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AN ELEMENTARY PROOF OF JAMES' CHARACTERISATION OF WEAK COMPACTNESS. II

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Abstract

In this paper we provide an elementary proof of James' characterisation of weak compactness for Banach spaces whose dual ball is weak* sequentially compact.

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1. Introduction

In the paper [5], the first author gave a simple proof of James' theorem on weak compactness for Banach spaces whose dual ball is weak* sequentially compact. This class of spaces is quite large because, in addition to all the separable Banach spaces (whose dual ball is weak* metrisable), it contains all Asplund spaces [4] (that is, spaces in which every separable subspace has a separable dual space) and all spaces that admit an equivalent smooth norm [2] (which includes all weakly compactly generated spaces [1]). In fact, it contains all Gâteaux differentiability spaces [4]. On the other hand, it does not contain $\ell_\infty(\mathbb{N})$. However, the proof in [5] still relied upon the Krein–Milman theorem, Milman's theorem and the Bishop–Phelps theorem. In this paper we obtain the same result but rely only upon the Hahn–Banach theorem and convexity. The idea of the proof comes from [6, Lemmas 4 and 5] and [3, Lemma 2]. For any x in a normed linear space X, we shall define $\widehat{x} \in X^{**}$ by $\widehat{x}(x^*) := x^*(x)$ for all $x^* \in X^*$. Then $x \mapsto \widehat{x}$ is a linear isometric embedding of X into X^{**}. In particular, if X is a Banach space, then $\widehat{X}$ is a closed linear subspace of X^{**}.

Let K be a weak* compact convex subset of the dual of a Banach space X. A subset B of K is called a *boundary* of K if for every $\widehat{x} \in \widehat{X}$ there exists a $b^* \in B$ such that $\widehat{x}(b^*) = \sup\{\widehat{x}(y^*) : y^* \in K\}$. We shall say B (I)-*generates* K if, for every countable cover $\{C_n\}_{n\in\mathbb{N}}$ of B by weak* compact convex subsets of K, the convex hull of $\bigcup_{n\in\mathbb{N}} C_n$ is norm dense in K.

2. Preliminaries

The main theorem relies upon the following prerequisite results.

Lemma 2.1. *Let* $0 < \beta$, $0 < \beta'$ *and suppose that* $\varphi : [0, \beta + \beta'] \to \mathbb{R}$ *is a convex function. Then*

$$\frac{\varphi(\beta) - \varphi(0)}{\beta} \leq \frac{\varphi(\beta + \beta') - \varphi(\beta)}{\beta'}.$$

Proof. The inequality given in the statement of the lemma follows by rearranging the inequality

$$\varphi(\beta) \leq \frac{\beta}{\beta + \beta'}\varphi(\beta + \beta') + \frac{\beta'}{\beta + \beta'}\varphi(0).$$ □

Lemma 2.2. *Let* V *be a vector space (over* $\mathbb{R}$*) and let* $\varphi : V \to \mathbb{R}$ *be a convex function. If* $(A_n)_{n\in\mathbb{N}}$ *is a decreasing sequence of nonempty convex subsets of* V, $(\beta_n)_{n\in\mathbb{N}}$ *is any sequence of strictly positive numbers,* $r \in \mathbb{R}$ *and*

$$\beta_1 r + \varphi(0) < \inf_{a\in A_1} \varphi(\beta_1 a),$$

then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ *in* V *such that, for all* $n \in \mathbb{N}$:

(i) $a_n \in A_n$; *and*
(ii) $\varphi(\sum_{i=1}^{n} \beta_i a_i) + \beta_{n+1} r < \varphi(\sum_{i=1}^{n+1} \beta_i a_i)$.

Proof. We proceed to prove the lemma in two parts. Firstly we prove that if $u \in V$ and $\beta_n r + \varphi(u) < \inf_{a\in A_n} \varphi(u + \beta_n a)$ for some $n \in \mathbb{N}$, then there exists an $a_n \in A_n$ such that

$$\beta_{n+1} r + \varphi(u + \beta_n a_n) < \inf_{a\in A_n} \varphi(u + \beta_n a_n + \beta_{n+1} a).$$

To see this, suppose that $u \in V$ and that $\beta_n r + \varphi(u) < \inf_{a\in A_n} \varphi(u + \beta_n a)$. Then there exists an $0 < \varepsilon$ such that

$$r + 2\varepsilon < \frac{\inf_{a\in A_n} \varphi(u + \beta_n a) - \varphi(u)}{\beta_n}. \tag{2.1}$$

So, choose $a_n \in A_n$ such that $\varphi(u + \beta_n a_n) < \inf_{a\in A_n} \varphi(u + \beta_n a) + \beta_{n+1}\varepsilon$. Let $a \in A_n$. Then $v := (\beta_n a_n + \beta_{n+1} a)/(\beta_n + \beta_{n+1}) \in A_n$ and so

$$\begin{aligned} r + 2\varepsilon &< \frac{\varphi(u + \beta_n v) - \varphi(u + 0v)}{\beta_n} && \text{by (2.1) and the fact that } v \in A_n \\ &\leq \frac{\varphi(u + (\beta_n + \beta_{n+1})v) - \varphi(u + \beta_n v)}{\beta_{n+1}} && \text{by Lemma 2.1.} \end{aligned}$$

Rearranging gives $\beta_{n+1}(r + \varepsilon) + [\varphi(u + \beta_n v) + \beta_{n+1}\varepsilon] < \varphi(u + \beta_n a_n + \beta_{n+1} a)$ for all $a \in A_n$. Since $\varphi(u + \beta_n a_n) < [\varphi(u + \beta_n v) + \beta_{n+1}\varepsilon]$, the desired inequality follows.

From this, we may inductively construct a sequence $(a_n)_{n\in\mathbb{N}}$. For the first step, we set $u := 0$ and then, by the hypothesis, $\beta_1 r + \varphi(0) < \inf_{a\in A_1} \varphi(\beta_1 a) = \inf_{a\in A_1} \varphi(0 + \beta_1 a)$. So, by the above, there is an $a_1 \in A_1$ such that $\beta_2 r + \varphi(\beta_1 a_1) < \inf_{a\in A_1} \varphi(\beta_1 a_1 + \beta_2 a)$.

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For the nth step, set $u := \sum_{i=1}^{n-1} \beta_i a_i$. Since $A_n \subseteq A_{n-1}$ and by the way a_{n-1} was constructed, $\beta_n r + \varphi(u) < \inf_{a \in A_{n-1}} \varphi(u + \beta_n a) \le \inf_{a \in A_n} \varphi(u + \beta_n a)$. So, by the first result again, there exists $a_n \in A_n$ such that

$$\beta_{n+1} r + \varphi\Big(\sum_{i=1}^{n} \beta_i a_i\Big) < \inf_{a \in A_n} \varphi\Big(\sum_{i=1}^{n} \beta_i a_i + \beta_{n+1} a\Big),$$

which completes the induction. The sequence $(a_n)_{n\in\mathbb{N}}$ has the properties claimed above. □

3. The main theorem

THEOREM 3.1. *Let K be a weak* compact convex subset of the dual of a Banach space X and let B be a boundary of K. Then B (I)-generates K.*

PROOF. After possibly translating K, we may assume that $0 \in B$. Let $\{C_n\}_{n\in\mathbb{N}}$ be weak* compact convex subsets of K such that $B \subseteq \bigcup_{n\in\mathbb{N}} C_n$ and suppose, for a contradiction, that $\mathrm{co}[\bigcup_{n\in\mathbb{N}} C_n]$ is not norm dense in K. Then there must exist $0 < \varepsilon$ and $y^* \in K$ such that

$$y^* \in K \backslash \Big(\mathrm{co}\Big[\bigcup_{n\in\mathbb{N}} C_n\Big] + \varepsilon B_{X^*}\Big), \quad \text{where } B_{X^*} := \{x^* \in X^* : \|x^*\| \le 1\}.$$

Since, for all $n \in \mathbb{N}$, $\mathrm{co}[\bigcup_{j=1}^{n} C_j]$ is weak* compact and convex, there exist $(\widehat{x}_n)_{n\in\mathbb{N}}$ in $\widehat{X}$ such that for every $n \in \mathbb{N}$, $\|\widehat{x}_n\| = 1$ and

$$\max\Big\{\widehat{x}_n(x^*) : x^* \in \mathrm{co}\Big[\bigcup_{j=1}^{n} C_j\Big]\Big\} + \varepsilon$$
$$= \max\Big\{\widehat{x}_n(x^*) : x^* \in \mathrm{co}\Big[\bigcup_{j=1}^{n} C_j\Big] + \varepsilon B_{X^*}\Big\} < \widehat{x}_n(y^*). \tag{3.1}$$

Now $(\widehat{x}_n(y^*))_{n\in\mathbb{N}}$ is a bounded sequence of real numbers and thus has a convergent subsequence $(\widehat{x}_{n_k}(y^*))_{k\in\mathbb{N}}$. Let $s := \lim_{k\to\infty} \widehat{x}_{n_k}(y^*)$. Then $\varepsilon \le s$ and, after relabelling the sequence $(\widehat{x}_n)_{n\in\mathbb{N}}$ if necessary, we may assume that $|\widehat{x}_n(y^*) - s| < \varepsilon/3$ for all $n \in \mathbb{N}$. Note that this relabelling does not disturb the inequality in (3.1).

We define $A_n := \mathrm{co}\{\widehat{x}_k : n \le k\}$ for all $n \in \mathbb{N}$ and note that:

(i) $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty convex subsets of $\widehat{X}$; and

(i) if $N < n$ and $b^* \in C_N$, then

$$g(b^*) < [g(y^*) - \varepsilon] \quad \text{for all } g \in A_n \tag{3.2}$$

since $\{\widehat{x}_k : n \le k\} \subseteq \{\widehat{x} \in \widehat{X} : \widehat{x}(b^* - y^*) < -\varepsilon\}$, which is convex.

Next, we define $p : \widehat{X} \to \mathbb{R}$ by

$$p(\widehat{x}) := \sup_{x^* \in K} \widehat{x}(x^*) \quad \text{for all } \widehat{x} \in \widehat{X}.$$

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Then p is a convex functional on $\widehat{X}$ such that $p(0) = 0$. Moreover, for all $g \in A_1$, we have $(s - \varepsilon/3) < g(y^*) \le p(g)$ since $\{\widehat{x_n}\}_{n\in\mathbb{N}} \subseteq \{\widehat{x} \in \widehat{X} : (s - \varepsilon/3) < \widehat{x}(y^*)\}$, which is convex, and $y^* \in K$.

Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty}(\sum_{i=n+1}^{\infty}\beta_i)/\beta_n = 0$. Now $\beta_1(s - \varepsilon/2) + p(0) < \beta_1(s - \varepsilon/3) \le \beta_1[\inf_{g\in A_1} p(g)] = \inf_{g\in A_1} p(\beta_1 g)$.

Therefore, by Lemma 2.2, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\widehat{X}$ such that $g_n \in A_n$ and

$$p\Big(\sum_{i=1}^{n}\beta_i g_i\Big) + \beta_{n+1}(s - \varepsilon/2) < p\Big(\sum_{i=1}^{n+1}\beta_i g_i\Big) \quad \text{for all } n \in \mathbb{N}.$$

Since $\|g_n\| \le 1$ for all $n \in \mathbb{N}$, we have $\sum_{i=1}^{\infty}\|\beta_i g_i\| \le \sum_{i=1}^{\infty}\beta_i < \infty$. As X is a Banach space, this implies that $g := \sum_{i=1}^{\infty}\beta_i g_i \in \widehat{X}$ and so there exists a $b^* \in B$ such that $p(g) = g(b^*)$. Then

$$\begin{aligned}\beta_n(s - \varepsilon/2) &< p\Big(\sum_{i=1}^{n}\beta_i g_i\Big) - p\Big(\sum_{i=1}^{n-1}\beta_i g_i\Big) \le p(g) - p\Big(\sum_{i=1}^{n-1}\beta_i g_i\Big)\\ &\le g(b^*) - \sum_{i=1}^{n-1}\beta_i g_i(b^*) = \sum_{i=n}^{\infty}\beta_i g_i(b^*).\end{aligned}$$

Since $B \subseteq \bigcup_{n\in\mathbb{N}} C_n$, $b^* \in C_N$ for some $N \in \mathbb{N}$. Thus, if $N < n$,

$$(s - \varepsilon/2) < \frac{1}{\beta_n}\Big(\sum_{i=n+1}^{\infty}\beta_i g_i(b^*)\Big) + g_n(b^*) < \frac{1}{\beta_n}\Big(\sum_{i=n+1}^{\infty}\beta_i g_i(b^*)\Big) + [g_n(y^*) - \varepsilon],$$

by (3.2), since $g_n \in A_n$. By taking the limit as n tends to infinity, we find that $(s - \varepsilon/2) \le (s - \varepsilon)$, which is impossible. Therefore, B (I)-generates K. □

Remark 3.2. If $\beta_n := 1/n!$ for all $n \in \mathbb{N}$, or $\beta_n := 1/2^{n^2}$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty}\frac{\sum_{i=n+1}^{\infty}\beta_i}{\beta_n} = 0.$$

We will say that a subset C of a Banach space X is *weakly compactly generated* if for every $0 < \varepsilon$ there exists a countable family $\{C_n^{\varepsilon}\}_{n\in\mathbb{N}}$ of weakly compact convex subsets of X such that $C \subseteq [\bigcup_{n\in\mathbb{N}} C_n^{\varepsilon}] + \varepsilon B_X$. Here, B_X denotes the closed unit ball in the Banach space X. Our first compactness result is based upon the following observation: for each $\mathcal{F} \in X^{***}$, there exists an $x^* \in X^*$ such that $\mathcal{F}|_{\widehat{X}} = \widehat{x^*}|_{\widehat{X}}$. In this way we see that the relative weak topology on $\widehat{X}$ coincides with the relative weak* topology on $\widehat{X}$. In particular, each weak* compact subset of $\widehat{X}$ is weakly compact (and, of course, *vice versa*).

Corollary 3.3. *Let C be a closed and bounded convex subset of a Banach space X. If C is weakly compactly generated and every continuous linear functional on X attains its supremum over C, then C is weakly compact.*

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PROOF. Let $K := \overline{\widehat{C}}^{w^*}$. To show that C is weakly compact, it is sufficient to show that for every $0 < \varepsilon$, $K \subseteq \widehat{X} + 2\varepsilon B_{X^{**}}$. To this end, fix $0 < \varepsilon$ and let $\{C_n^\varepsilon\}_{n\in\mathbb{N}}$ be any countable family of weakly compact convex subsets of X such that $C \subseteq [\bigcup_{n\in\mathbb{N}} C_n^\varepsilon] + \varepsilon B_X$. For each $n \in \mathbb{N}$, let $K_n^\varepsilon := K \cap [\widehat{C_n^\varepsilon} + \varepsilon B_{X^{**}}]$. Then $\{K_n^\varepsilon\}_{n\in\mathbb{N}}$ is a cover of $\widehat{C}$ by weak* closed convex subsets of K. Since $\widehat{C}$ is a boundary of K, $K \subseteq \overline{\text{co}} \bigcup_{n\in\mathbb{N}} K_n^\varepsilon \subseteq \widehat{X} + 2\varepsilon B_{X^{**}}$. □

The author in [5] used Theorem 3.1 to give a short proof of the following result.

COROLLARY 3.4 [5, Theorem 3]. *Let C be a closed and bounded convex subset of a Banach space X. If $(B_{X^*}, weak^*)$ is sequentially compact and every continuous linear functional on X attains its supremum over C, then C is weakly compact.*

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