QUADRUPLE INTEGRAL EQUATIONS AND OPERATORS OF FRACTIONAL INTEGRATION

by M. IFTIKHAR AHMAD

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Cooke [1] modified a technique used by Erdélyi and Sneddon [2] to solve triple integral equations of a certain type. In this paper, we extend this method to solve the quadruple integral equations

$$L_1(\alpha, \rho) \equiv \int_0^\infty \xi^{-2\alpha}\psi(\xi)J_\nu(\rho\xi)\,d\xi = F_1(\rho) \qquad (0 < \rho < a), \tag{1a}$$

$$L_2(\beta, \rho) \equiv \int_0^\infty \xi^{-2\beta}\psi(\xi)J_\nu(\rho\xi)\,d\xi = G_2(\rho) \qquad (a < \rho < b), \tag{1b}$$

$$L_3(\alpha, \rho) \equiv \int_0^\infty \xi^{-2\alpha}\psi(\xi)J_\nu(\rho\xi)\,d\xi = F_3(\rho) \qquad (b < \rho < c), \tag{1c}$$

$$L_4(\beta, \rho) \equiv \int_0^\infty \xi^{-2\beta}\psi(\xi)J_\nu(\rho\xi)\,d\xi = G_4(\rho) \qquad (\rho > c), \tag{1d}$$

where F_1, G_2, F_3 and G_4 are prescribed functions of ρ and $\psi(\xi)$ is to be determined. With no loss of generality we shall assume that $G_2(\rho) \equiv 0$, $G_4(\rho) \equiv 0$.

1. Operators. We recall here a few definitions and properties of the operators used in solving the integral equations (1). Cooke [1] has defined† the operators ${}_a^bI_{\eta,\alpha}$ and ${}_c^dK_{\eta,\alpha}$ by the formulae

$${}_a^bI_{\eta,\alpha}f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_a^b (x^2-u^2)^{\alpha-1}u^{2\eta+1}f(u)\,du \qquad (\alpha > 0), \tag{2}$$

$${}_a^bI_{\eta,\alpha}f(x) = \frac{x^{-2\eta-2\alpha-1}}{\Gamma(1+\alpha)}\frac{d}{dx}\int_a^b (x^2-u^2)^{\alpha}u^{2\eta+1}f(u)\,du \qquad (-1 < \alpha < 0), \tag{3}$$

$${}_c^dK_{\eta,\alpha}f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)}\int_c^d (u^2-x^2)^{\alpha-1}u^{-2\alpha-2\eta+1}f(u)\,du \qquad (\alpha > 0), \tag{4}$$

$${}_c^dK_{\eta,\alpha}f(x) = -\frac{x^{2\eta-1}}{\Gamma(1+\alpha)}\frac{d}{dx}\int_c^d (u^2-x^2)^{\alpha}u^{-2\alpha-2\eta+1}f(u)\,du \qquad (-1 < \alpha < 0). \tag{5}$$

For $\alpha = 0$, these are just the identity operators. Note that with these definitions ${}_0^xI_{\eta,\alpha}$ and ${}_x^\infty K_{\eta,\alpha}$ are simply the Erdélyi–Kober operators [5]. In these cases we will drop the indices on the left and write them as $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$. We also observe that (2), (3) make sense if $b < x$ and similarly (4), (5) are defined only if $c > x$.

† Cooke uses $\binom{b}{a}I_{\eta,\alpha}$, $\binom{d}{c}K_{\eta,\alpha}$, but our notation seems convenient.

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The modified operator $S_{\eta,\alpha}$ of the Hankel transforms is defined by

$$S_{\eta,\alpha}f(x) = 2^{\alpha}x^{-\alpha}\int_0^{\infty}\xi^{1-\alpha}J_{2\eta+\alpha}(x\xi)f(\xi)\,d\xi. \tag{6}$$

Sneddon [4] has shown the following relations between the Erdélyi–Kober and Hankel operators.

$$I_{\eta+\alpha,\beta}S_{\eta,\alpha} = S_{\eta,\alpha+\beta}, \tag{7}$$

$$K_{\eta,\alpha}S_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}, \tag{8}$$

$$S_{\eta+\alpha,\beta}S_{\eta,\alpha} = I_{\eta,\alpha+\beta}, \tag{9}$$

$$S_{\eta,\alpha}S_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}, \tag{10}$$

provided that the conditions for the existence of the various operations are satisfied. The inverse operators are

$${}^{b}_{a}I^{-1}_{\eta,\alpha} = {}^{b}_{a}I_{\eta+\alpha,-\alpha}, \tag{11}$$

$${}^{d}_{c}K^{-1}_{\eta,\alpha} = {}^{d}_{c}K_{\eta+\alpha,-\alpha}, \tag{12}$$

$$S^{-1}_{\eta,\alpha} = S_{\eta+\alpha,-\alpha}. \tag{13}$$

We require two lemmas also given by Cooke [1], which define the product of pairs of operators.

Lemma A. *Let* ${}^{b}_{a}I_{\eta,\alpha}$, ${}^{x}_{d}I^{-1}_{\eta,\alpha}$ *be operators as defined in* (2), (3) *and* (11). *Then*

$${}^{x}_{d}I^{-1}_{\eta,\alpha}\,{}^{b}_{a}I_{\eta,\alpha}f(x) = \frac{2\sin\pi\alpha}{\pi}x^{-2\eta}(x^2-d^2)^{-\alpha}\int_a^b\frac{(d^2-t^2)^{\alpha}t^{2\eta+1}f(t)}{x^2-t^2}\,dt \tag{14}$$

provided that $x > d \geqq b > a$.

Lemma B. *Let* ${}^{b}_{a}K_{\eta,\alpha}$, ${}^{d}_{x}K^{-1}_{\eta,\alpha}$ *be operators as defined in* (4), (5), *and* (12). *Then*

$${}^{d}_{x}K^{-1}_{\eta,\alpha}\,{}^{b}_{a}K_{\eta,\alpha}f(x) = \frac{2\sin\pi\alpha}{\pi}x^{2\eta+2\alpha}(d^2-x^2)^{-\alpha}\int_a^b\frac{(t^2-d^2)^{\alpha}t^{-2\alpha-2\eta+1}f(t)}{t^2-x^2}\,dt, \tag{15}$$

provided that $x < d \leqq a < b$.

2. Solution of the equations (1). We transform the equations (1) into a form to which the operational theory is applicable by substituting

$$\psi(\xi) = \xi A(\xi), \quad f_i(\rho) = (2/\rho)^2F_i(\rho); \tag{16}$$

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by means of this we get

$$L_1(\alpha, \rho) \equiv 2^{2\alpha}\rho^{-2\alpha}\int_0^\infty \xi^{1-2\alpha}A(\xi)J_\nu(\rho\xi)\,d\xi = f_1(\rho) \qquad (0<\rho<a), \tag{17a}$$

$$L_2(\beta, \rho) \equiv 2^{2\beta}\rho^{-2\beta}\int_0^\infty \xi^{1-2\beta}A(\xi)J_\nu(\rho\xi)\,d\xi = 0 \qquad (a<\rho<b), \tag{17b}$$

$$L_3(\alpha, \rho) \equiv 2^{2\alpha}\rho^{-2\alpha}\int_0^\infty \xi^{1-2\alpha}A(\xi)J_\nu(\rho\xi)\,d\xi = f_3(\rho) \qquad (b<\rho<c), \tag{17c}$$

$$L_4(\beta, \rho) \equiv 2^{2\beta}\rho^{-2\beta}\int_0^\infty \xi^{1-2\beta}A(\xi)J_\nu(\rho\xi)\,d\xi = 0 \qquad (\rho>c). \tag{17d}$$

Let I_1 denote the interval $(0, a)$, I_2 the interval (a, b), I_3 the interval (b, c) and I_4 the interval (c, ∞). For a function f in $L_2(0, \infty)$ we shall write $f_1+f_2+f_3+f_4$, where

$$f_i = f \quad \text{on} \quad I_i \quad \text{and} \quad = 0 \quad \text{on} \quad I_j \qquad (i, j = 1, 2, 3, 4;\ i \neq j)$$

and similarly for g. Using the S-operator defined in (6), we see that the integral equations (17) reduce to the form

$$S_{\frac{1}{2}\nu-\alpha,\, 2\alpha}A(\rho) = f(\rho), \tag{18}$$

$$S_{\frac{1}{2}\nu-\beta,\, 2\beta}A(\rho) = g(\rho). \tag{19}$$

Here f_1 and f_3 are prescribed, $g_2 = 0 = g_4$ but g_1, f_2, g_3 and f_4 are to be determined. Let us take as trial solution

$$A(\rho) = S_{\frac{1}{2}\nu+\beta,\, -\alpha-\beta}\, l(\rho). \tag{20}$$

Substituting this value of $A(\rho)$ in (18), (19) and using formulas (9), (10) we have

$$f = I_{\frac{1}{2}\nu+\beta,\, \alpha-\beta}\, l, \tag{21}$$

$$g = K_{\frac{1}{2}\nu-\beta,\, \beta-\alpha}\, l. \tag{22}$$

Also, we have

$$l = I^{-1}_{\frac{1}{2}\nu+\beta,\, \alpha-\beta} f \tag{23}$$

$$= K^{-1}_{\frac{1}{2}\nu-\beta,\, \beta-\alpha}\, g. \tag{24}$$

We proceed to determine l. The subscripts on all the operators will be dropped for brevity sake. All I's will be supposed to have subscripts $\frac{1}{2}\nu+\beta$, $\alpha-\beta$ understood and all K's to have $\frac{1}{2}\nu-\beta$, $\beta-\alpha$.

Evaluating (23) on I_1, we get

$$l_1 = {}_0^\rho I^{-1} f_1. \tag{25}$$

Taking (24) on I_4, we have

$$l_4 = {}_\rho^\infty K^{-1} g_4 = 0. \tag{26}$$

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Evaluate (22) on I_2; then

$$ {}_{\rho}^{b}Kl_2 + {}_{b}^{c}Kl_3 + {}_{c}^{\infty}Kl_4 = 0, $$

which gives

$$ l_2 = -{}_{\rho}^{b}K^{-1}\,{}_{b}^{c}Kl_3. \tag{27} $$

Applying Lemma B, we have

$$ l_2(\rho) = -\frac{2\sin\pi(\beta-\alpha)}{\pi}\rho^{\nu-2\alpha}(b^2-\rho^2)^{\alpha-\beta}\int_b^c \frac{(t^2-b^2)^{\beta-\alpha}t^{-\nu+2\alpha+1}l_3(t)}{t^2-\rho^2}\,dt. \tag{28} $$

Finally, evaluating (21) on I_3, we have

$$ l_3 = {}_{b}^{\rho}I^{-1}f_3 - {}_{b}^{\rho}I^{-1}\,{}_{0}^{a}Il_1 - {}_{b}^{\rho}I^{-1}\,{}_{a}^{b}Il_2. \tag{29} $$

Since f_3 and l_1 are known functions, the function

$$ d(\rho) = {}_{b}^{\rho}I^{-1}f_3(\rho) - {}_{b}^{\rho}I^{-1}\,{}_{0}^{a}Il_1 \tag{30} $$

is known. Applying Lemma A to the last term on the right-hand side of (29) and substituting (28), (30) in that equation, we obtain

$$ \begin{aligned} l_3(\rho) = d(\rho) &+ \frac{2\sin\pi(\alpha-\beta)}{\pi}\rho^{-\nu-2\beta}(\rho^2-b^2)^{\beta-\alpha} \\ &\times \int_a^b (b^2-y^2)^{\alpha-\beta}y^{\nu+2\beta+1}\left\{\frac{2\sin\pi(\beta-\alpha)}{\pi}y^{\nu-2\alpha}(b^2-y^2)^{\alpha-\beta}\right. \\ &\times \left.\int_b^c \frac{(t^2-b^2)^{\beta-\alpha}t^{-\nu+2\alpha+1}l_3(t)}{t^2-y^2}\,dt\right\}\frac{1}{\rho^2-y^2}\,dy. \end{aligned} \tag{31} $$

Inverting the order of integration, we get

$$ \begin{aligned} l_3(\rho) = d(\rho) &- \frac{4\sin^2\pi(\alpha-\beta)}{\pi^2}\int_b^c \left\{\rho^{-\nu-2\beta}(\rho^2-b^2)^{\beta-\alpha}\right. \\ &\times (t^2-b^2)^{\beta-\alpha}t^{-\nu+2\alpha+1}\int_a^b (b^2-y^2)^{2(\alpha-\beta)}y^{2\nu-2\alpha+2\beta+1} \\ &\times \left.\frac{1}{(t^2-y^2)(\rho^2-y^2)}\,dy\right\} l_3(t)\,dt. \end{aligned} \tag{32} $$

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Putting $4\pi^{-2}\sin^2\pi(\alpha-\beta) = -\lambda$, and the expression within the curly brackets equal to $K(\rho, t)$, we obtain

$$l_3(\rho) = d(\rho)+\lambda\int_b^c K(\rho, t)l_3(t)\,dt, \tag{33}$$

which is a Fredholm's integral equation of the second kind and can be solved by known methods. The equations (25), (26), (28) and (33) completely determine l and our problem is formally solved.

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Department of Mathematics
University of the Punjab (New Campus)
Lahore, Punjab
Pakistan