

Duality in topological algebra

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Aspects of duality relating to compact totally disconnected universal algebras are considered. It is shown that if P is a "basic" set of injectives in a variety of compact totally disconnected algebras then the category \bar{P} of P -copresentable objects is in duality with the class of all G -copresentable algebras on P , where $G : P \rightarrow \mathit{Ens}$ is the forgetful functor and an algebra is taken to mean a finite-product-preserving functor from P to Ens .

Introduction

Let U be the category of Hausdorff topological algebras for a given algebraic theory over the category of compactly generated Hausdorff spaces (see Schubert [6] for the concept of an algebraic theory, and see Borceux and Day [1] for its relative enrichment). We denote by $K_0U \subset U$ the category of compact totally disconnected algebras and we denote by FU the category of all finite discrete algebras; thus $K_0U \cong S(P(FU))$ formed in U (where P denotes the formation of products and S denotes the taking of *strong* subobjects).

Given a small "basic" category $P \subseteq FU$ of injectives in $P' = S(PP)$ formed in U , we consider the functor category $[P, \mathit{Ens}]$ of actions of P on the category Ens of small sets. Adjointness between P' and $[P, \mathit{Ens}]$ is examined and a duality is derived between the category of P -copresentable objects in U and the category of all G -copresentable algebras in $[P, \mathit{Ens}]$, where $G : P \rightarrow \mathit{Ens}$ is the forgetful functor.

Examples of varieties which yield such dualities include:

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- (1) boolean rings;
- (2) abelian torsion groups generated by \mathbb{Z}_n ;
- (3) vector spaces over a finite field;
- (4) any equational class of rings generated by finitely many finite fields of different characteristics (see Choe [3], Example 3).

We also recall from Choe [2, 3] that if the given algebraic theory is associative and distributive, then an injective in FU is an injective in K_0U , hence in $P' \subseteq K_0U$. It will be seen that most of our examples lie in this direction.

The basic references for category theory are Mac Lane [5] and Schubert [6], and powers A^X are denoted $\{X, A\}$.

1. The limit closure of P

With the notation of the Introduction we first observe that $P' = S(PP)$ in U , is closed under products and strong subobjects in U . Moreover, $A \in P'$, iff the canonical map $A \rightarrow \Pi_P\{U(A, P), P\}$ is a strong mono in U and the epireflection $U \rightarrow P$ is given by factoring the canonical map $A \rightarrow \Pi_P\{U(A, P), P\}$ into an epi followed by a strong mono in U .

PROPOSITION 1.1. *In P' the composite of two regular monos is regular.*

Proof. By Kelly [4], Proposition 5.10, it suffices to verify that the pushout of a regular mono in P' is a mono. But this follows from the fact that P is a cogenerating set of injectives in P' . //

Let \bar{P} be the full subcategory of P' determined by those A for which there exists a regular mono $A \rightarrow \Pi_P \lambda$ in P' .

PROPOSITION 1.2. *\bar{P} is reflective in P' .*

Proof. \bar{P} is closed under limits in P' by Proposition 1.1. Moreover, P is a small set of cogenerators of P' , hence of \bar{P} , so the special adjoint functor theorem applies. //

COROLLARY 1.3. *\bar{P} is the limit closure of P in U .*

Proof. Suppose $A \rightarrow \prod_{\lambda} P_{\lambda} \rightrightarrows \prod_{\mu} P_{\mu}$ is an equaliser presentation in U . Then it is an equaliser in \bar{P} by Proposition 1.2. Conversely, if $A \in \bar{P}$, then there is a regular mono $A \rightarrow \prod_{\mu} P_{\mu}$ in P' , hence there is a presentation $A \rightarrow \prod_{\mu} P_{\mu} \rightrightarrows \prod_{\mu} P_{\mu}$ in U . //

2. The duality

Now define $T : \bar{P}^{op} \rightarrow [P, Ens]$ by $TA(P) = P(A, P)$ and define $S : [P, Ens]^{op} \rightarrow \bar{P}$ by $SF = \int_P \{FP, P\}$. Then

$$(\varepsilon, \eta) : S^{op} \rightarrow T : \bar{P}^{op} \rightarrow [P, Ens],$$

and $P \cong STP$ for all $P \in P$, because

$$STP = \int_Q \{TP(Q), Q\} = \int_Q \{P(P, Q), Q\} \cong P$$

by the representation theorem.

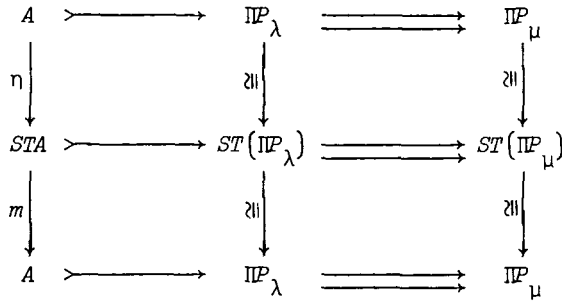
DEFINITION 2.1. The small category P , of injectives in P' , is said to be *basic* if it is closed under finite products and each map $f : \prod_{\lambda} P_{\lambda} \rightarrow P$ in \bar{P} , with $P_{\lambda}, P \in P$, factors through a finite subproduct.

REMARK 2.2. The category P is always basic if it is closed under products and U is uniformly pointed in the sense that each regular epimorphism f in U is the coequaliser of $\ker f$ and 0 (for example, groups, rings, finitely complete and cocomplete additive categories, and so on).

PROPOSITION 2.3. If P is a basic set of injectives in \bar{P} then $\eta : 1 \cong ST : \bar{P} \rightarrow \bar{P}$ or, in other words, $P \subset \bar{P}$ is a codense inclusion (that is, $A \cong \int_P \{\bar{P}(A, P), P\}$ for all $A \in \bar{P}$).

Proof. We know $A \in \bar{P}$ iff there exists an equaliser presentation $A \rightarrow \prod_{\lambda} P_{\lambda} \rightrightarrows \prod_{\eta} P_{\eta}$ in U . First consider the product $\prod_{\lambda} P_{\lambda}$. If this is viewed as a limit $\lim_{\downarrow} P'_{\nu}$ cofiltered over the finite subproducts of $\prod_{\lambda} P_{\lambda}$, then we have a bijection $\text{colim } P(P'_{\nu}, P) \cong \bar{P}(\lim_{\downarrow} P'_{\nu}, P)$. This implies

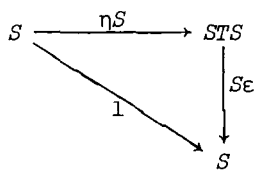
$\Pi P_\lambda \cong ST(\Pi P_\lambda)$. Now consider A in \bar{P} , and look at the following diagram:



The induced map m is a mono, thus is an isomorphism, since $m\eta = 1$. //

Let $G : P \rightarrow Ens$ denote the forgetful functor and let $G = \{F \in [P, Ens]; \text{there exist small sets } X \text{ and } Y \text{ and an equaliser presentation } F \twoheadrightarrow G^X \rightrightarrows G^Y\}$.

Then we have an induced adjunction $(\epsilon, \eta) : S^{OP} \dashv T : \bar{P}^{OP} \rightarrow G$, where $[P, Ens](-, G) : G^{OP} \rightarrow Ens$ reflects isomorphisms. But $[P, Ens](-, G) = US^{OP} : G^{OP} \rightarrow Ens$, where $U : \bar{P} \rightarrow Ens$ is the forgetful functor; thus S reflects isomorphisms. This implies, by the triangle identity



and the fact that η is an isomorphism, that ϵ is an isomorphism. To summarise, we have that $(\epsilon, \eta) : S^{OP} \dashv T : \bar{P}^{OP} \rightarrow G$ is a *category equivalence* and $G \subset [P, Ens]$ is *reflective*.

We conclude this section with the following observations about \bar{P} .

PROPOSITION 2.4. *If each $A \in FU$ has a presentation $A \rightarrow \Pi P_\lambda \rightrightarrows \Pi P_\mu$, $P_\lambda, P_\mu \in P$, in U , then $\overline{FU} = \bar{P}$.*

Proof. Clearly $P \subset FU \subset \bar{P}$, so $\bar{P} \subset \overline{FU}$. Now let A have

$A \rightarrow \prod V_\lambda \overset{\rightarrow}{\dashv} \prod V_\mu$ in U where $V_\lambda, V_\mu \in FU$. Then, since \bar{P} is reflective in U and each $V \in FU$ is in \bar{P} , we have $\bar{P} = \overline{FU}$. //

COROLLARY 2.5. *Suppose the algebraic theory under consideration is associative and distributive (in the sense of Choe [2]), and suppose each $A \in FU$ has a presentation $A \rightarrow \prod P_\lambda \overset{\rightarrow}{\dashv} \prod P_\mu$, $P_\lambda, P_\mu \in P$, in U . Then $K_0U = \overline{FU}$ and, since $K_0U \supset S(P(FU)) \supset S(PP) \supset \bar{P} = \overline{FU}$, we have $K_0U = \bar{P}$.*

3. Examples

EXAMPLE 3.1. Consider the category U of compactly generated Hausdorff boolean rings. We take $P = FU = \text{Ens} \delta \text{in}^{\text{op}}$ and observe that \bar{P} ($= K_0U$ in this case) is equivalent to G^{op} , where G is reflective in $[\text{Ens} \delta \text{in}^{\text{op}}, \text{Ens}]_x \cong \text{Ens}$. Since G contains the non-trivial object $2 (\cong G)$, we must have $G \cong \text{Ens}$. Thus $(K_0U)^{\text{op}} \cong \text{Ens}$. In other words, the category of compact totally disconnected boolean rings is equivalent to the category of complete atomic boolean algebras.

EXAMPLE 3.2. Let U_n be the category of compactly generated Hausdorff abelian groups A such that $na = 0$ for all $a \in A$. Then \mathbb{Z}_n is injective in FU_n and is a strong cogenerator; that is, each $P \in FU_n$ has an equaliser presentation $P \rightarrow \mathbb{Z}_n^m \rightarrow \mathbb{Z}_n^p$. Thus, if we take $P = \{1, \mathbb{Z}_n^2, \dots, \mathbb{Z}_n^m, \dots\} \subset FU_n$, we obtain $\bar{P} = K_0U_n$.

EXAMPLE 3.3. Let U be the category of compactly generated Hausdorff topological vector spaces over a finite discrete field Q . Let $P = \{1, Q, Q^2, \dots, Q^n, \dots\}$; then $\bar{P} = K_0U$.

References

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- [5] Saunders Mac Lane, *Categories for the working mathematician* (Graduate Texts in Mathematics, 5. Springer-Verlag, New York, Heidelberg, Berlin, 1971).
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