THE ANALYTIC CONTINUATION OF THE RIEMANN-LIOUVILLE INTEGRAL IN THE HYPERBOLIC CASE

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Introduction. In 1949 I published in the *Acta Mathematica* (vol. 81) a rather long paper: "L'intégrale de Riemann-Liouville et le problème de Cauchy." This work will be quoted in the sequel as *Acta* paper. Only minor local references to this paper will be made here, and knowledge of it is not required for the reading of the present article. The notations used here are slightly different from those used in my former paper.

In the *Acta* paper I introduce multiple integrals I^α and I^α_* of the Riemann-Liouville type depending on a parameter α and converging for sufficiently large values of α. I give the solution of the Cauchy problem for the wave equation in a unique formula, the same for space-time of odd or even dimensions, implying an analytic continuation with respect to the parameter α. When this analytic continuation is carried out, it leads to final formulae of quite different types for odd or even dimensions, the one relative to even dimensions obeying the Huygens principle.

The main difficulty concerning the analytic continuation was to prove that I^0 is the identity operator. My way of doing this was neither simple nor elegant. The principal aim of the present paper is to give a more satisfactory proof.

I hope that the present approach will be useful in other connections as well. Indeed, this method of analytic continuation has found unexpected applications in other fields. Here I only make reference to results of Gelfand and Grajew.[1]

1. Preliminaries. If the co-ordinates of a point x in m-dimensional space-time or Lorentz-space are denoted by $x^0, x^1, \ldots, x^{m-1}$, the metric form will be

$$(x, x) = (x^0)^2 - (x^1)^2 - \ldots - (x^{m-1})^2 = l_{ik}x^i x^k, \tag{1.1}$$

where the ordinary summation convention is used. The square of the distance of two points x and y is given by

$$R_{xy} = r^2{}_{xy} = (x - y, x - y) = l_{ik}(x^i - y^i)(x^k - y^k). \tag{1.2}$$

The scalar product (a, b) of two vectors a and b, with the respective components a^k and b^k, is defined by

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[1]See Appendix III of the book by I. M. Gelfand and M. A. Neumark, *Unitäre Darstellungen der klassischen Gruppen* (Berlin: Akademie-Verlag, 1957), also A.M.S. translations, Series **2**, vol. 9, pp. 123–154.

$$(a, b) = l_{ik}a^i b^k. \tag{1.3}$$

Two vectors whose scalar product vanishes are said to be *orthogonal* to each other. In what follows, *orthogonality and normality are always meant in this sense.*

According as the scalar square (a, a) of a vector a is (1) positive, (2) zero, (3) negative, the vector is said to be (1) *time-like*, (2) *light-like* or a *null vector*, (3) *space-like*. A time-like or a light-like vector a is called *positive* or *negative* according as its time component a^0 is positive or negative.

Time-like unit vectors u and *space-like unit vectors* v are defined by the relations $(u, u) = 1$ and $(v, v) = -1$ respectively.

The *light cone* or characteristic cone with vertex a is given by the equation $(x - a, x - a) = 0$. The *positive* and *negative* half-cones correspond to $x^0 - a^0 > 0$ or < 0 respectively. These half-cones will be called *positive and negative light cones* in the sequel.

Consider now a p-dimensional (curved) variety S whose points y are referred to p parameters $\lambda^1, \lambda^2, \ldots, \lambda^p$. The p-dimensional *volume element* dS of S, or alternately *surface element* if $1 < p < m$, can be defined in the following way (cf. *Acta* paper pp. 44–45). Let $ds^2 = (dy, dy) = \sum_{i,k} \gamma_{ik} d\lambda^i d\lambda^k$ be the square of the arc element in S. Form the determinant $\gamma = |\gamma_{ik}|$. Then

$$dS = \sqrt{|\gamma|}\, d\lambda^1 d\lambda^2 \ldots d\lambda^p. \tag{1.4}$$

An $(m-1)$-dimensional surface is said to be space-like if its normal is time-like. Let S be a space-like surface. Suppose that the negative light cone C^x with vertex x and the surface S enclose a bounded domain $D_S{}^x$. We shall consider functions defined in domains including $D_S{}^x$ and make the blanket hypothesis that the functions and all their derivatives with respect to the Cartesian co-ordinates which explicitly or implicitly enter into our computations exist and are continuous. We express this by saying that the functions are *well behaved*. The same phrase will be used in an appropriate sense in connection with the surface S and functions defined on S.

We form the *volume potential*

$$I^\alpha f(x) = \frac{1}{H_m(\alpha)} \int_{D_S{}^x} f(y) r_{xy}{}^{\alpha - m} dV, \tag{1.5}$$

where $dV = dy^0 dy^1 \ldots dy^{m-1}$ is the volume element of m-space and

$$H_m(\alpha) = \pi^{\frac{1}{2}(m-2)} 2^{\alpha-1} \Gamma(\tfrac{1}{2}\alpha) \Gamma(\tfrac{1}{2}(\alpha + 2 - m)). \tag{1.6}$$

The integral in (1.5) converges for $\alpha > m-2$ (cf. *Acta* paper, p. 31), or more generally for $Re\,\alpha > m$-2, if we admit complex values of α. Similarly, our subsequent assertions about convergence of integrals or analytic continuation of I^α or $I^{*\alpha}$ (see below) remain valid for complex α, if we replace all inequalities of the type $\alpha > \alpha_0$ by $Re\,\alpha > Re\,\alpha_0$.

Besides the volume potentials we also consider potentials of a *simple layer* and of a *double layer*.

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Let S^x be that part of the surface S which is interior to the cone C^x. Denote further by n the positive unit normal to S and let g and h be two functions defined on S. We write

$$(1.7) \qquad I_*^\alpha \overline{f, g, h}(x) = \frac{1}{H_m(\alpha)} \int_{D_s{}^x} f(y) r_{xy}{}^{\alpha-m} dV + \frac{1}{H_m(\alpha)} \int_{S^x} \left\{ g(y) r_{xy}^{\alpha-m} - h(y) \frac{d}{dn} r_{xy}^{\alpha-m} \right\} dS,$$

where dS is the surface element of S (cf. formula (1.4)).

The simple layer converges for $\alpha > m-2$, while the double layer, whose kernel has a stronger singularity, converges only for $\alpha > m$ (cf. *Acta* paper, pp. 48–49, and §4 of the present paper).

We will show that by virtue of our hypotheses about the behaviour of the surface S and the functions f, g, h the integral $I_*{}^\alpha$ can be continued analytically down to an arbitrary value $\alpha_0 \gtreqless 0$. Moreover, if $\alpha_0 < 0$, then

$$I_*^0 \overline{f, g, h}(x) = I^0 f(x) = f(x).$$

For a specification of the derivatives needed for different purposes cf. *Acta* paper, pp. 59–60, 64, 223.

Some simple facts concerning the analytic continuation of the ordinary Riemann-Liouville integral in *one dimension* will be needed in the sequel (cf. *Acta* paper, pp. 14–16).

Set

$$(1.9) \qquad J^\alpha f(0) = \frac{1}{\Gamma(\alpha)} \int_0^b f(t) t^{\alpha-1} dt.$$

If $f(t)$ is continuous in the closed interval $[0, b]$, this integral is convergent for $\alpha > 0$. If for $k \leqq n$ the derivatives $f^{(k)}(t)$ exist and are continuous in $[0, b]$, then *$J^\alpha f(0)$ has a holomorphic continuation to all $\alpha > -n$. Moreover, if p is an integer $0 \leqq p < n$, then*

$$(1.10) \qquad J^{-p} f(0) = (-1)^p f^{(p)}(0).$$

(As a matter of fact, only the case $p = 0$ will be used explicitly in the sequel.)

To prove this, set

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

Then we have for $\alpha > 0$, to begin with, and subsequently by analytic continuation, for all $\alpha > -n$

$$(1.11) \qquad J^\alpha f(0) = \frac{1}{\Gamma(\alpha)} \int_0^b [f(t) - P(t)] t^{\alpha-1} dt + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!(k+\alpha)} b^{k+\alpha}.$$

Indeed, the last integral is convergent and the whole expression (1.11) is holomorphic for $\alpha > -n$. $J^\alpha f(0)$ reduces to $(-1)^p f^{(p)}(0)$ at $\alpha = -p$, since $\Gamma(\alpha)$ has a simple pole with residue $(-1)^p/p!$ at this point.

The following extension will also be needed, and the corresponding result will be quoted as *the extended one-dimensional case*. Its verification is left to the reader.

Let $f(t)$ also depend on α. If $f(t)$ and its derivatives with respect to t up to the order n are continuous in the closed interval $[0, b]$ and moreover are holomorphic in α for $\alpha > -n$, then our above statement and its proof remain valid, except for some slight changes in the notations.

2. A co-ordinate system. We place the origin O at the point x and will eventually refer the domain $D_S{}^O$ to co-ordinates which are to be introduced here. We denote a *fixed* negative time-like unit vector by a and a *variable* space-like unit vector orthogonal to a by v. In a suitable Lorentz frame a and v can be written $a = (-1, 0, \ldots, 0)$ and $v = (0, v^1, \ldots, v^{m-1})$ with $\sum (v^k)^2 = 1$. If the vector v issues from the origin, its endpoint describes the unit sphere S_{m-2} lying in the $(m-1)$-plane orthogonal to a. We write out explicitly that

$$(a, a) = 1,\ (v, v) = -1,\ (a, v) = 0. \tag{2.1}$$

An arbitrary position vector y can be written

$$y = ta + \rho v,\ \rho \geqq 0. \tag{2.2}$$

We always suppose that also $t \geqq 0$. This inequality is obviously satisfied in the domain $D_S{}^O$.

The relation (2.2) can also be written

$$y = \tfrac{1}{2}(t + \rho)(a + v) + \tfrac{1}{2}(t - \rho)(a - v).$$

Furthermore, if we set

$$b = \tfrac{1}{2}(a + v),\ c = \tfrac{1}{2}(a - v), \tag{2.3}$$

then

$$y = (t + \rho)b + (t - \rho)c = (t + \rho)\left(b + \frac{t - \rho}{t + \rho}c\right).$$

Setting now

$$\tau = \frac{t - \rho}{t + \rho}, \quad \sigma = t + \rho, \tag{2.4}$$

we obtain

$$y = \sigma(b + \tau c). \tag{2.5}$$

The inverted formulae (2.4) are

$$\rho = \tfrac{1}{2}\sigma(1 - \tau),\ t = \tfrac{1}{2}\sigma(1 + \tau). \tag{2.6}$$

It follows from (2.1) and (2.3) that

$$(b, b) = 0,\ (c, c) = 0,\ (b, c) = \tfrac{1}{2}. \tag{2.7}$$

Hence b and c are (negative) *null vectors*.

The variables τ and σ and the angular variable v which varies on the sphere S_{m-2} and determines the vectors b and c will be our new co-ordinates. Here are the principal merits of τ and σ. The square of the Lorentz distance of a point from the vertex can be expressed and "separated" in τ and σ. The vertex of the cone C^0 is given by the single equation $\sigma = 0$, while the cone apart from the vertex is given by the equation $\tau = 0$. The derivatives $\partial^p f/\partial\tau^p$ of an arbitrary function $f(y)$ vanish at the vertex since they contain the factor σ^p.

We prove these assertions and complete them in certain respects. The square r^2 of the Lorentz distance is according to (2.2) and (2.1)

$$(y, y) = (ta + \rho v, ta + \rho v) = t^2 - \rho^2 = (t + \rho)^2 \frac{t - \rho}{t + \rho}.$$

Hence

$$r^2 = (y, y) = \sigma^2 \tau. \tag{2.8}$$

The same relation also follows from (2.5) and (2.7), since (2.7) gives $(b + \tau c, b + \tau c) = \tau$. The equation of the cone C^0 is $(y, y) = 0$. At the vertex $\sigma = 0$, while τ is indeterminate. On the cone, except at the vertex, $\tau = 0, \sigma > 0$. It follows from (2.4) that $0 < \tau \leqq 1$ inside the cone and that, in particular, $\tau = 1$ on the axis $y = \sigma a$ (or $\rho = 0$) and only there.

We always have $\sigma \geqq 0$, according to (2.4) and the inequalities $t \geqq 0$, $\rho \geqq 0$. The equation $\sigma = \text{const.} = \gamma > 0$, which, in view of (2.4), is equivalent to $t + \rho = \gamma$ is the equation of a *positive light cone* $C_{\gamma a}$, with the vertex γa. It is clear that the inequalities $0 \leqq \tau \leqq 1$ and $0 \leqq \sigma \leqq \gamma$ characterize the interior and the boundary of a double cone $D_{\gamma a}{}^O$ limited by the negative light cone C^0 and the positive light cone $C_{\gamma a}$.

From now on we make ample use of our hypothesis that the function $f(y)$ is well behaved (cf. p. 38). We have

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial \tau}[\sigma(b + \tau c)] = \sigma c, \quad \frac{\partial^p y}{\partial \tau^p} = 0, \quad p = 2, 3, \ldots . \tag{2.9}$$

From this it follows for any function $f(y)$

$$\frac{\partial f}{\partial \tau} = \sigma\left(\sum c^k \partial_k\right) f, \quad \text{where } \partial_k = \frac{\partial}{\partial y^k},$$

and, more generally, for any positive integer p,

$$\frac{\partial^p f}{\partial \tau^p} = \sigma^p \left(\sum c^k \partial_k\right)^p f. \tag{2.10}$$

This proves our assertion about the behaviour of the derivatives with respect to τ at the vertex.

3. The volume potential. If the surface element of the sphere S_{m-2} is denoted by dS_{m-2}, the volume element dV of the m-space can be written

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$dV = \rho^{m-2}d\rho dt dS_{m-2}$. From (2.6) we have that $\rho = \frac{1}{2}\sigma(1-\tau)$ and that the Jacobian

$$\frac{d(\rho, t)}{d(\sigma, \tau)} = \frac{1}{2}\sigma.$$

Making use of (2.5) and (2.8) we obtain after some simplifications

$$\frac{1}{H_m(\alpha)} f(y) r^{\alpha - m} dV \tag{3.1}$$
$$= \frac{2^{1-m}}{H_m(\alpha)} f[\sigma(b + \tau c)] \sigma^{\alpha-1} \tau^{\frac{1}{2}(\alpha - m)} (1-\tau)^{m-2} d\tau d\sigma dS_{m-2}.$$

In order to get $I^\alpha f(O)$, we have to integrate this expression over the domain $D_S{}^O$. However, it will be convenient to divide this domain into two parts and treat these parts separately. First we choose γ small enough, so that the double-cone $D_{\gamma a}{}^O$ should be contained in $D_S{}^O$. Then we divide the latter domain into $D_{\gamma a}{}^O$ and $D_S{}^O - D_{\gamma a}{}^O$ and show by rather different methods that the corresponding parts of the integral I^α, denoted incidentally by $I_I{}^\alpha$ and $I_{II}{}^\alpha$, are holomorphic for $\alpha > -1$ and that $I_I{}^0 = f(O)$, $I_{II}{}^0 = 0$, which gives that the original $I^0 f(O) = f(O)$. By this our *main objective* will be attained. The more difficult part of the proof, the one concerning I_I, will be carried out in the present section. The easy part I_{II} can be treated by a method similar to that used in §4 for a simple layer. Therefore it is postponed to §5. In the same section we apply our results concerning the analytic continuation of a simple and a double layer to carry out the "unlimited" analytic continuation of the volume potentials.

The integral of the right-hand side of (3.1) extended over the double-cone $D_{\gamma a}{}^O$ gives us the functional $I^\alpha f(O)$ relative to this special domain. A very great simplification arises here from the fact that the limits of integration with respect to τ and σ are fixed, τ varying between 0 and 1 and σ between 0 and γ. Thus we have in the present case

$$I^\alpha f(O) = \int_{S_{m-1}} dS_{m-2} \int_0^\gamma d\sigma \int_0^1 \ldots d\tau,$$

where the dots stand for the integrand given in the right-hand side of (3.1)

Besides the formula (1.6) for $H_m(\alpha)$ we shall need the relations

$$\int_0^1 t^{r-1}(1-t)^{s-1} dt = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \tag{3.2}$$

$$\Gamma(r) = \pi^{-\frac{1}{2}} 2^{r-1} \Gamma(\tfrac{1}{2}r)\Gamma(\tfrac{1}{2}r + \tfrac{1}{2}) \tag{3.3}$$

and the explicit expression for the total surface $|S_{m-2}|$ of the sphere S_{m-2},

$$|S_{m-2}| = \frac{2\pi^{\frac{1}{2}m - \frac{1}{2}}}{\Gamma(\frac{1}{2}m - \frac{1}{2})}. \tag{3.4}$$

We develop $f[\sigma(b + \tau c)]$ in a finite power series in τ with a remainder term. We have

$$f(y) = f[\sigma(b + \tau c)] = \sum_{p=0}^{N-1} \frac{\tau^p}{p!} \Phi_p(\sigma, v) + R_N(\tau), \tag{3.5}$$

where $\quad \Phi_p(\sigma, v) = \left\{ \frac{\partial^p}{\partial \tau^p} f[\sigma(b + \tau c)] \right\}_{\tau=0}$, and

$$R_N(\tau) = \frac{1}{(N-1)!} \int_0^\tau \frac{\partial^N}{\partial \bar{\tau}^N} f[\sigma(b + \bar{\tau} c)](\tau - \bar{\tau})^{N-1} d\bar{\tau}. \tag{3.6}$$

N is here a sufficiently large integer, to be specified later. Obviously $R_N(\tau) = O(\tau^N)$.

We first compute the expression

$$A(\alpha) = \frac{2^{1-m}}{H_m(\alpha)} \int_0^1 f[\sigma(b + \tau c)] \tau^{\frac{1}{2}(\alpha - m)} (1 - \tau)^{m-2} d\tau.$$

This integral and all the integrals which follow are convergent for $\alpha > m - 2$. On account of (3.5) we have

$$A(\alpha) = \sum_{p=0}^{N-1} A_p(\alpha) \Phi_p(\sigma, v) + \frac{2^{1-m}}{H_m(\alpha)} \int_0^1 R_N(\tau) \tau^{\frac{1}{2}(\alpha - m)} (1 - \tau)^{m-2} d\tau, \tag{3.7}$$

where by means of (3.2) with $r = \frac{1}{2}(\alpha + 2 - m) + p$, $s = m - 1$

$$A_p(\alpha) = \frac{2^{1-m} \Gamma(m-1) \Gamma(\frac{1}{2}(\alpha + 2 - m) + p)}{p! H_m(\alpha) \Gamma(\frac{1}{2}(\alpha + m) + p)}. \tag{3.8}$$

The most important term in (3.7) is $A_0(\alpha)\Phi_0(\sigma, v) = A_0(\alpha) f(\sigma b)$. In view of the expression (1.6) of $H_m(\alpha)$ we find

$$A_0(\alpha) = \frac{2^{1-m} \Gamma(m-1)}{\pi^{\frac{1}{2}(m-2)} 2^{\alpha - 1} \Gamma(\frac{1}{2}\alpha) \Gamma(\frac{1}{2}(\alpha + m))}. \tag{3.9}$$

Expressing $2^{\alpha-1}\Gamma(\frac{1}{2}\alpha)$ by means of (3.3), with $r = \alpha$, we find after some simplifications

$$A_0(\alpha) = K_0(\alpha) \cdot \frac{1}{\Gamma(\alpha)}, \text{ where } K_0(\alpha) = \frac{2^{1-m} \Gamma(m-1) \Gamma(\frac{1}{2}\alpha + \frac{1}{2})}{\pi^{\frac{1}{2}(m-1)} \Gamma(\frac{1}{2}(\alpha + m))}. \tag{3.10}$$

Since $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$, we have

$$K_0(0) = \frac{2^{1-m} \Gamma(m-1)}{\pi^{\frac{1}{2}(m-2)} \Gamma(\frac{1}{2}m)}. \tag{3.11}$$

According to (3.3), with $r = m - 1$, and to (3.4)

$$K_0(0) = \frac{\Gamma(\frac{1}{2}m - \frac{1}{2})}{2\pi^{(\frac{1}{2}m - \frac{1}{2})}} = \frac{1}{|S_{m-2}|}. \tag{3.12}$$

Our next step is to carry out the analytic continuation of the expression

$$A_0(\alpha) \cdot \int_0^\gamma \Phi_0(\sigma, v) \sigma^{\alpha - 1} d\sigma = K_0(\alpha) \cdot \frac{1}{\Gamma(\alpha)} \int_0^\gamma f(\sigma b) \sigma^{\alpha - 1} d\sigma. \tag{3.13}$$

The integral converges for $\alpha > 0$ and, according to what we know about the *one-dimensional case* (cf. p. 39), the analytic continuation of (3.13) is holomorphic for $\alpha > -1$. (For $\alpha = -1$ the function $\Gamma(\frac{1}{2}(\alpha + 1))$ has a pole.) For

$\alpha = 0$ the expression (3.13) becomes $K_0(0)\, f(O) = f(O)/|S_{m-2}|$. The integral of this constant value with respect to the angular variable is clearly $f(O)$. Hence, and this contains virtually our main result, the term corresponding to $p = 0$ in the I^α relative to the double-cone $D_{\gamma a}{}^O$ *yields exactly* $f(O)$ *for* $\alpha = 0$.

The terms in (3.7) with $p > 0$ are easy to handle. In analogy with (3.13) we have to consider the expression

$$A_p(\alpha) \,.\, \int_0^\gamma \Phi_p(\sigma, v)\sigma^{\alpha-1}d\sigma, \quad p \geqq 1, \tag{3.14}$$

where $A_p(\alpha)$ is given in (3.8). We first note that

$$\Gamma(\tfrac{1}{2}(\alpha + 2 - m) + p) = \Gamma(\tfrac{1}{2}(\alpha + 2 - m))P_p(\alpha),$$

where $P_p(\alpha)$ is a polynomial of degree p in α. Hence, after the same simplifications as those performed for $A_0(\alpha)$ we obtain

$$A_p(\alpha) = K_p(\alpha) \,.\, \frac{1}{\Gamma(\alpha)}\,, \text{ where } K_p(\alpha) = \frac{2^{1-m}\Gamma(m-1)\Gamma(\frac{1}{2}(\alpha+1))}{p!\,\pi^{(\frac{1}{2}m-\frac{1}{2})}\Gamma(\frac{1}{2}(\alpha+m)+p)} \,.\, P_p(\alpha).$$

According to (2.10) $\partial^p/\partial\tau^p$ contains the factor σ^p, $p \geqq 1$. Hence all the integrals of the type given in (3.14) converge if $\alpha > -1$. Moreover, $K_p(0)$ is finite, $1/\Gamma(0) = 0$, consequently all expressions (3.14) vanish for $\alpha = 0$.

Since $R_N = O(\tau^N\sigma^N)$, the remainder term can be treated in an analogous way. It is clear that for the present purposes N may be any integer such that $-\frac{1}{2}(1 + m) + N \geqq -1$ or $2N \geqq m - 1$.

Summing up, it is now shown that the integral $I^\alpha f(O)$ extended to the double cone can be analytically continued to all values $\alpha > -1$ and that for these values it is a holomorphic function of α. Moreover $I^0 f(O) = f(O)$, that is I^0 *is the identity operator in the case of the double cone.*

We could have gone a bit farther and established the possibility of the analytic continuation down to arbitrary negative values of α. However, one difficulty would have remained, the possible occurrence of (simple) poles at the negative odd integers = the poles of $\Gamma(\frac{1}{2}(\alpha + 1))$. As a matter of fact, none of these poles actually occurs. Their disappearance must be the effect of the integration with respect to the angular variable, considered here in a very summary way. On p. 64 of my *Acta* paper I indicate how the holomorphic character of the unlimited analytic continuation can be established by an indirect method. This will be carried out here in §5.

4. Simple layer and double layer. We now pass to the simple layer

$$\frac{1}{H_m(\alpha)} \int_{S^O} g(y) r^{\alpha-m} dS \tag{4.1}$$

considered in formula (1.7), where now the vertex coincides with the origin and r is written instead of r_{xy}. The integral converges for $\alpha > m - 2$ and has to be continued analytically for $\alpha \leqq m - 2$. The portion S^O of the surface S can be parametrized by the variables τ and v in the following way. Through

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every point of S passes a unique ray issuing from the origin. On such a ray τ and v are constant, hence also the vectors b and c corresponding to v are constant, while σ varies. If we write the point of intersection of the ray with the surface S in the form $y = \sigma_S(b + \tau c)$, the equations $\sigma = \sigma_S(\tau, v)$ or $y = \sigma_S(\tau, v)(b + \tau c)$ and the additional condition $0 \leqq \tau \leqq 1$ yield the required parametrization of S^O, because b and c depend only on v.

Since v is indeterminate on the axis $y = \sigma a$, where $\tau = 1$, we divide S^O into two parts $S_A{}^O$ and $S_B{}^O$ in the following way. With an arbitrary δ such that $0 < \delta < 1$ the first part will be given by $0 \leqq \tau \leqq \delta$ and the second by $\delta < \tau \leqq 1$.

That part of the simple layer which relates to $S_B{}^O$ is an entire function of α vanishing for all even integers $\leqq 0$. Indeed the corresponding part of the integral in (4.1) never ceases to converge and $H_m(\alpha)$ has poles at these integers owing to the factor $\Gamma(\frac{1}{2}\alpha)$ (cf. formula (1.6)).

In order to treat that part of (4.1) which is taken over $S_A{}^O$, we have to express the surface element dS in a convenient way. The angular variable v on the sphere S_{m-2} can be expressed by $m - 2$ local parameters $\phi^1, \phi^2, \ldots, \phi^{m-2}$. Thus, according to (1.4), we can write in summary notations

$$dS = G(\tau, v) \,.\, d\tau \,.\, \prod d\phi^i,\; dS_{m-2} = \Theta(v) \,.\, \prod d\phi^i,$$

hence $dS = H(\tau, v) d\tau dS_{m-2}$.

We set $\frac{1}{2}(\alpha + 2 - m) = \beta$ and can then write according to (1.6)

$$H_m(\alpha) = L_m(\alpha)\Gamma(\beta) \text{ where } L_m(\alpha) = \pi^{\frac{1}{2}m-1}2^{\alpha-1}\Gamma(\tfrac{1}{2}\alpha). \tag{4.2}$$

We also set $g(y) = g(\tau, v)$ and recall the relation $r^2 = \sigma^2\tau$ given in (2.8). Then we write that part of (4.1) which corresponds to $S_A{}^O$ in the form

$$U(\alpha) = \frac{1}{L_m(\alpha)} \int_{S_{m-2}} dS_{m-2} \frac{1}{\Gamma(\beta)} \int_0^\delta g(\tau, v) H(\tau, v) [\sigma_S(\tau, v)]^{\alpha-m} \tau^{\beta-1} d\tau. \tag{4.3}$$

Here $H(\tau, v)$, $\sigma_S(\tau, v)$, $g(\tau, v)$ are well-behaved even in τ and v by virtue of our hypothesis concerning the surface S and the function $g(y)$. Moreover σ_S is bounded away from 0. Hence we can apply our statement concerning the *extended one-dimensional case* (cf. p. 40), which gives that $(1/(\Gamma(\beta))\int_0^\delta \ldots$ is a holomorphic function of β, hence also of α, and this is then also true for $U(\alpha)$. Owing to the presence of the factor $\Gamma(\frac{1}{2}\alpha)$ in $L_m(\alpha)$, the function $U(\alpha)$ vanishes for $\alpha = 0$ and $\alpha =$ a negative even integer.

There is very little to change in the case of a double layer. It is easily seen that $dr/dn = r^{-1}(y, n)$ hence

$$\frac{dr^{\alpha-m}}{dn} = (\alpha - m) r^{\alpha-m-1} \frac{dr}{dn} = (\alpha - m) r^{\alpha-m-2}(y, n).$$

The scalar product (y, n) (cf. (1.3)) is a well behaved function in τ and v. Owing to the lowered exponent the double layer integral in (1.7) converges only for $\alpha > m$, thus the need of continuation begins already at m.

The part relative to $S_B{}^O$ is again an entire function of α and vanishes for all even integers $\leqq 0$. On the other hand, with $\beta' = \frac{1}{2}(\alpha - m) = \beta - 1$,

$$\frac{\alpha - m}{\Gamma(\beta)} r^{\alpha - m} = \frac{2(\beta - 1)}{\Gamma(\beta)} r^{\beta - 1} = \frac{2}{\Gamma(\beta - 1)} r^{\beta - 1} = \frac{2}{\Gamma(\beta')} r^{\beta'}.$$

Thus, when treating the part relative to $S_A{}^O$, we obtain a formula of the same type as (4.3), $\beta' = \beta - 1$ playing for the double layer the same role as β played for the simple layer, and the results are essentially the same.

Our findings can be summed up as follows. *Both the simple layer and the double layer potentials can be continued analytically to arbitrary values of α. They are holomorphic functions of α which vanish for all even integers* $\leqq 0$.

5. The volume potential (continued). Now we return to the volume potential and clarify the properties of the part which relates to the domain $D_S{}^O - D_{\gamma_\prime}{}^O$ (cf. the beginning of §3). We want to prove that this part of I^α, when continued analytically, is holomorphic for $\alpha > -1$, and vanishes for $\alpha = 0$.

In the same way as we did in the previous section with the portion of surface S^O, we now divide the domain $D_S{}^O - D_{\gamma a}{}^O$ into two parts according as $0 \leqq \tau \leqq \delta$ or $\delta < \tau \leqq 1$. The volume potential relative to the second part is again an entire function vanishing for all even integers $\leqq 0$. Thus we only have to investigate the first part. This can be written in the form (cf. (3.1.))

$$\frac{2^{1-m}}{H_m(\alpha)} \int_{S_{m-2}} dS_{m-2} \int_0^\delta \tau^{\frac{1}{2}(\alpha - m)} (1 - \tau)^{m-2} d\tau \int_\gamma^{\sigma_s} f[\sigma(b + \tau c)] \sigma^{\alpha - 1} d\sigma,$$

where σ_S, defined in the previous section, depends on τ and v, $\sigma_S = \sigma_S(\tau, v)$. We set

$$\int_\gamma^{\sigma_s} f[\sigma(b + \tau c)] \sigma^{\alpha - 1} d\sigma = F(\tau, v, \alpha),$$

where F is well behaved in τ (and v) and holomorphic in α, since σ is bounded away from 0. Writing, as in the case of a simple layer, $H_m(\alpha) = L_m(\alpha)\Gamma(\beta)$ we see by virtue of our findings in the extended one-dimensional case that

$$\frac{1}{L_m(\alpha)} \frac{1}{\Gamma(\beta)} \int_0^\delta F(\tau, v, \alpha) \tau^{\beta - 1} (1 - \tau)^{m-2} d\tau$$

can be continued analytically as a holomorphic function of α to any $\alpha > \alpha_0$, where α_0 is arbitrary. Thus it is a holomorphic function for $\alpha > -1$ any way, and again, owing to the presence of the factor $\Gamma(\alpha/2)$ in $L_m(\alpha)$, it vanishes for $\alpha = 0$.

This completes the proof of the fact stated in §3, that $I^0 f(O) = f(O)$, if I^0 is relative to the *original domain* $D_S{}^O$. This can clearly be expressed by the more inspiring formula

$$I^0 f(x) = f(x), \tag{5.1}$$

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or by the statement that I^0 *is the identity operator*. This is *our principal result.* The passage from (5.1) to the relation

$$I_*^0\overline{f, g, h}\,(x) = f(x) \tag{5.2}$$

follows from the properties of simple and double layers established in the previous section.

We conclude this paper by two additional remarks, the first concerning the unlimited analytic continuation of the volume potential, the second concerning the case of the infinite cone.

As indicated on p. 64 of our *Acta* paper, the unlimited holomorphic continuation of the volume potential can be reduced to that of simple layer and double layer potentials.

We consider the wave operator

$$\Delta = \frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{(\partial x^1)^2} - \dots - \frac{\partial^2}{(\partial x^{m-1})^2}\,.$$

Then, by virtue of Green's formula, we have in the notation (1.7) for $\alpha > m - 2$

$$I^\alpha f(x) = I_*^{\alpha+2}\,\overline{\Delta f, \frac{df}{dn}, f(x)} \tag{5.3}$$

(see *Acta* paper, pp. 46–47). The left-hand side converges for $\alpha > m - 2$. On the right-hand side the volume potential and the simple layer converge for $\alpha + 2 > m - 2$, that is for $\alpha > m - 4$, while the double layer converges only for $\alpha + 2 > m$, that is only for $\alpha > m - 2$. Thus, seemingly nothing is gained as far as the analytic continuation of $I^\alpha f(x)$ is concerned. But if we take into account our results about the unlimited holomorphic continuation of the simple and the double layer and the fact that $I^{\alpha+2}\Delta$ is holomorphic for $\alpha > m - 4$, the possibility of the holomorphic continuation of $I^\alpha f(x)$ down to $m - 4$ is established. The iteration of this procedure, that is the application of formula (5.3) to $I^{\alpha+2}\Delta f$, $I^{\alpha+4}\Delta^2 f, \dots,$ establishes the possibility of an unlimited holomorphic continuation of $I^\alpha f(x)$.

In the case of an infinite cone we suppose that $f(x)$ and its derivatives are not only well behaved, but also decrease rapidly enough at infinity. In this case formula (5.3) reduces to

$$I^\alpha f(x) = I^{\alpha+2}\Delta f(x), \tag{5.4}$$

and the iteration of this formula gives immediately the possibility of an unlimited holomorphic continuation. However, in order to establish the main relation $I^0 f(x) = f(x)$, we still have to go back to §3 and use the double-cone $D_{\gamma a}{}^O$. The treatment of the complementary expression $I_{II}{}^\alpha$ is in the infinite case still simpler than in the finite case.

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