epsilon_{n,m}}$ from condition (N3) for $\epsilon = \epsilon_{n,m}$. The decreasing nature of the sequence $(\epsilon_{n,m})_m$ implies that we can assume (with no loss of generality) that $E_{n,m} \subset E_{n,m+1}$ for all n and m. Moreover, since μ is non-atomic, we can assume $\mu(E_{n,m}) = 1 - 1/(2m^n)$ by choosing $E_{n,m}$ as a subset of $E_{\epsilon_{n,m}}$ if necessary. Thus, for every n and each $m \ge 1$,

$$\mu(E_{n,m+1} \setminus E_{n,m}) = \tfrac{1}{2m^n} - \tfrac{1}{2(m+1)^n} = \tfrac{(m+1)^n - m^n}{2m^n(m+1)^n} \le \tfrac{n}{2m^{n+1}}.$$

Let $D_{n,1} = E_{n,1}$ and $D_{n,m} = E_{n,m+1} \setminus E_{n,m}$ for all $m > 1$.

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Suppose the above construction has been made for all n. For each n, define a measurable function g_n by $g_n = \sum_m 1_{D_{n,m}} m^{(n+k)/r}$, where k is chosen so that $pk/r > 1$. Clearly, $g_n \geq 1$ on Ω for all n.

Let $r_n = \frac{rn}{n+k}$. Then (r_n) increases to r. Furthermore, it is easy to see that there is a constant $C < \infty$ which is independent of n so that

$$\mu(\{g_n > t\}) = \sum_{m > t^{r/(n+k)}} \mu(D_{n,m}) \leq \sum_{m > t^{r/(n+k)}} \frac{n}{2m^{n+1}} \leq Ct^{-rn/(n+k)}.$$

Thus, $\sup_n \|g_n\|_{r_n,\infty} < \infty$. Moreover, if $f \in \mathcal{F}$, then

$$\mu(\{|f|/g_n > t\}) = \sum_m \mu(D_{n,m} \cap \{|f| > tm^{(n+k)/r}\}).$$

Since $D_{n,m} \subset E_{\epsilon_{n,m}}$, (N3) and our choice of k imply that there is a constant

$$C = C(p, q, r) < \infty$$

that is independent of n so that

$$\mu(\{|f|/g_n > t\}) \leq t^{-p} \sum_m 2^{p/r} m^{np/r} m^{-p(n+k)/r} = t^{-p} 2^{p/r} \sum_m m^{-pk/r} = Ct^{-p}.$$

Since $r > 2$ and (r_n) is increasing to r, we can restrict our attention to all n such that $2 < r_n < r$. Without loss of generality, we may assume that $r_1 > 2$. Since μ is a probability measure, $\sup_n \|g_n\|_2 \leq C(r_1) \sup_n \|g_n\|_{r_n,\infty} < \infty$. By the Banach–Saks theorem for L_2, there exist a subsequence $(g_{n_k})_k$ and $g \in L_2$ so that $g = \lim_N G_N$, (in L_2), where $G_N = N^{-1} \sum_{j=1}^N g_{n_j}$. For any $m \geq 1$, define $G_{N,m} = N^{-1} \sum_{j=m}^{N+m-1} g_{n_j}$. By a simple application of the triangle inequality, it is easy to see that g is the L_2 limit of $(G_{N,m})_N$ for any m. By Fatou's lemma, $\mu(\{g > t\}) \leq \liminf_N \mu(\{G_{N,m} > t\})$ for any m. However, since (r_n) is increasing and μ is a probability measure, notice that $G_{N,m}$ is an average of functions whose $L_{r_m,\infty}$ norms are uniformly bounded in m by a constant independent of N. So there exists a constant $C(p,q) < \infty$ so that

$$\sup_m \|g\|_{r_m,\infty} < C(p,q).$$

Since μ is a probability measure and (r_m) increases to r, it follows that $g \in L_{r,\infty}$ with $\|g\|_{r,\infty} \leq C(p,q)$.

Fix $f \in \mathcal{F}$. Again, by applying Fatou's lemma,

$$\mu(\{|f|/g > t\}) \leq \liminf_N \mu(\{|f|/G_N > t\}).$$

By the arithmetic-geometric mean inequality, $G_N \geq (\prod_{j=1}^N g_{n_j})^{1/N}$. Therefore, by Hölder's inequality for weak-type spaces,

$$\begin{aligned}\mu(\{|f|/G_N > t\}) &\leq t^{-p} \||f|/G_N\|_{p,\infty}^p \\ &\leq t^{-p} \prod_{i=1}^N \||f|^{1/N}/g_{n_i}^{1/N}\|_{Np,\infty}^p = t^{-p} \prod_{i=1}^N \|f/g_{n_i}\|_{p,\infty}^{p/N}.\end{aligned}$$

By normalizing g in $L_{r,\infty}$, the condition $\sup_n \sup_{f \in \mathcal{F}} \|f/g_n\|_{p,\infty} < \infty$ implies there is a constant $C < \infty$ such that $\sup_{f \in \mathcal{F}} \|f/g\|_{p,\infty} \leq C$, and (N1) follows. ∎

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