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Conjugate Radius and Sphere Theorem

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Abstract. Bessa [Be] proved that for given n and i_0, there exists an $\varepsilon(n, i_0) > 0$ depending on n, i_0 such that if M admits a metric g satisfying $\mathrm{Ric}_{(M,g)} \geq n-1$, $\mathrm{inj}_{(M,g)} \geq i_0 > 0$ and $\mathrm{diam}_{(M,g)} \geq \pi - \varepsilon$, then M is diffeomorphic to the standard sphere. In this note, we improve this result by replacing a lower bound on the injectivity radius with a lower bound of the conjugate radius.

1 Introduction

Let (M, g) be an n-dimensional compact Riemannian manifold. Otsu [O] proved that for given n, $i_0 > 0$, and $k \in \mathrm{R}$, there exists an $\varepsilon(n, i_0) > 0$ depending on n, i_0 such that if M admits a metric g satisfying Ricci curvature $\mathrm{Ric}_{(M,g)} \geq n-1$, sectional curvature $K_{(M,g)} \geq k$, injectivity radius $\mathrm{inj}_{(M,g)} \geq i_0$ and diameter $\mathrm{diam}_{(M,g)} \geq \pi - \varepsilon$, then M is diffeomorphic to the standard sphere S^n. Bessa [Be] improved this result by removing the condition on the sectional curvature. He used the C^α-compactness theorem [AC] as a basic tool and remarked that a lower bound on the injectivity radius cannot be replaced by a lower bound on the volume in the case of manifolds with dimension bigger than or equal to 4.

We consider the conjugate radius of M, $\mathrm{conj}_{(M,g)}$ and investigate the case that a lower bound on the injectivity radius is replaced by a lower bound on the conjugate radius. Recall that the conjugate radius is defined to be the maximal radius r such that for every $q \in M$, the exponential map $\exp_q$ has maximal rank in the open ball of radius r centered at the origin of the tangent space T_qM.

In general, $\mathrm{inj}_{(M,g)}$ and $\mathrm{conj}_{(M,g)}$ have significant differences in each geometric contents. For example, consider the class of manifolds satisfying

$$\mathrm{Ric}_{(M,g)} \geq n-1, \quad \mathrm{inj}_{(M,g)} \geq i_0 \quad \text{and} \quad \mathrm{diam}_{(M,g)} \leq D,$$

then we know that the above class is C^α-compact [AC]. But if we replace the condition on the $\mathrm{inj}_{(M,g)}$ by the $\mathrm{conj}_{(M,g)}$, then it is not C^α-compact any longer since a collapsing may occur. Note also that if $K_M \leq K$, then $\mathrm{conj}_M \geq \frac{\pi}{\sqrt{K}}$.

The main purpose of this paper is to show the following theorem:

Theorem 1.1 *Given an integer n and $c_0 > 0$, there exists an $\varepsilon = \varepsilon(n, c_0) > 0$ such that if M admits a metric g satisfying*

$$\mathrm{Ric}_{(M,g)} \geq n-1, \quad \mathrm{conj}_{(M,g)} \geq c_0 \quad \textit{and} \quad \mathrm{diam}_{(M,g)} \geq \pi - \varepsilon,$$

then M is diffeomorphic to S^n.

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Theorem 1.2 *Given an integer n and $c_0 > 0$, there exists an $\varepsilon = \varepsilon(n, c_0) > 0$ such that if M admits a metric g satisfying*

$$\mathrm{Ric}_{(M,g)} \geq n-1, \ \mathrm{conj}_{(M,g)} \geq c_0 \quad \textit{and} \quad \lambda_{1(M,g)} \leq n+\varepsilon,$$

then M is diffeomorphic to S^n, where $\lambda_{1(M,g)}$ is the first eigenvalue of M.

Theorem 1.2 is an immediate consequence of Theorem 1.1 and the theorem due to Croke [Cr]. The proof of it is just similar to that of [Be], so we omit it.

We can also prove the following volume/diameter sphere theorem.

Theorem 1.3 *Given an integer n, there exists $\varepsilon > 0$ such that if M is an n-dimensional Riemannian manifold with $\mathrm{Ric}_{(M,g)} \geq n-1$, $\frac{\mathrm{vol}_{(M,g)}}{\mathrm{diam}_{(M,g)}} \geq \frac{\omega_n}{\pi} - \varepsilon$ and $e_{(M,g)} < \varepsilon$, then M is diffeomorphic to S^n, where ω_n is the volume of the standard sphere S^n and $e_{(M,g)}$ is the excess of M.*

The excess condition in Theorem 1.3 cannot be removed. This can be seen by using, for example, $M^4 = CP^2$ with metric normalized so that $\mathrm{Ric}_M = 3$.

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2 Preliminaries

We begin with the estimate of the Hessian of the distance function due to R. Brocks [Br].

Theorem 2.1 *Let M^n be a complete Riemannian manifold with $\mathrm{conj}_M \geq c_0$ and $\mathrm{Ric}_M \geq -(n-1)k^2$ and c be a geodesic on M^n. Let $A(t)$ be a Hessian of distance function along $c(t)$ from $c(0)$. Let $\alpha(t) = k^2 t + 2k \coth k(c_0 - t)/2$, $\gamma(t) = 2\sqrt{(n-1)\alpha(t)}\sqrt{t}$ for $0 \leq t \leq c_0$. Then*

$$\int_0^t \left\| A(\tau) - \frac{I}{\tau} \right\| d\tau \leq 2\gamma(t)$$

for all t such that $\gamma(t) \leq 1/7$.

Let M be a manifold as the above theorem and $c\colon [0, l_0] \to M$ be a geodesic on M, where $l_0 \leq c_0/2$. If $J(t)$ be a Jacobi field along $c(t)$ such that $J(0) = 0$ and $\langle J'(0), c'(0) \rangle = 0$, then we know that $A(t)$ can be written in normal coordinate as $B(t) + I/t$ with $B(t)$ smooth at $t = 0$.

More importantly, by the standard arguments for the estimate of the norm of Jacobi field (see [P1], [DSW], [DW]), we know that for $0 \leq t \leq l_0$,

$$e^{-\int_0^{l_0} \|B\|} \|J'(0)\| t \leq \|J(t)\| \leq e^{\int_0^{l_0} \|B\|} \|J'(0)\| t.$$

Now let's estimate $\int_0^{l_0} \|B\|$ in the case of $\mathrm{Ric}_M \geq -(n-1)\varepsilon^2$, $\mathrm{conj}_M \geq c_0 := C\varepsilon^{-1}$, where ε is a sufficiently small positive constant. To estimate $\int_0^{l_0} \|B\|$, it suffices to estimate $\gamma(t)$ in Theorem 2.1. For this purpose, first we estimate $\alpha(t) = \varepsilon^2 t + 2\varepsilon \coth \varepsilon(\frac{c_0 - t}{2})$ in

Theorem 2.1. For $0 \le t \le c_0/2$, we have $\frac{C}{4} = \frac{\varepsilon c_0}{4} \le \frac{\varepsilon(c_0 - t)}{2} \le \frac{\varepsilon c_0}{2} = \frac{C}{2}$ and

$$\begin{aligned} 2\varepsilon \coth \varepsilon \frac{(c_0 - t)}{2} &= \frac{2}{c_0 - t} 2\varepsilon \frac{c_0 - t}{2} \coth \varepsilon \frac{(c_0 - t)}{2} \\ &\le \frac{8}{c_0} \varepsilon \frac{(c_0 - t)}{2} \coth \varepsilon \frac{(c_0 - t)}{2} \\ &\le \frac{H}{c_0} = HC\varepsilon. \end{aligned}$$

for some constant $H > 0$, since we know that $f(x) = x \coth x$ is bounded for any x with $C/4 \le x \le C/2$. Now

$$\begin{aligned} \gamma(t) &= 2\sqrt{(n-1)\alpha(t)}\sqrt{t} \\ &= 2\sqrt{(n-1)\left\{(\varepsilon t)^2 + 2\left(\varepsilon \coth \varepsilon \left(\frac{(c_0 - t)}{2}\right)\right) \cdot t\right\}} \\ &\le 2\sqrt{(n-1)\big((\varepsilon t)^2 + HC\varepsilon t\big)} \end{aligned}$$

for $0 \le t \le c_0/2$. So we have

$$\gamma(t) \le 2\sqrt{(n-1)\big((\varepsilon l_0)^2 + HC\varepsilon l_0\big)} = \tau(l_0|\varepsilon)$$

for $t \le l_0 \le c_0/2$. Here, $\tau(l_0|\varepsilon)$ is a positive number converging to zero as $\varepsilon \to 0$ if we fix l_0.

To prove Theorem 1.1, one needs the following result of Calabi and Hartman [CH] on the smoothness of isometries.

Theorem 2.2 *Let (M, g) be a C^α-Riemannian manifold with respect to some coordinate. Then its geodesics are $C^{1,\alpha}$ with respect to the same coordinate. Moreover the $C^{1,\alpha}$-norm of the geodesics can be bounded by the C^α-norm of the metric.*

An immediate corollary of this result is the following lemma.

Lemma 2.3 *If (M_i, g_i, p_i) converges to (M, g, p) in C^α-topology, then for any geodesics $\{\gamma^i\}$ in $B_{p_i}(r) \subset M_i$ for some $r > 0$, a subsequence of $\{\gamma^i\}$ converges to a geodesic γ in M in $C^{1,\alpha'}$-topology, $\alpha' < \alpha$.*

3 Proof of Theorem 1.1

Consider a sequence of Riemannian manifolds $\{(M_i, g_i)\}$ satisfying

$$\mathrm{Ric}_{(M_i,g_i)} \ge n-1, \quad \mathrm{conj}_{(M_i,g_i)} \ge c_0, \quad \text{and} \quad \mathrm{diam}_{(M_i,g_i)} \ge \pi - \varepsilon_i,$$

where $\lim_{i\to\infty} \varepsilon_i = 0$. If $\tilde{M}_i$ is a universal covering of M_i, then we know that

$$\mathrm{Ric}_{(\tilde{M}_i,\tilde{g}_i)} \ge n-1, \quad \mathrm{conj}_{(\tilde{M}_i,\tilde{g}_i)} \ge c_0, \quad \text{and} \quad \mathrm{diam}_{(\tilde{M}_i,\tilde{g}_i)} \ge \pi - \varepsilon_i,$$

where $\tilde{g}_i$ is the induced metric on $\tilde{M}_i$ from M_i.

From now on, we will show that $\tilde{M}_i$ converges to S^n with the standard metric in the C^α-topology and it suffices to prove Theorem 1.1 since the only Riemannian manifold with diameter close to π which has S^n (with the metric close to the standard metric of S^n in the C^α-sense) as a Riemannian covering space is S^n itself.

Note that if $\mathrm{inj}_{\tilde{M}_i}$ is uniformly bounded below, then we have done by [Be]. So suppose that $\mathrm{inj}_{\tilde{M}_i}$ converges to zero. (In this case, we will induce a contradiction.)

Theorem 3.1 **([P2])** *Let M be an n-dimensional complete simply connected Riemannian manifold. Given $n, H, c_0 > 0$, there exists an $i_0(n, H, c_0) > 0$ depending on n, H, c_0 such that if $\mathrm{Ric}_M \geq (n-1)H^2$, $\mathrm{conj}_M \geq c_0$, then injectivity radius of M has a lower bound $i_0(n, H, c_0) > 0$.*

Proof Consider a sequence of simply connected manifolds $\{(M_i, g_i)\}$ satisfying

$$\mathrm{Ric}_{(M_i,g_i)} \geq (n-1)H^2, \quad \mathrm{conj}_{(M_i,g_i)} \geq c_0 \quad \text{and} \quad \mathrm{inj}_{(M_i,g_i)} \to 0.$$

Let γ_i be the shortest closed geodesic on $\tilde{M}_i$ and r_i be the length of γ_i as above. Then we know that γ_i are smooth and $r_i \to 0$ as $i \to \infty$.

We will show that there are no conjugate points on $\gamma_i(t)$ for $t \leq \pi/H$. Rescaling metrics by multiplying by r_1^{-2}, we obtain

$$\mathrm{Ric}_{(M_i,g_i')} \geq (n-1)H^2 r_i^2 \to 0, \quad \mathrm{conj}_{(M_i,g_i')} \geq c_0/r_i \to \infty$$

and $\mathrm{inj}_{(M_i,g_i')} \geq i_0$, where g_i' are the rescaled metrics. From the compactness theorem of [AC], we know that $M_i \to X$ for some smooth manifold X with C^α-metric. We also know that $\exp_i \colon T_{p_i}M_i \to M_i$ is nonsingular for $B(0, c_0/r_i)$ and $\|d\exp_i\|$ is estimated in previous sections. Let id be an identity map from $\mathrm{R}^n = T_{p_i}M_i$ to R^n. From Section 2, we get that for any fixed l_0,

$$\frac{\|d\exp_i\|}{\|\mathrm{id}_*\|} = e^{\int_0^{l_0}\|B_i\|} \to 1$$

as $\varepsilon_i \to 0$. So $\exp_i$ converges to id in Hölder sense for every compact set.

We know that exp is a covering map from $T_pM = \mathrm{R}^n$ to M if $\mathrm{conj}_M = \infty$ [BC]. By the same reason, we know that $\exp_p = \lim_{i\to\infty} \exp_i = \mathrm{id}$ is a covering map so R^n is the universal covering space of the limit space X. So X is a flat manifold.

We know that the homomorphism

$$i \colon \pi_1\big(B(p_i, r_i)\big) \to \pi_1\big(B(p_i, c_0/2)\big)$$

are inclusions. Consider the universal covering space of $B(p_i, c_0/2)$, $B\widetilde{(p_i, c_0/}2)$. We know that γ_i do not represent 0 in $\pi_1\big(B(p_i, r_i)\big)$. So we define T_i as $\langle\gamma_i\rangle$-orbit of $c_0/2$-ball centered at a lifting of p_i, $\tilde{p}_i$ in $B\widetilde{(p_i, c_0/}2)$.

Let $\bar{\gamma}_i = \mathrm{R}$ be the lifting of γ_i. We also know that the injectivity radii of $B\widetilde{(p_i, c_0/}2)$ are bounded below [BK]. Then by compactness theorem, we know that $(T_i, \tilde{p}_i)$ converges

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to (X,p) in C^α-topology uniformly since $\pi_1\big(B(p_i,c_0/2)\big)$ act on $B(\widetilde{p_i,c_0}/2)$ cocompactly. Rescaling the metrics by multiplying r_i^{-2} as above, we easily know that

$$\frac{\int_0^{r_i}\|B_i\|}{r_i}\to 0.$$

(The above values are invariant under rescaling. For the rescaled metrics, we know that $r_i=1$ and (M_i,g_i') converges to flat manifold so $\int_0^1\|B_i'\|\to 0$, where B_i' is the B_i for g_i'.) We also know that the rotational parts of holonomy along γ_i depends only on $\int\|B_i\|$ since M_i are simply connected and γ_i are smooth at $t=0$. From Section 3, we obtain that the rotational part of parallel translation along $\bar\gamma:=\lim_{i\to\infty}\bar\gamma_i=\mathrm{R}\subset T_pX$ from $\bar\gamma(0)$ to $\bar\gamma(r)$ is $\lim_{i\to\infty}l_i\int_0^{r_i}\|B_i\|$, where $l_ir_i=r$ and T_pX has the pull-back metric by exp. Since

$$\frac{l_i\int_0^{r_i}\|B_i\|}{l_ir_i}\to 0,$$

we know that the parallel translation along $\bar\gamma$ is the same as that of T_pX with Euclidean metric. So we may consider X as $\mathrm{R}\times F$ for some C^α-manifold, F in infinitesimal tubular neighborhood of R. This means that there exist geodesics in T_i such that the distance from p_i to the first conjugate point converges to infinity as $i\to\infty$. But $\mathrm{Ric}_M\ge(n-1)H^2>0$ implies that $t_0\le\pi/H$, which is a contradiction. ■

From this theorem, we know that $\tilde M_i$ have a lower bound on injectivity radius so we completes the proof of Theorem 1.1.

Remark 3.2 We may wonder that the condition of $\mathrm{Ric}_M\ge(n-1)H^2$ can be replaced by $\mathrm{Ric}_M\ge-(n-1)H^2$. But considering Berger's spheres, we know that the positive Ricci curvature condition is essential. This theorem can be considered as a Ricci curvature version of Klingenberg's theorem for the lower bound on injectivity radius [CE].

4 Proof of Theorem 1.3

Consider a sequence of manifolds $\{(M_i,g_i)\}$ such that $\mathrm{Ric}_{(M_i,g_i)}\ge n-1$, $\frac{\omega_n}{\pi}-\varepsilon_i\le\frac{\mathrm{vol}_{(M_i,g_i)}}{\mathrm{diam}_{(M_i,g_i)}}$ and $e_{(M_i,g_i)}<\varepsilon_i$, where $\lim_{i\to\infty}\varepsilon_i=0$, $\varepsilon_i>0$.

Passing to a subsequence, if necessary, we assume that $\mathrm{diam}_{(M_i,g_i)}\le\pi/2$ for all i or $\mathrm{diam}_{(M_i,g_i)}>\pi/2$ for all i.

Case 1

$\mathrm{diam}_{(M_i,g_i)}>\frac{\pi}{2}$ for all i:

Let p_i, q_i be the points satisfying $\max_x e_{p_i,q_i}(x)=e_{(M_i,g_i)}$ and d_i be the distance between p_i and q_i.

From $\mathrm{diam}_{(M_i,g_i)}\le e_{(M_i,g_i)}+d_i$, it follows immediately that $d:=\lim_{i\to\infty}d_i=\lim_{i\to\infty}\mathrm{diam}_{(M_i,g_i)}\ge\frac{\pi}{2}$. So we can choose α_i, β_i so that $\alpha_i+\beta_i=d_i$ and $\alpha_i\uparrow\pi/2$,

$\beta_i \uparrow d - \pi/2$. Using the volume comparison theorem, we have

$$\begin{aligned}\frac{\mathrm{vol}_{(M_i,g_i)}}{\mathrm{diam}_{(M_i,g_i)}} &\le \frac{1}{d_i}\left\{\mathrm{vol}\left(B_{p_i}\left(\alpha_i + \frac{\epsilon_i}{2}\right)\right) + \mathrm{vol}\left(B_{q_i}\left(\beta_i + \frac{\epsilon_i}{2}\right)\right)\right\} \\ &\le \frac{1}{d_i}\left\{\omega_{n-1}\int_0^{\alpha_i+\frac{\epsilon_i}{2}} \sin^{n-1} t\, dt + \omega_{n-1}\int_0^{\beta_i+\frac{\epsilon_i}{2}} \sin^{n-1} t\, dt\right\} \\ &= \frac{\omega_{n-1}}{d_i}\left\{\int_0^{\alpha_i} \sin^{n-1} t\, dt + \int_0^{\beta_i} \sin^{n-1} t\, dt\right\} + \delta_i \\ &\le \frac{\omega_{n-1}}{d_i}\left\{\frac{\alpha_i}{\pi}\int_0^{\pi} \sin^{n-1} t\, dt + \frac{\beta_i}{\pi}\int_0^{\pi} \sin^{n-1} t\, dt\right\} + \delta_i \\ &= \frac{\omega_n}{\pi} + \delta_i,\end{aligned}$$

where $\delta_i \to 0$ as $i \to \infty$. Since $\frac{\mathrm{vol}_{(M_i,g_i)}}{\mathrm{diam}_{(M_i,g_i)}} \ge \frac{\omega_n}{\pi} - \varepsilon_i$, we obtain by letting $i \to \infty$, that $\frac{1}{d-\pi/2}\int_0^{d-\pi/2} \sin^{n-1} t\, dt = \frac{1}{\pi}\int_0^{\pi} \sin^{n-1} t\, dt$.

Now since $f(x) = \frac{\int_0^x \sin^{n-1} r\, dr}{x}$ is strictly increasing function of $x(\le \frac{\pi}{2})$, we have $d - \pi/2 = \pi/2$ or $d = \pi$. So $\mathrm{vol}_{(M_i,g_i)} \to \omega_n$ and the result follows from the Appendix 1 of [CCo2] (*cf.* [CCo1]).

Case 2

$\mathrm{diam}_{(M_i,g_i)} \le \frac{\pi}{2}$ for all i:

Note that there exists a space M such that $M_i \to M$ in the Gromov-Hausdorff topology. Let $l = \mathrm{diam}_M$ then $\mathrm{diam}_{(M_i,g_i)} =: l_i \to l \le \frac{\pi}{2}$ and we have

$$\frac{\mathrm{vol}_{(M_i,g_i)}}{\omega_n} \le \frac{\int_0^{l_i} \sin^{n-1} r\, dr}{\int_0^{\pi} \sin^{n-1} r\, dr} \le \frac{l_i}{\pi} \to \frac{l}{\pi}.$$

Thus by the limit argument, we obtain

$$\frac{\int_0^{l} \sin^{n-1} r\, dr}{l} = \frac{\int_0^{\pi} \sin^{n-1} r\, dr}{\pi}.$$

Now as in the case 1, we have $l = \frac{\pi}{2}$. So, we observed that $\mathrm{diam}_{(M_i,g_i)} \to \frac{\pi}{2}$. Under the same setting as in Case 1, choose α_i, β_i so that $\alpha_i \uparrow \pi/3$, $\beta_i \uparrow \pi/6$. Then we have

$$\begin{aligned}\frac{\mathrm{vol}_{(M_i,g_i)}}{\mathrm{diam}_{(M_i,g_i)}} &\le \frac{\omega_{n-1}}{d_i}\left\{\int_0^{\alpha_i} \sin^{n-1} t\, dt + \int_0^{\beta_i} \sin^{n-1} t\, dt\right\} + \delta_i \\ &\le \frac{\omega_{n-1}}{d_i}\left\{\frac{\alpha_i}{\pi}\int_0^{\pi} \sin^{n-1} t\, dt + \frac{\beta_i}{\pi}\int_0^{\pi} \sin^{n-1} t\, dt\right\} + \delta_i \\ &= \frac{\omega_n}{\pi} + \delta_i.\end{aligned}$$

By letting $i \to \infty$, we know that the above inequalities are equalities. Consequently, we have a contradiction to the strict increasing property of $f(x) = \frac{\int_0^x \sin^{n-1} r\, dr}{x} (0 \le x \le \frac{\pi}{2})$.

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