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Duality in topological algebra

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Aspects of duality relating to compact totally disconnected universal algebras are considered. It is shown that if P is a "basic" set of injectives in a variety of compact totally disconnected algebras then the category $\overline{P}$ of P-copresentable objects is in duality with the class of all G-copresentable algebras on P, where $G : P \to Ens$ is the forgetful functor and an algebra is taken to mean a finite-product-preserving functor from P to Ens.

Introduction

Let U be the category of Hausdorff topological algebras for a given algebraic theory over the category of compactly generated Hausdorff spaces (see Schubert [6] for the concept of an algebraic theory, and see Borceux and Day [1] for its relative enrichment). We denote by $K_0U \subset U$ the category of compact totally disconnected algebras and we denote by FU the category of all finite discrete algebras; thus $K_0U \supseteq S(P(FU))$ formed in U (where P denotes the formation of products and S denotes the taking of *strong* subobjects).

Given a small "basic" category $P \subseteq FU$ of injectives in $P' = S(PP)$ formed in U, we consider the functor category $[P, Ens]$ of actions of P on the category Ens of small sets. Adjointness between P' and $[P, Ens]$ is examined and a duality is derived between the category of P-copresentable objects in U and the category of all G-copresentable algebras in $[P, Ens]$, where $G : P \to Ens$ is the forgetful functor.

Examples of varieties which yield such dualities include:

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(1) boolean rings;

(2) abelian torsion groups generated by $\mathbb{Z}_n$;

(3) vector spaces over a finite field;

(4) any equational class of rings generated by finitely many finite fields of different characteristics (see Choe [3], Example 3).

We also recall from Choe [2, 3] that if the given algebraic theory is associative and distributive, then an injective in FU is an injective in K_0U , hence in $P' \subseteq K_0U$. It will be seen that most of our examples lie in this direction.

The basic references for category theory are Mac Lane [5] and Schubert [6], and powers A^X are denoted $\{X, A\}$.

1. The limit closure of P

With the notation of the Introduction we first observe that $P' = S(PP)$ in U , is closed under products and strong subobjects in U . Moreover, $A \in P'$, iff the canonical map $A \to \Pi_P\{U(A, P), P\}$ is a strong mono in U and the epireflection $U \to P$ is given by factoring the canonical map $A \to \Pi_P\{U(A, P), P\}$ into an epi followed by a strong mono in U .

PROPOSITION 1.1. *In* P' *the composite of two regular monos is regular.*

Proof. By Kelly [4], Proposition 5.10, it suffices to verify that the pushout of a regular mono in P' is a mono. But this follows from the fact that P is a cogenerating set of injectives in P' . //

Let $\overline{P}$ be the full subcategory of P' determined by those A for which there exists a regular mono $A \to \Pi P_\lambda$ *in* P' .

PROPOSITION 1.2. $\overline{P}$ *is reflective in* P' .

Proof. $\overline{P}$ is closed under limits in P' by Proposition 1.1. Moreover, P is a small set of cogenerators of P' , hence of $\overline{P}$, so the special adjoint functor theorem applies. //

COROLLARY 1.3. $\overline{P}$ *is the limit closure of* P *in* U .

Proof. Suppose $A \to \Pi P_\lambda \overset{\to}{\to} \Pi P_\mu$ is an equaliser presentation in $\mathcal{U}$. Then it is an equaliser in $\overline{P}$ by Proposition 1.2. Conversely, if $A \in \overline{P}$, then there is a regular mono $A \to \Pi P_\mu$ in P', hence there is a presentation $A \to \Pi P_\mu \overset{\to}{\to} \Pi P_\mu$ in $\mathcal{U}$. //

2. The duality

Now define $T : \overline{P}^{\mathrm{op}} \to [P, \mathcal{E}ns]$ by $TA(P) = P(A, P)$ and define $S : [P, \mathcal{E}ns]^{\mathrm{op}} \to \overline{P}$ by $SF = \int_P \{FP, P\}$. Then

$$(\varepsilon, \eta) : S^{\mathrm{op}} \dashv T : \overline{P}^{\mathrm{op}} \to [P, \mathcal{E}ns] ,$$

and $P \cong STP$ for all $P \in P$, because

$$STP = \int_Q \{TP(Q), Q\} = \int_Q \{P(P, Q), Q\} \cong P$$

by the representation theorem.

DEFINITION 2.1. The small category P, of injectives in P', is said to be *basic* if it is closed under finite products and each map $f : \Pi P_\lambda \to P$ in $\overline{P}$, with $P_\lambda, P \in P$, factors through a finite subproduct.

REMARK 2.2. The category P is always basic if it is closed under products and $\mathcal{U}$ is uniformly pointed in the sense that each regular epimorphism f in $\mathcal{U}$ is the coequaliser of $\ker f$ and 0 (for example, groups, rings, finitely complete and cocomplete additive categories, and so on).

PROPOSITION 2.3. *If P is a basic set of injectives in $\overline{P}$ then $\eta : 1 \cong ST : \overline{P} \to \overline{P}$ or, in other words, $P \subset \overline{P}$ is a codense inclusion* (*that is,* $A \cong \int_P \{\overline{P}(A, P), P\}$ *for all* $A \in \overline{P}$).

Proof. We know $A \in \overline{P}$ iff there exists an equaliser presentation $A \to \Pi P_\lambda \overset{\to}{\to} \Pi P_\eta$ in $\mathcal{U}$. First consider the product ΠP_λ. If this is viewed as a limit $\lim P'_\nu$ cofiltered over the finite subproducts of ΠP_λ, then we have a bijection $\operatorname{colim} P(P'_\nu, P) \cong \overline{P}(\lim P'_\nu, P)$. This implies

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$\Pi P_\lambda \cong ST(\Pi P_\lambda)$. Now consider A in $\overline{P}$, and look at the following diagram:

$$\begin{array}{ccccc} A & \rightarrowtail & \Pi P_\lambda & \rightrightarrows & \Pi P_\mu \\ \eta \downarrow & & \wr\!\!\parallel \downarrow & & \wr\!\!\parallel \downarrow \\ STA & \rightarrowtail & ST(\Pi P_\lambda) & \rightrightarrows & ST(\Pi P_\mu) \\ m \downarrow & & \wr\!\!\parallel \downarrow & & \wr\!\!\parallel \downarrow \\ A & \rightarrowtail & \Pi P_\lambda & \rightrightarrows & \Pi P_\mu \end{array} .$$

The induced map m is a mono, thus is an isomorphism, since $m\eta = 1$. //

Let $G : P \to \mathcal{E}ns$ denote the forgetful functor and let

$G = \{F \in [P, \mathcal{E}ns];$ there exist small sets X and Y and an equaliser presentation $F \rightarrowtail G^X \rightrightarrows G^Y\}$.

Then we have an induced adjunction $(\varepsilon, \eta) : S^{op} \dashv T : \overline{P}^{op} \to G$, where $[P, \mathcal{E}ns](-, G) : G^{op} \to \mathcal{E}ns$ reflects isomorphisms. But $[P, \mathcal{E}ns](-, G) = US^{op} : G^{op} \to \mathcal{E}ns$, where $U : \overline{P} \to \mathcal{E}ns$ is the forgetful functor; thus S reflects isomorphisms. This implies, by the triangle identity

$$\begin{array}{ccc} S & \xrightarrow{\eta S} & STS \\ & {\scriptstyle 1} \searrow & \downarrow {\scriptstyle S\varepsilon} \\ & & S \end{array}$$

and the fact that η is an isomorphism, that ε is an isomorphism. To summarise, we have that $(\varepsilon, \eta) : S^{op} \dashv T : \overline{P}^{op} \to G$ is a *category equivalence* and $G \subset [P, \mathcal{E}ns]$ is *reflective*.

We conclude this section with the following observations about $\overline{P}$.

PROPOSITION 2.4. *If each* $A \in FU$ *has a presentation* $A \to \Pi P_\lambda \rightrightarrows \Pi P_\mu$, $P_\lambda, P_\mu \in P$, *in* U, *then* $\overline{FU} = \overline{P}$.

Proof. Clearly $P \subset FU \subset \overline{P}$, so $\overline{P} \subset \overline{FU}$. Now let A have

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$A \to \Pi V_\lambda \overset{\to}{\to} \Pi V_\mu$ in $\mathcal{U}$ where $V_\lambda, V_\mu \in F\mathcal{U}$. Then, since $\overline{P}$ is reflective in $\mathcal{U}$ and each $V \in F\mathcal{U}$ is in $\overline{P}$, we have $\overline{P} = \overline{F\mathcal{U}}$. //

COROLLARY 2.5. *Suppose the algebraic theory under consideration is associative and distributive (in the sense of* Choe [2]*), and suppose each* $A \in F\mathcal{U}$ *has a presentation* $A \to \Pi P_\lambda \overset{\to}{\to} \Pi P_\mu$, $P_\lambda, P_\mu \in P$, *in* $\mathcal{U}$. *Then* $K_0\mathcal{U} = \overline{F\mathcal{U}}$ *and, since* $K_0\mathcal{U} \supset S(P(F\mathcal{U})) \supset S(PP) \supset \overline{P} = \overline{F\mathcal{U}}$, *we have* $K_0\mathcal{U} = \overline{P}$.

3. Examples

EXAMPLE 3.1. Consider the category $\mathcal{U}$ of compactly generated Hausdorff boolean rings. We take $P = F\mathcal{U} = \mathcal{E}ns\mathcal{f}in^{op}$ and observe that $\overline{P}$ $(= K_0\mathcal{U}$ in this case$)$ is equivalent to G^{op}, where G is reflective in $[\mathcal{E}ns\mathcal{f}in^{op}, \mathcal{E}ns]_\times \cong \mathcal{E}ns$. Since G contains the non-trivial object 2 $(\cong G)$, we must have $G \cong \mathcal{E}ns$. Thus $(K_0\mathcal{U})^{op} \cong \mathcal{E}ns$. In other words, the category of compact totally disconnected boolean rings is equivalent to the category of complete atomic boolean algebras.

EXAMPLE 3.2. Let $\mathcal{U}_n$ be the category of compactly generated Hausdorff abelian groups A such that $na = 0$ for all $a \in A$. Then $\mathbb{Z}_n$ is injective in $F\mathcal{U}_n$ and is a strong cogenerator; that is, each $P \in F\mathcal{U}_n$ has an equaliser presentation $P \to \mathbb{Z}_n^m \to \mathbb{Z}_n^p$. Thus, if we take $P = \left\{1, \mathbb{Z}_n^2, \ldots, \mathbb{Z}_n^m, \ldots\right\} \subset F\mathcal{U}_n$, we obtain $\overline{P} = K_0\mathcal{U}_n$.

EXAMPLE 3.3. Let $\mathcal{U}$ be the category of compactly generated Hausdorff topological vector spaces over a finite discrete field Q. Let $P = \{1, Q, Q^2, \ldots, Q^n, \ldots\}$; then $\overline{P} = K_0\mathcal{U}$.

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