Ergod. Th. & Dynam. Sys. (1982), **2**, 491–512
Printed in Great Britain

Ergodic behaviour of Sullivan's geometric measure on a geometrically finite hyperbolic manifold

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(*Received* 1 *September* 1981 *and revised* 24 *September* 1982)

Abstract. Sullivan's geometric measure on a geometrically finite hyperbolic manifold is shown to satisfy a mean ergodic theorem on horospheres and through this that the geodesic flow is Bernoulli.

Sullivan [3] has shown that on the limit set $\Lambda(\Gamma)$ of a discrete subgroup of hyperbolic space $\mathscr{H}^n$ there is a unique 'geometric measure' μ. Sullivan has investigated much of the geometric nature of this measure. Of most importance to us here, μ gives rise to a natural Borel probability measure m_μ on $T(M)$, the unit tangent bundle of $M = \mathscr{H}^n/\Gamma$. This measure is not only invariant for the geodesic flow $\{g_t\}_{t\in\mathbb{R}}$ on $T(M)$, but Sullivan shows that if Γ is geometrically finite i.e. has a finite-sided fundamental domain, g_t is ergodic for m_μ. What we will demonstrate here is that a mean ergodic theorem holds for m_μ on horospheres (theorem 17) and that the geodesic flow is in fact a Bernoulli flow in this geometrically finite case.

Just as in the case of M compact, where m_μ is the Lebesgue measure, these two facts are intimately related (see [1]). The expanding and contracting horospheres form stable and unstable foliations for g_t, and so the behaviour of the measure on these foliations governs the dynamics of g_t. Thus our argument follows the basic format of the proof that a weakly mixing Anosov flow is Bernoulli. As the measure on foliations is not smooth and because, even though Sullivan gives us much information, we have only weak information about the measure, we shall work carefully to carry out the standard arguments.

We begin by describing μ, and hence m_μ. For any $\gamma \in \Gamma$, μ transforms by the rule $\gamma^*(\mu) = |\gamma'|^d \mu$ where d is the Hausdorf dimension of $\Lambda(\Gamma)$. Up to normalization Sullivan shows this makes μ unique.

For $\xi \in \partial(\mathscr{H}^n)$ and $x \in \mathscr{H}^n$ let $H = H(\xi, x)$ be the horosphere passing through x and ξ. Let P_H be the projection of $\partial(\mathscr{H}^n)$ onto H.

Let $\mu_H = |P'_H|^d P^*_H(\mu)$, where H is geometrically $\mathbb{R}^{n-1}$. This projects μ onto each horosphere with R.N. derivative $|P'_H|^d$. $T(\mathscr{H}^n)$ is foliated in three ways, first by the geodesic flow lines $g(x)$, $x \in T(\mathscr{H}^n)$, second by expanding (unstable) horospheres

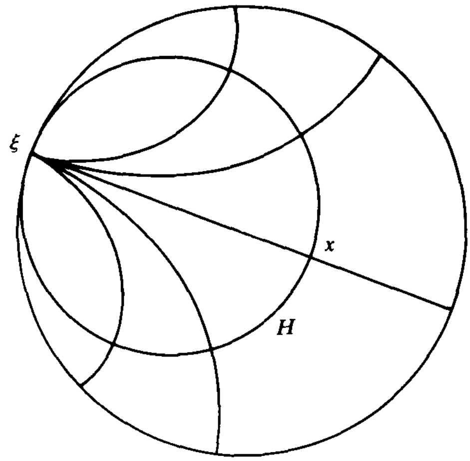

FIGURE 1

$H_u(x)$, and third by contracting horospheres, $H_s(x)$. It is important to note that g and H_u commute, (as do g and H_s) in the sense that all the H_u-leaves intersecting a given g-leaf are the same as all g leaves intersecting a given H_u-leaf. On the other hand H_u and H_s do not commute and in fact any set made of both full H_u and H_s leaves is either $\varnothing$ or all of $T(M)$.

For $x \in T(\mathscr{H}^n)$ there are two points $g_\infty(x)$ and $g_{-\infty}(x)$ in $S^{n-1} = \partial(\mathscr{H}^n)$, the forward and backward limits of the geodesic through x. $H_u(x)$ is the sphere through x based at $g_{-\infty}(x)$ and $H_s(u)$, the sphere based at $g_\infty(x)$. Sullivan defines m_μ from μ differentially, for $x \in \mathscr{S} = \{x \mid g_\infty(x), g_{-\infty}(x) \in \Lambda(\Gamma)\}$.

$$dm_\mu = \frac{d\mu_{H_u(x)}\, d\mu_{H_s(x)}\, dg}{|g_{-\infty}(x) - g_\infty(x)|^{2d}}.$$

It will be of value for us to look closely at this expression. First, identify M with a choice of fundamental domain D, and select an origin $x \in T(D)$. Any point $y \in T(D)$ can be reached from x by a unique series of movements, first on $g(x)$ to a point $x_1(y)$, then on $H_s(x_1(y))$ to a point $x_2(y)$, and lastly $y \in H_u(x_2(y))$.

For any continuous f and choice of origin $x \in \mathscr{S}$,

$$\int f(y)\, dm_\mu(x) = \int_{g(x)} \int_{H_s(x_1)} \int_{H_u(x_2)} \frac{f(y)}{|g_{-\infty}(x) - g_\infty(x)|^{2d}}\, d\mu_{H_u(x_2)}\, d\mu_{H_s(x_1)}\, dg.$$

As written this depends on our choice of origin x and on the order of integration. As

$$g_t^*(\mu_{H_s(x)}) = e^{\delta t}\mu_{H_s(g_{-t}(x))} \quad \text{and} \quad g_t^*(\mu_{H_u(x)}) = e^{-\delta t}\mu_{H_u(g_{-t}(x))},$$

$$\int f(y)\, dm_\mu(x) = \int f(y)\, dm_\mu(g_t(x))$$

for all t. Further, for any $\gamma \in \Gamma$, and $y_1, y_2 \in \partial(\mathscr{H}^n)$, as

$$|\gamma(y_1) - \gamma(y_2)|^2 = |\gamma'(y_1)|\,|\gamma'(y_2)|\,|y_1 - y_2|^2,$$

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for any x' with $\gamma(g(x)) = g(x')$ and continuous f that

$$\int f\,dm_\mu(x) = \int f\,dm_\mu(x'),$$

i.e. m_μ is Γ as well as g_t invariant. This certainly says $m_\mu(x)$ is well defined on $T(M)$. We get more though. For any $x' \in \mathcal{S}$, we can select a sequence $\gamma_i \in \Gamma$ with

$$\gamma_i(g_{-\infty}(x)) \xrightarrow{i} g_{-\infty}(x') \quad \text{and} \quad \gamma_i(g_\infty(x)) \xrightarrow{i} g_\infty(x').$$

Hence there is a sequence of points

$$x_i = \gamma_i(g_{t_i}(x)) \xrightarrow{i} x, \quad \text{for all } x_i \in \mathcal{S}.$$

Certainly

$$\int f\,dm_\mu(x_i) = \int f\,dm_\mu(x),$$

but further, $\mu_{H_u(x_i)}$, $\mu_{H_s(x_i)}$ and $dg(x_i)$ all converge weakly to $\mu_{H_u(x')}$, $\mu_{H_s(x')}$ and $dg(x')$. Thus dm_μ is independent of choice of origin.

Now to see that it is also independent of the order of integration. Let $B_u(x, r)$ be the Euclidean ball of radius r about x in $H_u(x)$, and for any set A, $B_u(A, r) = \bigcup_{x\in A} B_u(x, r)$. Similarly define $B_s(x, r)$ and set

$$G_r(x) = \bigcup_{t\in(-r,r)} g_t(x), \quad \text{and} \quad G_r(A) = \bigcup_{x\in A} G_r(x).$$

A 'cell'

$$C_{u,x}(r_1, r_2, r_3) = G_{r_3}\Big(\bigcup_{y\in B_u(x,r_2)} P_{H_s(y)} P^{-1}_{H_s(x)}(B_s(x, r_1))\Big)$$

and similarly

$$C_{s,x}(r_1, r_2, r_3) = G_{r_3}\Big(\bigcup_{y\in B_s(x,r_2)} P_{H_u(y)} P^{-1}_{H_u(x)}(B_u(x, r_1))\Big).$$

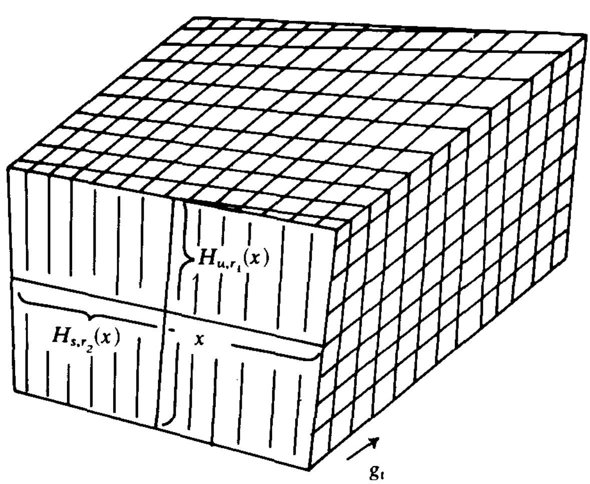

FIGURE 2

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Now certainly for r_1, r_2, r_3 small enough that $C_{u,x}(r_1, r_2, r_3)$ has a homeomorphic lift to $T(\mathscr{H}^n)$,

$$m_\mu(C_{u,x}(r_1, r_2, r_3)) = 2r_3 \int_{B_u(x,r_2)} \mu_{H_s(y)}(P_{H_s(y)}P^{-1}_{H_s(x)}(B_s(x, r_1)))\, d\mu_{H_s(x)}$$

$$= 2r_3\mu_{H_s(x)}(B_s(x, r_1)) \int_{B_u(x,r_2)} |(P_{H_s(y)} \cdot P^{-1}_{H_s(x)})'|\, d\mu_{H_u(x)}.$$

For fixed r_1, $|(P_{H_s(y)} \cdot P^{-1}_{H_s(x)})'|$ becomes uniformly close to 1 as $r_2 \to 0$. Thus

$$m_\mu(C_{u,x}(r_1, r_2, r_3)) = 2r_3\mu_{H_s(x)}(B_s(x, r_1))\mu_{H_u(x)}(B_u(x, r_2)) + 0(m_\mu(C_{u,x}(r_1, r_2, r_3))).$$

We now will define m_μ in terms of open covers by cells and show that, in these terms, order of integration does not matter. Let

$$V_u(x, r) = \mu_{H_u(x)}(B_u(x, r)) \quad \text{and} \quad V_s(x, r) = \mu_{H_s(x)}(B_s(x, r)).$$

LEMMA 1. *Both $V_u(x, r)$ and $V_s(x, r)$, for fixed x, are continuous functions of r.*

Proof. Suppose not. Then on $\Lambda(\Gamma)$ there must exist countably infinitely many spheres of dimension less than $n-1$ each of positive μ-measure. Take those of minimal dimension. Two such intersect in, at most, a sphere of lower dimension, hence of measure zero. Thus these form a countable collection of spheres which are within μ-measure zero of disjoint. Γ permutes these spheres. As μ is non-atomic, their dimension is not zero. If Γ acting on these spheres has an infinite cycle, it has a dissipative part on $\Lambda(\Gamma)$, which conflicts with minimality of Γ acting on $\Lambda(\Gamma)$.

If Γ has only finite cycles, it is not ergodic, again a conflict. □

Let $A_u(r, K)$ be that set of $x \in \mathscr{S}$ with

$$V_u(x, 5r) < KV_u(x, r).$$

Similarly define $A_s(r, K)$. Clearly as K increases $A_u(r, K)$ increases to all of $\mathscr{S}$. Further, $A_u(r, K)$ is open and

$$g_{-t}(A_u(r, K)) = A_u(e^{-t}r, K).$$

Let $A'_u(r, \varepsilon, \delta)$ be the set of all x with

$$V_u(x, r(1+\delta)) - V_u(x, r(1-\delta)) < \varepsilon V_u(x, r(1+\delta)).$$

Similarly define $A'_s(r, \varepsilon, \delta)$. Here also, for any ε, as $\delta \to 0$, $A'_u(r, \varepsilon, \delta)$ increases to all of $\mathscr{S}$, A'_u is open and

$$g_{-t}(A_u(r, \varepsilon, \delta)) = A_u(e^{-t}r, \varepsilon, \delta).$$

LEMMA 2. *For some $K_0 > 0$, for any $\varepsilon > 0$ there is a $\delta(\varepsilon)$ so that for all $x \in \mathscr{S}$ there are sequences $r_i(x) \to 0$, $\bar{r}_i(x) \to 0$ with*

$$x \in A_u(r_i, K_0) \cap A_s(\bar{r}_i, K_0) \cap A'_u(r_i, \varepsilon, \delta) \cap A'_s(\bar{r}_i, \varepsilon, \delta).$$

Proof. Pick K_0 so large $A_u(1, K_0)$ is non-empty. Now choose δ so small

$$\mathcal{O} = A_u(1, K_0) \cap A'_u(1, \varepsilon, \delta) \neq \varnothing.$$

As Γ acts minimally on $\Lambda(\Gamma)$, for all $x \in \mathscr{S}$, there are infinitely many $t_i(x)$,

$$x \in g_{-t_i}(\mathcal{O}) = A_u(e^{-t_i}, K_0) \cap A'_u(e^{-t_i}, \varepsilon, \delta).$$

The other half is symmetric to this. □

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Fix this K_0. Now for any ε and $\delta<\delta(\varepsilon)$, call a cell $C_{u,x}(r_1, r_2, r_3)$ with

$$x \in A_u(r_1, K_0) \cap A'_u(r_1, \varepsilon, \delta) \cap A_s(r_2, K_0) \cap A'_s(r_2, \varepsilon, \delta)$$

an 'ε, δ-good cell'. Now for any continuous function f with compact support define

$$\int f\, dm^*_\mu = \lim_{\varepsilon\to 0}\left(\lim_{\operatorname{dia}(C_j)\to 0}\left(\sup_{\substack{\text{disjoint}\\ \text{collections}\\ \text{of } \varepsilon,\, \delta\text{-good cells}\\ C_j,\ \delta \le \delta(\varepsilon)}}\left(\sum_j f(x_j) m_\mu(C_j)\right)\right)\right).$$

LEMMA 3. $m^*_\mu = m_\mu$.

Proof. Clearly $m^*_\mu \le m_\mu$. For any $\varepsilon>0$, we have $\varepsilon, \delta(\varepsilon)$ good cells of arbitrarily small diameter about every point. By the Vitali covering lemma [**6**], there is a disjoint covering of supp (f) to within m_μ-measure zero by such cells. Hence

$$m_\mu(f) \le m^*_\mu(f). \qquad \square$$

For a fixed $\delta>0$, once dia $(C_{u,x}(r_1, r_2, r_3))$ is sufficiently small,

$$C_{s,x}(r_2(1-\delta), r_1(1-\delta), r_3(1-\delta)) \subset C_{u,x}(r_1, r_2, r_3).$$

If we now define

$$d\bar{m}_\mu = \frac{d\mu_{H_s(x)}\, d\mu_{H_u(x)}\, dg}{|g_{-\infty}(x) - g_\infty(x)|^{2d}},$$

i.e. change the order of integration, we can follow all the above reasoning for m_μ up to constructing

$$\int f\, d\bar{m}^*_\mu = \lim_{\varepsilon\to 0}\left(\lim_{\operatorname{dia} C_j\to 0}\left(\sup_{\substack{\text{disjoint}\\ \varepsilon,\, \delta\text{-good cells}\\ C_j}}\left(\sum_j f(x_j) \bar{m}_\mu(\bar{C}_j)\right)\right)\right)$$

and conclude

$$\int f\, d\bar{m}_\mu = \int f\, d\bar{m}^*_\mu.$$

But now once dia (C_j) are small enough, set

$$\bar{C}_j = C_{u,x_j}\left(r_{j,2}\left(1-\frac{\delta}{2}\right), r_{j,1}\left(1-\frac{\delta}{2}\right), r_{j,3}\left(1-\frac{\delta}{2}\right)\right)$$

where $C_j = C_{s,x_j}(r_{j,1}, r_{j,2}, r_{j,3})$. The $\bar{C}_j$ form a disjoint set of $\varepsilon, \delta/2$-good cells for $\bar{m}^*_\mu$. As

$$\begin{aligned} m_\mu(C_j) &= 2r_{j,3}\mu_{H_s(x_j)}(B_s(x, r_{j,1}))\mu_{H_u(x_j)}(B(x, r_{j,2})) + 0(m_\mu(C_j)) \\ &= \bar{m}_\mu(\bar{C}_j) + 0(m_\mu(C_j)) - 3\varepsilon m_\mu(C_j). \end{aligned}$$

Thus $m^*_\mu \le \bar{m}^*_\mu$. The other inequality follows symmetrically. Hence

LEMMA 4. *dm_μ can be computed by integration in any order.* $\square$

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Our first, and most basic step, is to prove that g_t is weakly mixing. Such arguments normally require absolute continuity of the foliation measures as we move along the foliations. We do not have this. Instead we rely on the definition of the foliation measures μ_H as geodesic projections of μ with a smooth R.N. derivative $|P'_H|^d$. We will not use the ergodicity of g_t and hence will obtain an alternative proof of Sullivan's result. Because of the importance of this step, we will be quite careful in the proof.

THEOREM 5. *$\{g_t\}_{t\in\mathbb{R}}$ acting on $(T(M), B, m_\mu)$ is weakly mixing.*

Proof. We will show that for all $\alpha \neq 0$, and all uniformly continuous $f \in L^1(T(M))$,

$$\bar{f}(x) = \lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=0}^{n-1} g_{i\alpha}(f(x))$$

is m_μ-almost everywhere constant, hence g_α is ergodic for all α and g_t is weakly mixing. □

We know $\bar{f} \in L^1(m_\mu)$,

$$\bar{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g_{i\alpha}(f(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g_{-i\alpha}(f(x))$$

almost everywhere by the Birkhoff ergodic theorem. A number of facts are now evident from the uniform continuity of f:

(i) $\bar{f}$ is constant on all stable horospheres;

(ii) $\bar{f}$ is constant μ_{H_u}-almost everywhere on m_μ-almost every unstable horosphere; and

(iii) $\varphi_x(t) = \bar{f}(g_t(x))$ is uniformly continuous in t, uniformly for all x, and α-periodic.

These imply:

(iv) $\bar{f}$ is constant on all stable horospheres, and on m_μ-almost every unstable horosphere H_u, $\bar{f}$ is μ_{H_u}-almost everywhere constant, and now using (iii);

(v) for μ-almost everywhere $\xi \in \bigwedge(\Gamma)$, $\bar{f}$ is μ_{H_u}-almost everywhere on *all* unstable horospheres based at ξ, and the set of all points x where $\bar{f}(x)$ differs from its $\mu_{H_u(x)}$-almost everywhere constant value is g_t-invariant.

Lift $\bar{f}$ to $\bar{F}$ on $T(\mathscr{H}^n)$. Let $A = \{\xi \in \bigwedge(\Gamma) |$ for every unstable horosphere based at ξ, $\bar{f}$ is μ_{H_u}-almost everywhere constant$\}$. We know $\mu(A) = 1$.

Let $\xi_1, \xi_2 \in A$, and pick a horosphere $H_{u,1}$ based at ξ_1, with $\bar{F}(H_{u,1}) = \lambda$-almost everywhere.

For any $\xi \in \bigwedge(\Gamma)$ there is a unique stable horosphere $H_{s,3}(\xi)$ based at ξ tangent to $H_{u,1}$.

There is in turn a unique $H_{u,2}(\xi)$ based at ξ_2 tangent to $H_{s,0}(\xi)$. Let these points of tangency be $x_1(\xi)$ and $x_2(\xi)$.

Fix a $\xi_3 \in \bigwedge(\Gamma)$ and let $H_{u,2} = H_{u,2}(\xi_3)$, with $\bar{F}(H_{u,2}) = \lambda'$-almost everywhere. We now demonstrate $\lambda = \lambda'$. Let $N \subset \partial(\mathscr{H}^n)$ be a small neighbourhood of ξ_3. We know $\mu(N) > 0$. Project N along geodesics to $N_{u,1} \subset H_{u,1}$, a neighbourhood of $x_1(\xi_3)$ in $H_{u,1}$. Hence $\mu_{H_{u,1}}(N_{u,1}) > 0$, and so for a subset of $x_1(\xi) \in N_{u,1}$ of full measure,

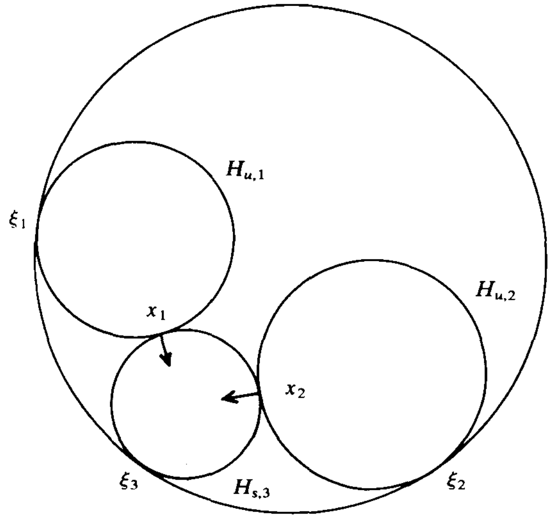

FIGURE 3

$\bar{F}(x_1(\xi))=\lambda$. By (i) $\bar{F}(H_{s,3}(\xi))=\lambda$ for μ-almost everywhere $\xi\in N$. Let $\overline{H_{s,3}}(\xi)$ be the stable horosphere based at ξ tangent to $H_{u,2}$. By the same reasoning

$$\bar{F}(\overline{H_{s,3}}(\xi))=\lambda' \quad \text{for } \mu\text{-almost everywhere } \xi\in N.$$

Now $\overline{H_{s,3}}(\xi)=g_{t(\xi)}(H_{s,3}(\xi))$ where $t(\xi)\to 0$ as $\xi\to\xi_0$.

Thus allowing the dia $(N)\to 0$ we conclude, by (iii) that $\lambda=\lambda'$ proving our claim. Similarly

$$\lambda_t=\bar{F}(g_t(H_{u,1}))=\bar{F}(g_t(H_{u,2}))$$

μ_{H_u}-almost everywhere.

The set of points $S\subset N$ with $H_{u,2}(\xi)=H_{u,2}$ is the intersection of N with an $(n-2)$-sphere, so $\mu(S)=0$. Hence in N there are points $\xi_i\to\xi_0$, $\xi_i\in\bigwedge(\Gamma)$ with $H_{u,2}(\xi_i)=g_{t_i}(H_{u,2})$, $t_i\neq 0$, $t_i\to 0$. We conclude

$$\lambda_t=\bar{F}(g_t(H_{u,1}))=\bar{F}(g_t(g_{t_i}(H_{u,2})))=\lambda_{t+t_i}.$$

Hence λ_t, which we know is uniformly continuous in t, is invariant under the shifts $t_i\neq 0$, $t_i\to 0$. Hence $\lambda_t=\lambda$ for all t and we are done. □

COROLLARY 6. *For any two sets A and $B\subset T(M)$, and any $\varepsilon>0$, there is a set of $t\subset\mathbb{R}$ of full density with*

$$m_\mu(g_t(A)\cap B)=m_\mu(A)m_\mu(B)(1\pm\varepsilon).$$

Proof. This is equivalent to weakly mixing (see [**5**]). □

The rest of our work proceeds along the following lines. We use the above corollary to prove a weak ergodic theorem for horospheres. This in turn will imply a mean ergodic theorem on horospheres which then will prove not only that g_t is weakly mixing, but that it is in fact a Bernoulli flow.

To continue we must look a bit more carefully at the geometry and measure size of cells. Some of what follows repeats some of our earlier work but is included to make the arguments here self-contained.

LEMMA 7. *For any $\varepsilon>0$ there is a $C(\varepsilon)$ so that for any $x\in T(M)$, $y\in g_tB_s(x,r_1)$ if $r_1\cdot r_2<C(\varepsilon)$ then $H_u(y)\cap C_{u,x}(\infty,r_2,\infty)$ has a connected component containing y which as a set is geometrically uniformly within $e^tr_2\varepsilon$ of $B_u(x,e^tr_2)$.*

Proof. For $r_1=1$, such a $C(\varepsilon)$ exists by smoothness and homogeneity of the foliations and projections. As

$$g_t(C_{u,x}(r_1,r_2,r_3))=C_{u,g_t}(x)(e^{-t}r_1,e^tr_2,r_3),$$

picking t so that $e^{-t}=r_1$, the result extends by conjugation by g_t. □

COROLLARY 8. *For any $\varepsilon>0$ there is a $C(\varepsilon)$ so that if*

$$y\in g_tB_s(x,r_1(1-\varepsilon))\subset G_{r_3(1-\varepsilon)}B_s(x,r_1(1-\varepsilon)),$$

then for any r_2 with $e^{r_3}r_1r_2<C(\varepsilon)$ then

$$H_u(y)\cap C_{u,x}(r_1,r_2,r_3)$$

is geometrically within $\varepsilon\, e^tr_2$ of $B_u(y,r_2\,e^t)$.

Proof. If $r_2\,e^{r_3}<\varepsilon$ and $r_1,r_3\geq 1$, by lemma 3,

$$H_u(y)\cap(C_{u,x}(r_1,r_2,r_3))=H_u(y)\cap(C_{u,x}(\infty,r_2,\infty))$$

near y. Conjugation by g_t extends the result to small r_1,r_3. □

For $r_1r_2\,e^{r_3}<C(\varepsilon)$, then, a cell looks to within ε like a geometric cube with cross-sections which are balls in the three foliations. We want such cells also to be measurably almost cubes i.e. that on both stable and unstable leaves a cell is not only geometrically close to a ball but usually measurably close as well.

Continuity (lemma 1) of $V_u(x,\cdot)$ and $V_s(x,\cdot)$ is critical in this and, as we shall see knowing more, the uniform continuity of $V_u(x,\cdot)$ in two variables would give us stronger results. It is easily seen that uniform continuity holds for $d>n-2$, but ought to be generally true.

LEMMA 9. *For any $r_1,r_3>0$, $\varepsilon>0$, for almost every x there are $r_2=r_2(x,r_1,r_3,\varepsilon)$, $\delta=\delta(x,r_1,r_3,\varepsilon)$ so that for*

$$C=C_{u,x}(r_1,r_2,r_3)$$

$$\mu_{H_s(x)}(B_s(x,r_2)\backslash B_s(x,r_2(1-\delta)))<\frac{\varepsilon}{3}\mu_{H_s(x)}(B_s(x,r_2(1-\delta))$$

and for any $y\in G_{r_3(1-\delta)}B_s(x,r_2(1-\delta))$, $y\in g_tB_s(x,r_2(1-\delta))$,

$$\mu_{H_u(y)}((H_u(y)\cap C)\Delta B_u(y,r_2\,e^t))<\frac{\varepsilon}{3}\mu_{H_u(y)}(C\cap H_u(y)).$$

Proof. We know for some $\bar\delta>0$, there is a set $A(r_2)$ of positive measure in $T(M)$ with

$$* \qquad V_u(x,r(1+\bar\delta))-V_u(x,r(1-\bar\delta))<\frac{\varepsilon}{6}V_u(x,r(1-\bar\delta))$$

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for some $r < r_2$. As we have seen before

$$g_{-t}(A(r)) = A(r)$$

hence

$$m_\mu(A(r)) = 1 \quad \text{and} \quad m_\mu\Big(\bigcap_{r\to 0} A(r)\Big) = 1.$$

Thus for almost everywhere x there are infinitely many $r \to 0$ satisfying *. The same is true of V_s.

Pick $\delta < \bar{\delta}/2$ so small that

$$V_s(x, r_2(1+\delta)) - V_s(x, r_2(1-\delta)) < \frac{\varepsilon}{3} V_s(x, r_2(1-\delta)).$$

Let r_1 be such that $x \in A(r_1)$, $e^{r_3} r_1 r_2 < C(\delta)$, and furthermore for any $y \in B_s(x, r_2)$, $|(P_{H_u(y\,\cdot)} P^{-1}_{H_u(x)})'|$ is uniformly within $\varepsilon/6$ of 1 on $H_{u,r_1}(x)$. The result now follows. □

An ε-cell will be a cell $C_{u,x}(r_1, r_2, r_3)$ made by, first, picking any r_1, then r_3 so that $e^{r_3} < 1 + \varepsilon/3$, and then r_2 using the above lemma.

It follows that any H_s-leaf intersects an ε-cell in ε-near disks all of which are projections of the same set, $B_s(x, r_1)) \subseteq \Lambda(\Gamma)$. Any H_u-leaf intersecting an ε-cell more than $\delta(x, r_2, r_3, \varepsilon) < \varepsilon$ from the boundary intersects it in an ε-near disk both geometrically and measurably. Hence if $C = C_{u,x}(r_1, r_2, r_3)$ is an ε-cell,

$$m_\mu(C) = \mu_{H_s(x)}(H_{s,r_1}(x))\mu_{H_u(x)}(H_{u,r_2}(x))2r_3(1 \pm \varepsilon).$$

(By $a = b(1 \pm \varepsilon)$ we mean $a \in \{b(1-\varepsilon), b(1+\varepsilon)\}$.)

We are now ready to prove a mean ergodic theorem for horospheres. First a weak ergodic theorem.

THEOREM 10. *For any Borel set A, for any $\varepsilon > 0$, there is a set of full density $T \subset \mathbb{R}$ so that for $r = e^t$, $t \in T$. For all but ε of the $y \in T(M)$,*

$$\frac{\mu_{H_u(y)}(B_u(y, r) \cap A)}{\mu_{H_u(y)}(B_u(y, r))} = m_\mu(A)(1 \pm \varepsilon).$$

Proof. We show this first for A a cell $C_{s,x}(r_1, r_2, r_3)$ where $r_1, r_2 \le 1$ and $e^{r_3} < (1 + \varepsilon/20)$. Subdivide $B_s(x, r_2)$ into open subsets $A_1, \ldots, A_k$ with m_μ-measure zero boundary on each of which $f(y) = \mu_{H_u(y)}(H_u(y) \cap C)$ is within a fraction $\varepsilon/20$ of constant. We prove the result first for

$$A_i' = G_{r_3}\Big(\bigcup_{y\in A_i} P_{H_u(y)} P^{-1}_{H_u(x)}(B_u(x, r_1))\Big) = G_{r_3}\Big(\bigcup_{y\in A_i} (H_u(y) \cap C)\Big).$$

Pick δ_1 so small that there is a set $\bar{A}_i \subset A_i$, a δ_1-neighbourhood of $\bar{A}_i$ is contained in A_i, and

$$m_\mu(G_{r_3(1-\delta_1)} \bigcup_{y\in A_i} P_{H_u(y)} P^{-1}_{H_u(x)}(B_u(x, r_1(1-\delta_1)))) > (1 - \varepsilon/20) m_\mu(A_i'),$$

(as $\delta_1 \searrow 0$, $\bar{A}_i \nearrow A_i$), call this shrinking of A_i', $\bar{A}_i'$. Let δ_2 be so small that for any $y \in \bar{A}_i'$, $C_{u,y}(\delta_2, \delta_2, \delta_2) \subset A_i'$.

From the smallness of r_3 we conclude

$$m_\mu(A_i') = \mu_{H_u(x)}(A_i)\mu_{H_s(x)}(B_s(x, r))2r_3(1 \pm \varepsilon/20),$$
$$m_\mu(\bar{A}_i') = \mu_{H_u(x)}(\bar{A}_i)\mu_{H_s(x)}(B_s(x, r_1(1-\delta_1))2(r_3(1-\delta_1)(1 \pm \varepsilon/20)).$$

Let $\bar{\varepsilon} = (\varepsilon/40)m_\mu(A_i')$ and we can find through almost every y an $\bar{\varepsilon}$-cell

$$C(y) = C_{u,y}(r_4, 1, \delta_2 r_3)$$

where $r_4 = r_4(y)$ depends on y.

Select δ_4 and r_4 so small that for all $y \in B$, $m_\mu(B) > 1 - \varepsilon/20$, $r_4(y) \geq r_4$, $\delta_4 < \delta(y, 1, \delta_2 r_3)$.

Pick a compact set $K \subset T(M)$ so large that for $y \in \bar{B} \subset B$, $m_\mu(\bar{B}) > 1 - \varepsilon/10$, $C(y) \subset K$. Cover K with $\{b_1, \dots, b_k\}$, a finite collection of $\bar{\varepsilon}$ r_3-balls, and let $P = \bigvee_{i=1}^{ek} \{b_i, b_i^c\}$ be the finite partition they generate. Thus by lemma 9 for $y \in \bar{B}$, there are sets $C_P^+(y)$, $C_P^-(y) \in P$ approximating $C(y)$ outside and in, geometrically and measurably, to within $\bar{\varepsilon} r_3$.

As g_t is weakly mixing, there is a set T of full density in $\mathbb{R}^+$ with $\delta_4 e^t > r_2$, $r_4 e^{-t} < \delta_2$ and so that for all $S \subset P$,

$$m_\mu(g_t(S) \cap D) = m_\mu(S)m_\mu(D)(1 \pm \varepsilon/20)$$

for $D = A_i'$ or $\bar{A}_i'$.

It follows that

$$m_\mu(g_t(C(y)) \cap D) = m_\mu(C(y))m_\mu(D)(1 \pm \varepsilon/10)$$

for such D and all $y \in \bar{B}$, as $C(y)$ can be so well approximated by $C_P^+(y)$ and $C_P^-(y)$ in P.

We now want to examine the nature of

$$I = g_t(C(y)) \cap A_i' = C_{u, g_t(y)}(e^{-t} r_4(y), e^t, \delta_2 r_3) \cap A_i'.$$

What is significant here, of course, is that $g_t(C(y))$ consists of a huge u-leaf and tiny s-leaf and geodesic leaf in comparison to A_i'.

Let $y' = g_t(y) \in g_t(\bar{B})$ and suppose

$$z \in C_{u,y'}(e^{-t} r_4(y), e^t(1-\delta_4), \delta_2 t) \cap \bar{A}_i'.$$

It follows that for some

$$z' \in B_u(y', e^t(1-\delta_4)) \cap g_r(\bar{A}_i), \; r \subset (r_3(1-\delta_2), r_3(1+\delta_2)),$$

$$\bar{C}(z') = G_{\delta_2 t}\left(\bigcup_{w \in P_{H_u(z')} P_{H_u(x)}^{-1}(B_u(x, r_1))} (P_{H_s(w)} P_{H_s(y')}^{-1}(B_s(y, e^{-t} r_4)) \right) \subset I.$$

Now $\bar{C}(z')$ is, in shape, within $\varepsilon/20$ of

$$C_{u,z'}(e^{-t} r_4(y), r_1, \delta_2 t)$$

and measure within $\varepsilon/20$ of

$$\mu_{H_u(y')}(B_u(y', r_1))\mu_{H_s(x)}(B_s(x, e^{-t} r_4(y)))2\delta_2 t.$$

Now for z too close to the u-boundary of $g_t(C(y))$ or the $S \times g$-boundary of A_i', there is still a little cell $\bar{C}(z')$ with the same bounds geometrically and in measure as the others, but it is only partially in I. Such a cell must lie in $g_t(C(y)) \cap (A_i' \backslash \bar{A}_i')$ and hence is at most $\varepsilon/20$ of I.

Let N be the number of full cells $\bar{C}(z')$ in I, and we conclude

$$m_\mu(I) = N\mu_{H(x)}(B_u(x, r_1))\mu_{H_s(y')}(B_s(y', e^{-t}r_4(y))2\delta_2 r_3(1 \pm 3\varepsilon/20).$$

Now through the centre of each $\bar{C}(z')$ is an $\varepsilon/20$-near disk

$$\hat{C}(z) = P_{H_u(z')}P^{-1}_{H_u(x)}(B_u(x, r_1)) \subset B_u(y', e^t) \cap A'_i$$

and of course by lemma 9

$$\mu_{H_u(y')}(\hat{C}(z)) = \mu_{H_u(y')}(B_u(y', r_1))(1 \pm \varepsilon/20),$$

and such disks constitute all but $3\varepsilon/20$ of $B_u(y', e^t) \cap A'_i$, and

$$\begin{aligned}\mu_{H_u(y')}(B_u(y', e^t) \cap A'_i) &= N\mu_{H_u(x)}(B_u(x, r_1))(1 \pm 3\varepsilon/10)\\ &= m_\mu(I)(\mu_{H_s(y')}(B_s(y', e^{-t}r_4(y)))2\delta_2 r_3)^{-1}(1 \pm 3\varepsilon/5)\\ &= \mu_{H_u(y')}(B_u(y', e^t))m_\mu(A'_i)(1 \pm \varepsilon)\end{aligned}$$

and the result holds on A'_i.

The result extends to finite unions, hence to $\bigcup_i A'_i$. This is within measure zero of our original cell A. The result obviously holds on sets of m_μ-measure zero from the form of dm_μ.

For A a bounded open set, slice A by H_u-leaves into thin sections approximable very well (in terms of ε) inside and out in measure by finite unions of cells of the above form and the result follows for A.

Let $A_i \nearrow T(M)$ be bounded open sets. The result holds, then, for A^c_i, which are ever more distant ends of cusps, and as $m_\mu(A^c_i) \to^i 0$, for any open A the result is true of

$$(A \cap A_i) \nearrow_i A \quad \text{and} \quad (A \cap A_i) \cup A^c_i \searrow^i A,$$

and hence is true of A. As we now have open and closed sets, we get all sets by approximation. □

Now weak mixing does not imply mixing, as each are statements about individual numbers of a sequence, but our weak ergodicity will translate to a mean ergodic theorem as both are statements about averages. This mean ergodic theorem for horospheres then will give us back stronger dynamics on g_t.

COROLLARY 11. *For any set A and $\varepsilon > 0$, if R is sufficiently large, there is a set of measure greater than $1-\varepsilon$, so that for x in this set, for a subset $R(x) \subset (0, R)$ with* $\ln(R(x))$ *of density greater than $1-\varepsilon$,*

$$\frac{\mu_{H_u(x)}(B_u(x, r) \cap A)}{\mu_{H_u(x)}(B_u(x, r))} = m_\mu(A)(1 \pm \varepsilon).$$

Proof. Choose R so large that, with ε^2 in the last lemma, the good subset of R^+ on $(0, R)$ has density greater than $(1-\varepsilon^2)$ and apply Fubini to $T(M) \times (0, R)$. □

Some preliminaries before the ergodic theorem. We will say $V_u(\cdot, x)$ or $V_s(\cdot, x)$ are 'ε, δ-even' at r if

$$V(r(1+\delta), x) - V(r, x) \leq \varepsilon V(r, x).$$

If $V_u(\cdot, x)$ is ε, δ-even at r then $V_u(\cdot, g_t(x))$ is ε, δ-even at $e^t r$, and similarly for V_s.

LEMMA 12. *For any $\varepsilon>0$, there is a δ so that for almost all of $T(M)$, $V_u(x,\cdot)$ is ε, δ-even at a set R of r with* $\ln(R)$ *of density* $(1-\varepsilon)$.

Proof. As $V_u(x,\cdot)$ is continuous, choose δ so small that all but ε of $T(M)$ is ε, δ-even at 1. Conjugation by g_t now gives the result by ergodicity. □

To prove the mean ergodic theorem for horospheres we need a couple more tools, a finite Vitali covering lemma and a trivial version of the ergodic theorem for small sets.

LEMMA 13. *Let m be a Borel probability measure on $\mathbb{R}^n$. Suppose for each $x\in$* supp (m) *we have a ball $B(x,R(x))$ so that*

$$m(B(x,3R(x))<Km(B(x,R(x)).$$

There is, then a disjoint collection $b_1, b_2\ldots$ of the $B(x,R(x))$ so that

$$m((\bigcup b_i)^c)<1-1/K.$$

Proof. Select b_α inductively, α a countable ordinal, requiring

$$\text{radius}(b_\alpha)>\tfrac{1}{2}\Big(\sup_{B(R(x),x)\subset(\bigcup_{\alpha'<\alpha}b_{\alpha'})}(R(x))\Big).$$

Continue to select, transfinitely if necessary, to a maximal such sequence.

Let $b_\alpha=B(R(x_\alpha),x_\alpha)$. We claim

$$\text{supp}(m)\subseteq\bigcup_\alpha B(3R(x_\alpha),x_\alpha).$$

Suppose not, i.e. $y\notin B(3R(x_\alpha),\alpha_\alpha)$ for any α. By maximality, for some x_α,

$$B(R(y),y)\cap B(R(x_\alpha),x_\alpha)\neq\varnothing.$$

Let x_α be the least such. It follows that $R(y)>2R(x_\alpha)$. But as

$$B(R(y),y)\subset\bigcup_{\alpha'<\alpha}B(R(x_{\alpha'}),x_{\alpha'})$$

we have a conflict. Thus

$$m\Big(\bigcup_\alpha b_\alpha\Big)=\sum m(b_\alpha)>1/K\sum_\alpha m(B(3R(x_\alpha),x_\alpha)>1/K. \qquad \square$$

COROLLARY 14. (*Finite Vitali lemma*). *Let m be a Borel probability measure on $\mathbb{R}^n$. Suppose for all $x\in$* supp (m) *we have N balls $B(R_i(x),x)$, $i=1,\ldots,N$ with*

$$m(B(3R_i(x),x))<Km(B(R_i(x),x)).$$

Further, if $R_i=\sup_x R_i(x)$, then

$$m(B((R_i(x)+R_{i+1}),x)<(1+\varepsilon)m(B(R_i(x),x)).$$

We can find, then a countable disjoint sequence $b_1, b_2\ldots$ of the $B(R_i(x),x)$ with

$$m\Big(\bigcup_i b_i\Big)>1-(1-1/K)^N-\varepsilon.$$

Proof. Apply lemma 13 sequentially, first with $R_1(x)$ to get

$$b_{1,1}, b_{1,2}\ldots, b_{1,j}=B(R_1(x_{1,j}),x_{1,j}).$$

Set

$$\bar{b}_{1,j} = B(R_1(x_{1,j}) + R_2, x_{1,j})$$

and re-apply the lemma to supp $(m)|_{\bigcup_j \bar{b}_{1,j}}$ using $R_2(x)$. Set

$$\bar{b}_{2,j} = B(R_2(x_{2,j}) + R_3, x_{2,j})$$

and continue. After N steps we conclude

$$m\left(\left(\bigcup_j \bigcup_{i=1}^{N} \bar{b}_{i,j}\right)^c\right) < (1 - 1/K)^N$$

and we are done. □

LEMMA 15. *For any $\varepsilon > 0$ there is a δ so that for any set A, $m_\mu(A) < \delta$, for all r, for all but ε of the $x \in T(M)$,*

$$\mu_{H_u(x)}(B_u(x, r) \cap A) < \varepsilon \mu_{H_u(x)}(B_u(x, r)).$$

Proof. First choose $\bar{\delta}$ so small that for all but $\varepsilon/10$ of the $x \in T(M)$, $V_u(x, \cdot)$ is $\varepsilon/10$, δ-even at $r = 1$. Choose T and r_2 so small that for all but a further $\varepsilon/10$ of $T(M)$, for any $y \in G_T B_s(x, r_1)$, $V_u(y, \cdot)$ is still $\varepsilon/5$, δ-even.

Around each such x we have an open cell

$$C(x) = H_{u,\bar{\delta}} G_T B_s(x, r_1).$$

Let $C(x_1), \ldots, C(x_n)$ be a finite collection of these covering all but $\varepsilon/10$ of the remaining x, hence all but $3\varepsilon/10$ of $T(M)$.

Let $P = \bigvee_{i=1}^{n} \{C(x_i), C(x_i)^c\}$ be the partition they generate. Set

$$m = \min_{\substack{p \in P \\ y \in G_T B_s(x_i, r_1)}} (m_\mu(p), \mu_{H_u(y)}(B_u(y, 1)), (\mu_{H_s(x_i)} \times dg)(G_t B_s(x_i, r_1))).$$

Set $\delta = (m\varepsilon/10)^2$. Fix $\bar{r}$, let $t = \ln(\bar{r})$ and now let $p \in g_t(P)$, $P \subset g_t(C(x_i))$. We know

$$\int_{B_s(x_i, r_1/\bar{r}(g_t(x_i)))} \int_{-T}^{T} \int_{B_u(y, \bar{r})} \chi_A \, d\mu_{H_u(y)} \, dg \, d\mu_{H_s(g_t(x_i))}$$
$$= m_\mu(A \cap B_u(G_T(B_s(x_i, r_1/\bar{r}(g_t(x_i))), \bar{r})) \leq m_\mu(A) \leq m^3 \left(\frac{\varepsilon}{10}\right)^2.$$

Thus for all but $(m^2/\bar{r})\varepsilon/10$ in measure of the $y \in G_T B_s(x_i, r_1/\bar{r})$,

$$\int_{B_u(y, \bar{r})} \chi_A \, d\mu_{H_u(y)} < rm\varepsilon/10.$$

This is all but a fraction $\varepsilon/10$ of such a p.

Hence for any such y, if $y' \in B_u(y, \bar{\delta}\bar{r})$,

$$\int_{B_u(y', \bar{r}(1-\bar{\delta}))} \chi_A \, d\mu_{H_u(y')} < rm\varepsilon/10 < 3\varepsilon/10 \mu_{H_u(y')}(B_u(y', \bar{r}(1 - \bar{\delta})))$$

by $\varepsilon/5$, $\bar{\delta}$-evenness, and the result holds at y' for $r = r(1 - \bar{\delta})$.

Such y' constitute all but a subset of measure at most $m\varepsilon/10$ of $g_t(C(x_i)) \supset p$ hence for all but at most an $\varepsilon/10$'th fraction of p. As this is true for all but $3\varepsilon/10$ in measure of the $p \in P$ we are done. □

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We now describe Sullivan's estimates for $V_u(x, \cdot)$. Let $\xi \in \Lambda(\Gamma)$, and $V_\partial(\xi, \cdot)$ be the μ volume on $\Lambda(\Gamma)$ of a ball about ξ. Let 0 be a fixed origin for $\mathscr{H}^n$, and $v(t)$ be that point a distance t along the geodesic from 0 to ξ, then the hyperplane perpendicular to this geodesic at $v(t)$ intersects $\partial(\mathscr{H}^n)$ in a ball of radius, say, $r(t) = r$.

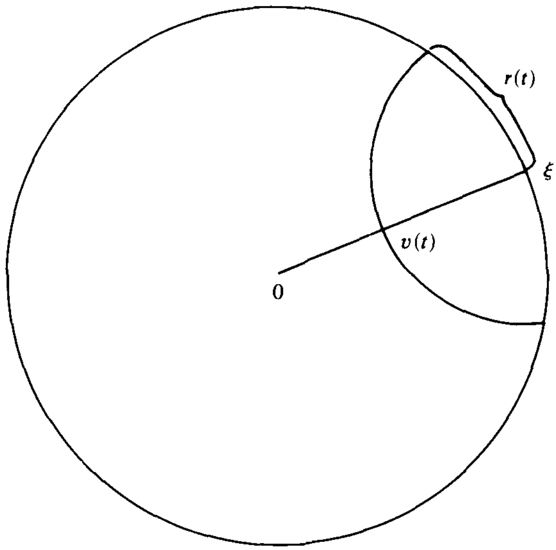

FIGURE 4

Now $v(t)$ is either in the thick part of M or on some cusp of order $k = 1, \ldots, n-1$.

Sullivan shows that for any compact $K \supseteq$(thick part) there is a constant $C(K)$, so that if $v(t)$ is in K

$$\frac{r^d}{C(K)} \geq V_\partial(\xi, r) \geq C(K) r^d,$$

and if $v(t)$ is in a cusp of order k,

$$\frac{r^d \exp((k-d)t)}{C} \geq V_\partial(\xi, r) \geq C r^d \exp((k-d)t).$$

(The 'thick part' is what remains after cutting off the cusps at standard positions.) As t is of the order of $\ln r$, on a cusp of order k

$$\frac{r^{2d-k}}{C} \geq V_\partial(\xi, r) \geq C r^{2d-k}.$$

Note that in order for m to have cusps of order k, $d \geq k/2$ so $V_\partial(\xi, r) \to 0$ as $r \to 0$ always. As $r \to 0$, we pass through various regions where $V_\partial(\xi, r)$ is controlled by different exponential rates.

To see that this same picture holds when μ is projected onto horospheres, make sure we choose 0 so that it is on a geodesic between two points of $\Lambda(\Gamma)$. Choose r so small that $|p'_{Hu(0)}|$ is uniformly within ε of constant on $B_\partial(\xi, r)$. Thus on $B_u(x, r')$

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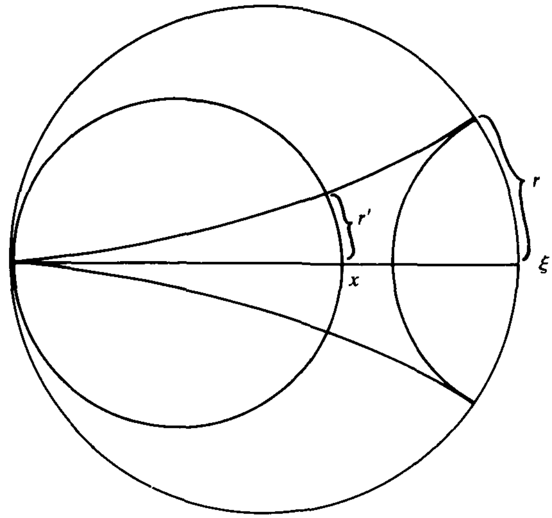

FIGURE 5

we get the same behaviour as at ξ, once r' is small enough. This behaviour remains nearly true in a neighbourhood of 0. But now in $T(M)$ the fact that, for r sufficiently small, $V_u(x, r)$ decays to 0 through intervals like r^d or r^{2d-k} is a g_t invariant property as

$$V_u(e^t r, g_t(x)) = e^t V_u(r, x).$$

As Sullivan has shown g_t is ergodic we have such bounds almost everywhere, and as $e^t r \nearrow \infty$ as $t \to \infty$, we have it for all r. It is important to remember that $v(t)$, by the ergodic theorem, lies in the various parts of $T(M)$ with proportions like m_μ.

LEMMA 16. *There is a constant K so that for almost every x,*

$$V_u(3r, x) < K V_u(r, x).$$

Proof. We show this for V_∂ and hence the result for V_u follows. We use Sullivan's estimates on V_∂. In M, $v(t(3r))$ and $v(t(r))$ are a bounded distance apart. If both are in a cusp we have the result, and if one is in the thick part, then the other is in a fixed compact region containing the thick part, and again we are done. □

THEOREM 17. (*Mean ergodic theorem*). *For any Borel set A and any ε, there is an R so that for all $r \geq R$, for all but ε of the $x \in T(M)$*

$$\frac{\mu_{H_u(x)}(B_u(x, r) \cap A)}{\mu_{H_u(x)}(B_u(x, r))} = m_\mu(A)(1 \pm \varepsilon).$$

Proof. Pick N so large that for K from lemma 16

$$\left(1 - \frac{1}{K 3^{d+1} m_\mu(A)}\right)^N < \frac{\varepsilon m_\mu(A)}{10}.$$

With $\varepsilon/10 m_\mu(A)^2$ in lemma 12 we get a δ_1 so that for all but $\varepsilon/10$ of $T(M)$, if $m_\mu(S) < \delta_2$, then the density of S in $B_u(x, r)$ is less than $\varepsilon/10 m_\mu(A)$.

Choose δ_2 so small that all but $\delta_1/2N$ of $T(M)$ is $\varepsilon/10m_\mu(A)^2$, δ_2-even, using lemma 12 and the weak ergodic theorem holds to within $\varepsilon/10m_\mu(A)^2$ on $B_u(x, r)$. Our control on densities of such r allows this. By deleting a further $\varepsilon m_\mu(A)/10$ of $T(M)$ leaving G, the 'good x'. Select inductively $R_1(x), R_2(x) \cdots R_N(x)$, such good radii,

$$\sup_G R_{i+1}(x) < \delta_2 m_\mu(A) R_i(x).$$

Choose

$$R > \frac{\sup_G R_1(x)}{\delta_2 m_\mu(A)}.$$

Now if x is $\varepsilon m_\mu(A)/10$, $\sup_G R_1(x)/R$-even at $r \geq R$, and all but a fraction $\varepsilon m_\mu(A)^2/10$ of $B_u(x, r)$ is in G, we can apply the finite Vitali lemma with

$$m = m_\mu|_{B_u(x,r)\cap G}$$

to get a disjoint set of balls $b_1, b_2, \ldots$, covering all but a fraction

$$\frac{\varepsilon m_\mu(A)}{10} + \left(1 - \frac{1}{K3^{d+1}}\right)^N + \frac{\varepsilon m_\mu(A)}{10} = \frac{3\varepsilon m_\mu(A)}{10}$$

of $B_u(x, r)$, on each of which the fraction in A is within $\varepsilon/10$ of $m_\mu(A)$. A fraction at most $\varepsilon/10m_\mu(A)$ of these balls extends outside $B_u(x, r)$ and we are done. □

It is interesting to note that the behaviour of $V_u(x, \cdot)$ is intimately related to the existence of a pointwise ergodic theorem.

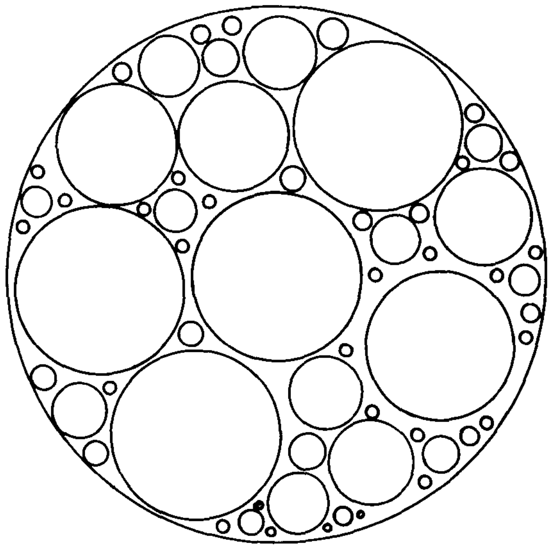

FIGURE 6

THEOREM 18. *If for all ε, there is a δ so that for almost every x, $V_u(x, \cdot)$ is ε, δ-even for all r, then for any set A*

$$\frac{\mu_{H_u(x)}(B_u(x,r)\cap A)}{\mu_{H_u(x)}(B_u(x,r))} \xrightarrow{r} m_\mu(A).$$

Proof. This is a variation of an unpublished proof by P. Shields of the Birkhoff ergodic theorem. Suppose we do not have pointwise convergence on A. There is, then, an $\bar{\varepsilon}>0$ and a set of positive measure B so that for $x\in B$, there are $r_i(x)\nearrow\infty$ and

$$\frac{\mu_{H_u(x)}(B_u(x,r_i(x))\cap A)}{\mu_{H_u(x)}(B_u(x,r_i(x))}>(1+\bar{\varepsilon})m_\mu(A).$$

(Replace A by A^c if always less than $(1-\bar{\varepsilon})m_\mu(A)$.) In this case we say x is $\bar{\varepsilon}$-bad at $r_i(x)$. By uniform evenness, for $x\in B$,

$$x'\in B_u(x,\delta(\bar{\varepsilon}/2m_\mu(A))r_i(x)),$$

x' is $\bar{\varepsilon}/2$-bad at $r_i(x)$. Now

$$\delta(\bar{\varepsilon}/2m_\mu(A))r_i(x)\nearrow^i\infty,$$

and by the mean ergodic theorem, almost every x has

$$B\cap H_u(x)\neq\varnothing.$$

Hence almost every x is $\bar{\varepsilon}/2$-bad for arbitrarily large r. Following the same format as lemma 17, only now constructing coverings with $\bar{\varepsilon}/2$-bad balls, we can, once r is large enough, cover all but $\bar{\varepsilon}/20$ of $B_u(y,r)$, for all but $\bar{\varepsilon}/20$ of the $y\in T(M)$ with disjoint $\bar{\varepsilon}/2$-bad balls. It follows that for all such y, $B_u(y,r)$ is itself $\bar{\varepsilon}/4$-bad. This conflicts with the weak ergodic theorem. □

Whether V_u is uniformly continuous on $\operatorname{supp}(m_\mu)$ now becomes an interesting question. Even if the answer is, in general, no, is there still a pointwise theorem?

We now return to the dynamics of g_t, to show that it is a Bernoulli flow. Simply retracing the mean ergodic theorem backward would lead to a proof of g_t mixing. We must work a little differently for the stronger result.

What we will do is construct sequences of partitions P_i, $\bar{P}_i$ so that for $\varepsilon_i=2^{-i}$:

(i) $$\text{for } i\le j,\ \bigvee_{i=1}^{j} P_i \subset^{\varepsilon_i} P_j;$$

(ii) $$P_i\subset^{\varepsilon_i}\bar{P}_i;$$

(iii) $$\bigvee_i P_i = B;\ \text{and}$$

(iv) $$\text{for almost every } f\in\bigvee_{i=0}^{-\infty} g_t(P_i).$$

($P\subset^{\varepsilon} Q$ if for each set A in P there is a set $\bar{A}$ in Q with $m_\mu(A\Delta B)<\varepsilon$.)

$$\bar{d}\Big(\bigvee_{i=0}^{\infty} g_t(\bar{P}_i)/f,\ \bigvee_{i=0}^{\infty} g_t(\bar{P}_i)\Big)=0.$$

Since this is not one of the standard conditions given that are equivalent to a flow being Bernoulli, we must verify the following lemma (see [**4**] for a treatment of the Bernoulli property).

LEMMA 19. *If P_i, $\bar{P}_i$ are as above, then for all i, $(g_t, \bigvee_{j=1}^{i} P_j)$ is ε-block independent, hence g_t is Bernoulli (see* [**2**]).

Proof. As $P_i \subset^{\varepsilon_i} \bar{P}_i$, we can approximate P_i by unions of sets in $\bar{P}_i$ and by the ergodic theorem, for almost every $f \in \bigvee_{i=0}^{-\infty} g_t(P_i)$,

$$\bar{d}\left(\bigvee_{t=0}^{\infty} g_t(P_j)/f, \bigvee_{t=0}^{\infty} g_t(P_j)\right) < \varepsilon_j.$$

Let $\bigotimes_{i=1}^{N} (\bigvee_{t=0}^{T} g_t(P_j))$ be the N-fold independent product of $\bigvee_{t=0}^{T} g_t(P_j)$. We conclude, for $T = T(j)$ large enough,

$$\bar{d}\left(\bigotimes_{i=1}^{N}\left(\bigvee_{t=0}^{T} g_t(P_j)\right), \bigvee_{t=0}^{NT} g_t(P_j)\right) < 2\varepsilon_j,$$

i.e. P_j satisfies $2\varepsilon_j$-block independence.

As $\bigvee_{k=1}^{i} P_k \subset^{\varepsilon_j} P_j$, once T is large enough, we can code all but $2\varepsilon_j$ of the names in $\bigvee_{t=0}^{T} (g_t(\bigvee_{k=1}^{i} P_k))$ from names in $\bigvee_{t=0}^{T} g_t(P_j)$ with at most a $2\varepsilon_j$ $\bar{d}$-error. Hence for $T > T(j)$

$$\bar{d}\left(\bigotimes_{i=1}^{N}\left(\bigvee_{t=0}^{T} g_t\left(\bigvee_{k=1}^{i} P_k\right)\right), \bigvee_{t=0}^{NT} g_t\left(\bigvee_{k=1}^{i} P_k\right)\right) < 6\varepsilon_j$$

and $(g_t, \bigvee_{k=1}^{i} P_k)$ is ε-block independent for all ε, and hence Bernoulli. As

$$\bigvee_{k=1}^{\infty} P_k = B,$$

g_t is Bernoulli by Ornstein's monotone theorem. □

All that remains is to construct P_i and $\bar{P}_i$. The P_i will be made mostly of disjoint $\varepsilon_i/10$-cells $C_{u,\cdot}(\cdot,\cdot,\cdot)$ and $\bar{P}_i$, $C_{s,\cdot}(\cdot,\cdot,\cdot)$.

Use the standard Vitali covering lemma to cover most of A_1 and A_2 by disjoint disks $B_s(x_i, r_i)$ and $B_u(x_j, r_j)$ with

$$\frac{d(V_s(x_i))}{dr}(r_i) < K.$$

The cells of P_i in q will be

$$G_{r_3}\left(\bigcup_{y \in B_u(x_j, r_j)} P_{H_s(y)} P_{H_s(x_i)}^{-1}(B_s(x_i, r_i))\right).$$

This is not exactly a cell, but as $|(P_{H_s(y)} P_{H_s(x_i)}^{-1})'|$ is uniformly within $\varepsilon_i/20$ of 1 on q, these almost cells are, both in geometry and measure, within $\varepsilon_i/20$ of

$$C_{u, P_{H_s(x_j)} P_{H_s(x_i)}^{-1}(x_i)}(r_i, r_j, r_3).$$

Furthermore, on H_s-cross sections these almost cells have at most $2K\varepsilon$ of their mass within ε of their boundary, if ε is small enough, as this property projects with small distortion.

Build $\bar{P}_i$ similarly, but with almost cells

$$G_{r_3}\Bigg(\bigcup_{y\in B_s(x_j,r_j)} P_{H_u(y)}P^{-1}_{H_u(x_i)}(B_u(x_i,r_i))\Bigg)$$

so small, and covering so much of $T(M)$ that $P_i \subset^{\varepsilon_i} \bar{P}_i$.

All that remains is to verify the near $\bar{d}$-independence of

$$\bigvee_{i=0}^{-\infty} g_t(P_i) \quad \text{and} \quad \bigvee_{i=0}^{\infty} g_t(\bar{P}_i).$$

We first identify the fibres $f\in\bigvee_{i=0}^{-\infty} g_t(P_i)$. As g_t is weakly mixing, f intersects a g_t-orbit in at most one point, as for $t<0$, g_t shrinks H_u-leaves, f intersects an H_u-leaf in at most one point. We wish to see that f is a bounded simply connected region on an H_s-leaf. Take a ball $B_s(x, 1)$ and let

$$f_{r,s}(x)\in \bigvee_{i=-r}^{-s} g_t(P_i)$$

be that atom containing x. Consider

$$B_s(x,1)\cap f_{0,s}(x) \quad \text{as } s\to\infty.$$

This set decreases in discrete jumps as $g_s(x)$ moves from set to set in P_i, and only then if $g_s(x)$ is sufficiently close to a boundary in the H_s-leaf of an atom of P_i to cut

$$g_s(B_s(x,1)) = B_s(g_s(x), e^{-s}).$$

Now once s is large enough, the probability that the set is so cut is less than $2kKe^{-s}$, where P_i has k cells.

A g_t-orbit passes through the sets of P_i in intervals. These intervals have a minimum length, say 3α. Thus each time $B_s(x,1)\cap f_{0,s}(x)$ decreases, $g_{[s/\alpha]\alpha}(x)$ is still within $e^{-[s/\alpha]\alpha}$ of the H_s-boundary of P_i. But the set of x within $e^{-n\alpha}$ of the H_s-boundary of P_i is summable in n. Thus, for almost every x, $B_s(x,1)\cap f_{0,s}(x)$ decreases only finitely many times, and for r small enough, $B_s(x,r)\subset f_{0,\infty}(x)$, and almost every f is an open set. Similarly, almost every $f'\in\bigvee_{t=0}^{\infty} g_t(\bar{P}_i)$ is an open simply connected region on an H_u-leaf.

It is now clear that verifying g_t weakly Bernoulli would be very difficult, as these partitions, P_i, are the most natural to the system, but as $H_u(x)\times H_s(x)$ does not span the tangent space at x, the past and future do not become independent as we separate them in time.

As we are only after $\bar{d}$-independence, we can 'fatten' the future fibres and apply the mean ergodic theorem on horospheres in the following form to get $\bar{d}$-independence.

LEMMA 20. *Let A_1, A_2 be bounded open regions on H_u-leaves. If r_3 is sufficiently small, let*

$$\bar{A}_1 = G_{r_3}(A_1), \quad \bar{A}_2 = G_{r_3}(A_2).$$

If A_1 and A_2 are not on a bad set of H_u-leaves of measure zero, once r is large enough, for all but ε of the $y\in T(M)$

$$\frac{\operatorname{card}(B_s(y,r)\cap\bar{A}_1)}{\operatorname{card}(B_s(y,r)\cap\bar{A}_2)} = \frac{\mu_{H_u}(A_1)}{\mu_{H_u}(A_2)}(1\pm\varepsilon).$$

Proof. Pick r_1, r_1' small enough that

$$\bar{A}_1 \subset C_{u,x_1}(r_1, r_2, r_3) \quad \text{and} \quad \bar{A}_2 \subset C_{u,x_2}(r_1', r_2, r_3)$$

are $\varepsilon/10$-cells. This can be done for almost every x, hence for any A_1, A_2 which has a good x on its H_u-leaf. Let

$$\hat{A}_1 = G_{r_3}\Big(\bigcup_{y\in A_1} P_{H_s(y)} P^{-1}_{H_s(x_1)}(B_s(x_1, r_1))\Big)$$

and

$$\hat{A}_2 = G_{r_3}\Big(\bigcup_{y\in A_2} P_{H_s(y)} P^{-1}_{H_s(x_2)}(B_s(x_2, r_2))\Big)$$

be the corresponding thickenings of $\bar{A}_1, \bar{A}_2$. Now

$$m_\mu(\hat{A}_1) = \mu_{H_u(x_1)}(A_1)\mu_{H_s(x_1)}(B_s(x_1, r_1))2r_3(1\pm\varepsilon/10),$$
$$m_\mu(\hat{A}_2) = \mu_{H_u(x_2)}(A_2)\mu_{H_s(x_2)}(B_s(x_2, r_1'))2r_3(1\pm\varepsilon/10).$$

Now for r sufficiently large, for all but ε of the $y \in T(M)$,

$$\frac{\mu_{H_s(y)}(B_s(y,r)\cap\hat{A}_1)}{\mu_{H_s(y)}(B_s(y,r)\cap\hat{A}_2)} = \frac{\mu_{H_u(x_1)}(A_1)\mu_{H_s(x_1)}(B_s(x_1,r_1))}{\mu_{H_u(x_2)}(A_2)\mu_{H_s(x_2)}(B_s(x_2,r_1'))}(1\pm\varepsilon/2)$$

by the mean ergodic theorem.

The analysis of theorem 10 tells us, that for r sufficiently large, $B_s(y,r)\cap\hat{A}_1$ consists for all but $\varepsilon/20$ of the space of full near-cells of the form

$$P_{H_s(y)}P^{-1}_{H_s(x_1)}(B_s(x_1, r_1)),\ y \in \bar{A}_1,$$

with some partial disks near the boundary of $B_s(y, r)$ constituting a fraction, at most $\varepsilon/20$, of their total number.

Each disk has at its centre a point of $B_s(y,r)\cap\bar{A}_1$, and so

$$\mu_{H_s(y)}(B_s(y,r)\cap\hat{A}_1) = \operatorname{card}(B_s(y,r)\cap\bar{A}_1)\mu_{H_s(x_1)}(B_s(x_1,r_1))(1\pm\varepsilon/10).$$

The same holds on $\hat{A}_2$ and we are done. □

We prove this only for open sets but the result clearly extends to any bounded Borel sets on H_u-leaves.

LEMMA 21. *Let $\mathscr{A}$ be any measurable partition whose atoms are open bounded regions on H_u leaves. For almost every $y \in T(M)$, the Radon–Nickodym derivative*

$$\frac{d(m_\mu(\mathscr{A}|G_{r_3}B_s(y,r))}{d\mu_{H_s}(y)\times dg}(f) = \operatorname{card}(f\cap G_{r_3}B_s(y,r)).$$

Proof. This is the same analysis as in the above lemma. If we fatten f slightly in the H_s and G directions, its intersection with $G_{r_3}B_s(y,r)$ becomes mostly a collection of small cells, one for each intersection point, all of nearly the same volume. As the size of thickening goes to zero, all the approximations improve to the result. □

https://doi.org/10.1017/S0143385700001735 Published online by Cambridge University Press

COROLLARY 22. *For any $\varepsilon > 0$, r_3 sufficiently small, for all but ε of the $y \in T(M)$, and for a set S of all but ε of $\bigvee_{t=R}^{\infty} g_t(\bar{P}_i)$, if $f_i \in S$,*

$$\frac{\text{card}\,(G_{r_3}(f_1) \cap B_s(R, y))}{\text{card}\,(G_{r_3}(f_2) \cap B_s(R, y))} = \frac{\mu_{H_u}(f_1)}{\mu_{H_u}(f_2)}(1 \pm \varepsilon),$$

and such $f \in S$ cover all but ε of $H_{s,1}(y)$.

Proof. As the f_i are regions on H_u-leaves, the result is true for almost any particular pair. To get the result on all pairs from a large set we take the usual tack, getting a uniform r_2 for most past atoms in the argument of lemma 17 and then a value r_1 bounded below for most past atoms. Next approximate most of these fattened atoms very well by a finite partition made mostly of small sets. The mean ergodic theorem to within $\varepsilon/20$ holds on all sets in this partition for r large enough, and hence uniformly over most fattened past atoms. The result is now completed as in lemma 17. That atoms not in S occupy a small fraction of $B_s(y, r)$ follows from the mean ergodic theorem. □

COROLLARY 23. *For any $\varepsilon > 0$, r_3 small enough, once r is large enough, for all but ε of the $y \in T(M)$, for any set $A \subset \bigvee_{t=0}^{\infty} g_t(\bar{P}_i)$,*

$$\frac{\mu_{H_s(y)} \times dg(G_{r_3}B_s(y, r) \cap A)}{\mu_{H_s(y)} \times dg(G_{r_3}B_s(y, r))} = m_\mu(A) \pm \varepsilon,$$

i.e. the conditional measure of $\bigvee_{t=0}^{\infty} g_t(\bar{P}_i)$ on $G_{r_3}B_s(y, r)$ is weakly within ε of m_μ.

Proof. We have identified the R.N. derivative

$$d(m_\mu(\mathscr{A}|G_{r_3}B_s(y, r))) = \text{card}\,(f \cap G_{r_3}B_s(y, r))\, dH_s\, dg.$$

Integrating over $f \in A$,

$$m_\mu(A|G_{r_3}B_s(y, r)) = \int_A \text{card}\,(f \cap G_{r_3}B_s(y, r))\, dH_s\, dg$$

$$= \int_A \mu_{H_u}(f)\, dH_s\, dg \pm \varepsilon = m_\mu(A) \pm \varepsilon. \qquad \square$$

COROLLARY 24. *For any r_3 small enough, for any r, for almost all the $y \in T(M)$,*

$$\bar{d}\left(\bigvee_{t=0}^{\infty} g_t(P_i)/G_{r_3}B_s(y, r), \bigvee_{t=0}^{\infty} g_t(P_i)\right) = 0.$$

Proof. For all but ε of the $y \in T(M)$, once T is large enough,

$$m_\mu\left(\bigvee_{t=T}^{\infty} g_t(\bar{P}_i)|G_{r_3}B_s(y, r)\right) \quad \text{and} \quad m_\mu\left(\bigvee_{t=T}^{\infty} g_t(\bar{P}_i)\right)$$

are weakly within ε, as they are as close as

$$m_\mu\left(\bigvee_{t=0}^{\infty} g_t(\bar{P}_i)|G_{r_3}H_{s,e^T r}(g_{-T}(y))\right) \quad \text{and} \quad m_\mu\left(\bigvee_{t=0}^{\infty} g_t(\bar{P}_i)\right).$$

But this weak closeness implies

$$\bar{d}\left(\bigvee_{t=0}^{\infty} g_t(\bar{P}_i)/G_{r_3}B_s(y, r), \bigvee_{t=0}^{\infty} g_t(\bar{P}_i)\right) < \varepsilon,$$

as the gap $(0, T)$ is insignificant in $\bar{d}$. Let $\varepsilon \to 0$. □

https://doi.org/10.1017/S0143385700001735 Published online by Cambridge University Press

One last lemma completes our basic work.

LEMMA 25. *For any $\varepsilon > 0$, there is a δ so that if $r_3 < \delta$, for any measurable partition $\mathcal{Q}$, for almost every $f \in \mathcal{Q}$,*

$$\bar{d}\left(\bigvee_{t=0}^{\infty} g_t(\bar{P}_i)/f, \bigvee_{t=0}^{\infty} g_t(\bar{P}_i)/G_{r_3}(f)\right) < \varepsilon.$$

Proof. Let $\mu_f = m_\mu(|f|)$ be the conditional measure on f. Couple $f \times G_{r_3}(f)$ by the measure

$$\nu(A \times B) = (\mu_f \times dg)(G_{r_3}(A) \cap B).$$

Any point $(x, x') \in \operatorname{supp}(\nu)$ must satisfy

$$g_t(x) = x' \quad \text{for some } |t| < r_3.$$

If δ is small enough,

$$m_\mu(g_t(\mathcal{Q})\,\Delta\mathcal{Q}) < \varepsilon$$

and so for almost every $(x, x') \in \operatorname{supp}(\nu)$,

$$\bar{d}((\mathcal{Q}\text{-name of } x), (\mathcal{Q}\text{-name of } x')) < \varepsilon$$

and we are done. □

COROLLARY 26. *For any $r > 0$, and almost any $x \in T(M)$,*

$$\bar{d}\left(\bigvee_{t=0}^{-\infty} g_t(\bar{P}_i)/B_s(x, r), \bigvee_{t=0}^{-\infty} g_t(\bar{P}_i)\right) = 0.$$

THEOREM 27. *For almost every $f \in \bigvee_{t=0}^{-\infty} g_t(P_i)$,*

$$\bar{d}\left(\bigvee_{t=0}^{\infty} g_t(\bar{P}_i)/f, \bigvee_{t=0}^{\infty} g_t(\bar{P}_i)\right) = 0.$$

Proof. By the standard Vitali lemma, almost every $f \in \bigvee_{t=0}^{-\infty} g_t(P_i)$ can be almost completely covered by disks $B_s(R, y)$ satisfying corollary 22. □

With this we conclude g_t is a Bernoulli flow.

The author expresses his appreciation to D. Sullivan for much help in clarifying the issues in this argument. The work was supported by the Sloan Foundation.

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