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BOUNDARY POINTS OF THE NUMERICAL RANGE OF AN OPERATOR

BY

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1. **Introduction.** The purpose of this note is to investigate boundary points of the numerical range of an operator in terms of inner and outer center points. Some applications on commutators are given.

Throughout this note, an operator will always mean a bounded linear operator on a Hilbert space X. Let us first give some familiar notation and terminology as follows: for an operator T, $W(T)$ is the numerical range; $\sigma(T)$ is the spectrum; $\Sigma(T)$ is the convex hull of $\sigma(T)$; $\sigma_p(T)$ is the point spectrum; $\sigma_{ap}(T)$ is the approximate point spectrum; $\sigma_r(T)$ is the residual spectrum; $\sigma_c(T)$ is the continuous spectrum; $w(T)$ is the numerical radius and $r(T)$ is the spectral radius. If M is a subset of the complex plane, ∂M is the boundary, $\bar{M}$ is the closure and conv M is the convex hull of M. We remark that $\partial M=\partial\bar{M}$ when M is convex; in particular, $\partial W(T)=\partial\bar{W}(T)$. Also the distance between a point u and M is denoted by $\text{dist}(u, M)$.

Let M be a compact convex subset of the complex plane; $u \in \partial M$ is said to be a *bare point* of M if there is a closed disc K such that $u \in \partial K$ and $M \subset K$. If M is convex, $u \in M$ and u is not in the interior of any line segment in M, then u is called an *external point* of M. Let $B(T)$ (resp. $E(T)$) denote the set of bare (resp. extremal) points of $\bar{W}(T)$. Then it is well known ([4, p. 231] and [7, Lemma 3]) that

$$E(T) = \bar{B}(T) \quad \text{and} \quad \sum(T) \subset \bar{W}(T) = \text{conv}\, E(T).$$

As is well known, $W(T)$ is convex and the relation

$$\bar{W}(uT+v) = u\bar{W}(T)+v$$

holds for all scalars u and v. By the definition of a bare point we see that if $u \in B(T)$, then one can always find a scalar u_0 such that

$$0 \neq |u-u_0| = w(T-u_0).$$

Also, if $u \in \partial W(T)$, the convexity of $W(T)$ implies that there is a scalar u_0 such that

$$0 \neq |u-u_0| = \text{dist}(u_0, W(T)) = \text{dist}(0, W(T-u_0)).$$

DEFINITION If $u \in B(T)$ and u_0 is the center of a closed disc K such that $u \in \partial K$ and $W(T) \subset K$, then u_0 is said to be an *inner center point* with respect to u, in short i.c.p. If $u \in \partial W(T)$ and u_0 is the center of a closed disc K such that $u \in \partial K$ and

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$\partial W(T) \cap K=\{u\}$, then u_0 is said to be an *outer center point* with respect to u, in short o.c.p.

Recall that an operator T is *hyponormal* if $TT^* \leq T^*T$, *normaloid* if $\|T\|=r(T)$ (i.e., $\|T\|=w(T)$), *transaloid* if $uT+v$ is normaloid for all scalar u and v (i.e., $T+v$ is normaloid for all v), *convexoid* if $\overline{W}(T)=\Sigma(T)$, and *spectraloid* if $w(T)=r(T)$.

2. **Boundary points of the numerical range.** The following theorem lists some (essentially known) criteria for an operator to be convexoid (see also Theorems 2 and 3 in [7]).

THEOREM 1. *The following statements are equivalent:*
(1) *T is convexoid;* (2) $E(T) \subset \sigma_{ap}(T)$; (3) $B(T) \subset \sigma_{ap}(T)$; (4) $E(T) \subset \sigma(T)$; (5) $B(T) \subset \sigma(T)$.

Proof. (1) $\Rightarrow$ (2) Because every $u \in E(T)$ is in $\partial\sigma(T)$, and the boundary of the spectrum of an operator is included in the approximate point spectrum [3, Problem 63]. That (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. (3) $\Rightarrow$ (1) For $E(T)=\bar{B}(T) \subset \bar{\sigma}_{ap}(T)=\sigma_{ap}(T)$, $\Sigma(T) \subset \overline{W}(T)=\operatorname{conv} E(T) \subset \Sigma(T)$ and hence the equality holds. That (5)$\Rightarrow$(1) can be argued as in the case (3) $\Rightarrow$ (1).

LEMMA 1. (1) $\sigma(T) \backslash W(T) \subset \sigma_c(T)$; (2) $\sigma(T) \cap \partial W(T) \subset \sigma_c(T) \cup \sigma_p(T)$.

Proof. (1) If $u \notin W(T)$, then $u \notin \sigma_p(T)$ since $\sigma_p(T) \subset W(T)$. Suppose that $u \in \sigma_r(T)$, then $\bar{u} \in \sigma_p(T^*) \subset W(T^*)$ and hence $u \in W(T)$, a contradiction. (2) This is well known and the proof depends on the normality of u ($u \in \sigma_p(T)$ is a *normal eigenvalue* if $\ker(T-u)=\ker(T-u)^*$ [4, p. 233]).

LEMMA 2. *If T is convexoid and $E(T) \cap \sigma_c(T)=\varnothing$, then $W(T)$ is closed.*

Proof. Since T is convexoid, $E(T) \subset \sigma(T)$ by Theorem 1. Then $E(T) \backslash W(T) \subset \sigma(T) \backslash W(T) \subset \sigma_c(T)$ by Lemma 1. But $E(T) \cap \sigma_c(T)=\varnothing$ by hypothesis; therefore $E(T) \backslash W(T)=\varnothing$, thus $E(T) \subset W(T)$ and so $\overline{W}(T)=\operatorname{conv} E(T) \subset W(T)$.

We remark that if T is an operator such that if $E(T) \cap W(T) \subset \sigma_p(T)$ and $W(T)$ is closed, then $E(T)=E(T) \cap W(T) \subset \sigma_p(T)$, therefore $E(T) \cap \sigma_c(T)=\varnothing$ trivially. A hyponormal operator T has the property that $E(T) \cap W(T) \subset \sigma_p(T)$ (see [9]). Accordingly, for such a T, $W(T)$ is closed if and only if $E(T) \cap \sigma_c(T)=\varnothing$.

THEOREM 2. *If T is transaloid and $B(T)$ is closed, then $W(T)$ is closed if and only if $E(T) \cap \sigma_c(T)=\varnothing$.*

Proof. Since a transaloid operator is convexoid [4, Satz 9], in view of Lemma 2 and the remark above, we need only show that $B(T) \cap W(T) \subset \sigma_p(T)$ when T is transaloid. But this was proved in [5, Theorem 1].

The following question was asked by S. Hildebrandt [4, p. 247]: does the relation $E(T) \cap W(T) \subset \sigma_p(T)$ hold if T is convexoid? If T is convexoid, $E(T) \cap W(T) \subset \sigma(T) \cap \partial W(T)$ by Theorem 1. For any operator T, $\sigma(T) \cap \partial W(T) \subset \sigma_c(T) \cup \sigma_p(T)$ by Lemma 1. We will answer this question by giving some necessary and

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sufficient conditions in order that $u \in \sigma_p(T)$. Before proceeding, we need some well known facts which can be easily verified, i.e., $\|(T-v)^{-1}\| \le \text{dist}(v, W(T))^{-1}$ for $v \notin \overline{W}(T)$, and $r((T-v)^{-1}) = \text{dist}(v, \sigma(T))^{-1}$ for $v \notin \sigma(T)$.

THEOREM 3. *Let T be a convexoid operator, $u=(Tx, x) \in E(T) \cap W(T)$, $\|x\|=1$, and let u_0 be an o.c.p. with respect to u. Then the following statements are equivalent:* (1) $Tx=ux$; (2) $\|(T-u_0)x\|^{-1}=r((T-u_0)^{-1})$; (3) $\|(T-u_0)x\|^{-1}=\|(T-u_0)^{-1}\|$.

Proof. First we note that $u \in \sigma(T) \cap \partial W(T)$. Also since

$$\|(T-u_0)^{-1}\| \le \text{dist}(u_0, W(T))^{-1} = \text{dist}(u_0, \sigma(T))^{-1}$$
$$= r((T-u_0)^{-1}) \le \|(T-u_0)^{-1}\|,$$

hence $r((T-u_0)^{-1})=\|(T-u_0)^{-1}\|=\text{dist}(u_0, W(T))^{-1}$. (1) $\Rightarrow$ (2) $\|(T-u_0)x\|^{-1}=|u-u_0|^{-1}=\text{dist}(u_0, W(T))^{-1}=r((T-u_0)^{-1})$. (2) $\Rightarrow$ (3) Because $r((T-u_0)^{-1})=\|(T-u_0)^{-1}\|$. (3) $\Rightarrow$ (1) For $\|(T-u_0)x\|=\text{dist}(u_0, W(T))=|u-u_0|=|((T-u_0)x, x)|$, i.e., $(T-u_0)x$ and x are proportional, i.e., $(T-u_0)x=vx$ for some scalar v, i.e., $Tx=ux$ since $(Tx, x)=u$.

We see from the proof that for an arbitrary operator T, if $\sigma(T) \cap \partial W(T) \neq \varnothing$, then the existence of a normaloid operator $(T-u_0)^{-1}$ for some scalar u_0 is guaranteed. It is well known that $\|(T-v)x\| \ge \text{dist}(v, W(T))$ for arbitrary operator T, scalar v and unit vector x [8, p. 470]. In fact, the following corollary is readily verified from the proof above and [4, Satz 2, (i)].

COROLLARY 1. *Let T be an arbitrary operator.*

(1) *If $u=(Tx, x) \in \partial W(T) \cap W(T)$ and u_0 is an o.c.p. with respect to u, then $\|(T-u_0)x\|=\text{dist}(u_0, W(T))$, iff $Tx=ux$, iff $T^*x=\bar{u}x$;*

(2) *If $u=(Tx, x) \in B(T) \cap W(T)$ and u_0 is an i.c.p. with respect to u, then $\|(T-u_0)x\|=w(T-u_0)$, iff $Tx=ux$, iff $T^*x=\bar{u}x$.*

3. **Some applications.** We will give some applications on commutators to demonstrate that the numerical range of an operator is a useful tool in operator theory.

Recall that if T and S are operators, the operator $TS-ST$ is called a commutator. It is well known that 0 is not necessarily in $\overline{W}(TS-ST)$, but $0 \in W(TS-ST)$ if $\dim X < \infty$.

THEOREM 4. *Let T and S be arbitrary operators.*

(1) *If $\partial W(T) \cap \sigma_p(T) \neq \varnothing$, then $0 \in W(TS-ST)$, and $0 \in W((T-u_0)^{-1}S-S(T-u_0)^{-1})$ for some scalar u_0;*

(2) *If $\partial W(T) \cap \sigma(T) \neq \varnothing$, then $0 \in \overline{W}(TS-ST)$, and $0 \in \overline{W}((T-u_0)^{-1}S-S(T-u_0)^{-1})$ for some scalar u_0.*

Proof. (1) If $u \in \partial W(T) \cap \sigma_p(T)$, then u is normal [4, Satz 2]. Choose a unit vector x with $Tx=ux$ (hence $T^*x=\bar{u}x$), then

$$|((TS-ST)x, x)| = |(Sx, \bar{u}x)-(ux, S^*x)| = 0.$$

Hence $0 \in W(TS-ST)$. Next, let u_0 be an o.c.p. with respect to u. Then, $\|(T-u_0)^{-1}\| = \text{dist}(u_0, W(T))^{-1}$ as in the proof of Theorem 3, and hence $\|(T-u_0)^{-1}\|=|u-u_0|^{-1}$. But it is easily seen that $(T-u_0)^{-1}x=(u-u_0)^{-1}x$. Thus, $(u-u_0)^{-1} \in \sigma_p((T-u_0)^{-1})$, and hence $(T^*-\bar{u}_0)^{-1}x=(\bar{u}-\bar{u}_0)^{-1}x$ by [10, Lemma 1]. Using the same technique as we have just done, we may omit the remainder of the proof. (2) We may apply [10, Theorem 3], after some routine arguments, to obtain the first desired result. However, we will show that this is indeed a consequence of item (1). Since $\sigma(T) \subset \overline{W}(T)$, we have $\partial W(T) \cap \sigma(T) \subset \partial\sigma(T) \subset \sigma_{ap}(T)$. Then (2) follows at once from (1), after a suitable change of Hilbert space. For if $T \mapsto T^0$ is the representation discussed in [1], [2], then $\overline{W}(T)=W(T^0)$ and $\sigma_{ap}(T)=\sigma_p(T^0)$ for all operators T.

In conclusion we shall present some remarks. A vector $x \in X$ is said to be *maximal* for an operator T if $\|Tx\|=\|T\| \, \|x\|$ [10, p. 335]. If $\sigma_p(T) \cap \partial W(T) \neq \varnothing$ for any operator T, then there exists a scalar u_0 such that $(T-u_0)^{-1}$ has a maximal vector (see the proof of (1) in Theorem 4). It is clear that if $u \in B(T)$ and u_0 is an i.c.p. with respect to u, then $T-u_0$ is normaloid if and only if $|u-u_0|=\|T-u_0\|$. From this we have the following: if T is transaloid and if $B(T) \cap W(T) \neq \varnothing$ (in particular, if $W(T)$ is closed), then there exists a scalar u_0 such that $T-u_0$ has a maximal vector (for if $u=(Tx, x) \in B(T) \cap W(T)$ and u_0 is an i.c.p. with respect to u, then $\|T-u_0\|=|u-u_0|=|((T-u_0)x, x)| \leq \|(T-u_0)x\| \leq \|T-u_0\|$). Also, if T is transaloid and if $B(T) \cap W(T) \neq \varnothing$ (in particular, if $W(T)$ is closed), then T has a proper invariant subspace, because the assumptions imply that a normal eigenvalue exists (see the proof of Theorem 2 and [4, Satz 2, (i)]).

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