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Accelerating self-modulated nonlinear waves in weakly and strongly magnetized relativistic plasmas

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It is known that a nonlinear Schrödinger equation describes the self-modulation of a large amplitude circularly polarized wave in relativistic electron–positron plasmas in the weakly and strongly magnetized limits. Here, we show that such an equation can be written as a modified second Painlevé equation, producing accelerated propagating wave solutions for those nonlinear plasmas. This solution even allows the plasma wave to reverse its direction of propagation. The acceleration parameter depends on the plasma magnetization. This accelerating solution is different to the usual soliton solution propagating at constant speed.

Key words: plasma waves, plasma nonlinear phenomena, plasma properties

1. Introduction

One of the nonlinear effects present in a relativistic hot magnetized electron–positron plasma is the self-modulation of circularly polarized electromagnetic waves or Alfvén waves. The large amplitude of the electromagnetic wave, and a background magnetic field, can modify the relativistic motion of particles in a significant way. Thus, the self-modulation depends on the magnetization of the plasma. This nonlinear process has been thoroughly studied for the weakly magnetized plasma and strongly magnetized plasma cases in Asenjo *et al.* (2012) and López *et al.* (2013), respectively.

Interestingly, the self-modulation gives origin to soliton plasma wave solutions, propagating at constant speed (Asenjo *et al.* 2012; López *et al.* 2013). However, there exists another kind of accelerating plasma wave solution that deserves to be explored for relativistic plasmas. This work is devoted to showing that there are nonlinear plasma waves that accelerate due to the self-modulation of the magnetized plasma system.

Relativistic plasma wave modes, presenting accelerating behaviour, have been recently introduced in Li, Li & Wang (2016), Winkler, Vásquez-Wilson & Asenjo (2023) and Minovich *et al.* (2014). Differently to those works, here, we study the case of self-modulation of a circularly polarized electromagnetic or Alfvén wave in a weakly or strongly magnetized relativistic electron–positron plasma with finite temperature. For these cases, the nonlinear Schrödinger equation that models such systems has been found

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to be (Asenjo *et al.* 2012; López *et al.* 2013)

$$\mathrm{i}\frac{\partial a}{\partial t} + P\frac{\partial^2 a}{\partial z^2} + Q|a|^2 a = 0, \tag{1.1}$$

where $a = a(t, z)$ is the time- and space-dependent complex modulational amplitude of the circularly polarized electromagnetic wave, under the approximation of a slowly time-varying modulation (Asenjo *et al.* 2012; López *et al.* 2013). Here, $P = c^2/(2\omega)$ is independent of the magnetization, where c is the speed of light, and ω is the frequency of the wave. Differently, Q depends on the limit case of magnetization of the plasma. In the weakly magnetized case, when $\omega \gg \Omega_c$ (where is Ω_c is the plasma cyclotron frequency), we have that (Asenjo *et al.* 2012)

$$Q = \frac{3\lambda\omega_p^2\Omega_c^2}{\omega^3 f^5}, \tag{1.2}$$

where $\lambda = e^2/m^2c^4$ (with the electron charge e and mass m), ω_p is the plasma frequency of the electron–positron plasma and f is the thermodynamic function relating the plasma density enthalpy per unit of mass and unit of number density (being a function of temperature). On the other hand, for the strongly magnetized plasma case, when $\omega \ll \Omega_c$, we find that (López *et al.* 2013)

$$Q = \frac{f\lambda\omega_p^2\omega^3}{4\Omega_c^4}. \tag{1.3}$$

It is clear that it is the background magnetic field, through the cyclotron frequency, the physical quantity that induces the nonlinear behaviour of the waves.

Furthermore, any other intermediate case for plasma magnetization (when ω is of the same order as Ω_c) does not produce a simple nonlinear Schrödinger equation such as (1.1). In these cases, the nonlinear effects cannot be straightforwardly analysed from relativistic factors of the plasma fluid velocities. Thus, the plasma system is more difficult to study as there are no analytical solutions for the plasma velocities (Asenjo *et al.* 2012; López *et al.* 2013).

2. Non-accelerating soliton

Equation (1.1) is usually solved in terms of a soliton amplitude propagating with constant speed v. This solution can be found by requiring that the amplitude of the electromagnetic wave has the form $|a(t, z)| = |a(z - vt)|$. Straightforwardly, it is found that such solution has the form of a soliton (Asenjo *et al.* 2012; López *et al.* 2013)

$$a(t, z) = \mathrm{sech}\left(\sqrt{\frac{Q}{2P}}(z - vt)\right)\exp\left(\mathrm{i}\frac{v}{2P}z - \mathrm{i}\left(\frac{v^2}{4P} - \frac{Q}{2}\right)t\right). \tag{2.1}$$

3. Accelerating solution

However, (1.1) can be also solved for an accelerating wave, i.e. we look for solution with amplitude in the form $|a(t, z)| = |a(z - vt - \beta t^2/2)|$, where v and β play the roles of an initial velocity and the acceleration of the propagation.

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Similar to the case of the previous section, this kind of solution can be studied by assuming the form of the plasma wavepacket as

$$a(t,z) = f(\xi)\exp(\mathrm{i}\eta(t,z)), \tag{3.1}$$

where f is a function to be determined through (1.1), and depending on the accelerated argument

$$\xi = \sqrt{\frac{Q}{2P}}\left(z - vt - \frac{\beta}{2}t^2\right). \tag{3.2}$$

Notice that we have assumed, in analogy with the constant velocity soliton solution, the same factor $\sqrt{Q/2P}$ for the argument. Also, the phase function $\eta(t,z)$ must be determined.

Using (3.1) in (1.1), we can find that when the acceleration is

$$\beta = Q\sqrt{\frac{QP}{2}}, \tag{3.3}$$

and the phase is

$$\eta(t,z) = \frac{Q}{2}\sqrt{\frac{Q}{2P}}\left(tz + \frac{v}{Q}\sqrt{\frac{2}{QP}}z - \frac{v^2}{Q^2}\sqrt{\frac{Q}{2P}}t - vt^2 - \frac{Q}{3}\sqrt{\frac{QP}{2}}t^3\right), \tag{3.4}$$

then, the function f fulfils a modified second Painlevé equation (Clarkson 2003)

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \xi f + 2f^3 = 0. \tag{3.5}$$

Solutions of this equation can be generally studied numerically, but they are no longer solitons. For the current case, they describe an accelerated propagation of a electromagnetic plasma wave train, with non-constant amplitude and with acceleration $\beta = \sqrt{Q^3P/2}$ along its direction of propagation (the velocity v is arbitrary). The acceleration of this nonlinear plasma wave depends on the magnetization of the plasma through Q, given in (1.2) and (1.3). We emphasize that this kind of accelerated behaviour for the plasma is only possible in the weakly and strongly magnetized limits.

3.1. *Case for $v = 0$*

In order to show the accelerating behaviour of this nonlinear plasma solution, we display in figure 1 the density plot for the numerical solution of (3.5) for $f(\xi = 0) = 1$, $d_\xi f(\xi = 0) = 0$ and $v = 0$. This plot shows the magnitude of function f in the $t - z$ space, in terms of normalized time $t' = Qt/\sqrt{2}$, and normalized distance $z' = \sqrt{Q/2P}z$. The solution shows curved (parabolic) trajectories for any part of the wave (maxima or minima). This can be explicitly seen through the red dashed lines, that are used as examples. Those lines correspond to $\xi = z' - t'^2/2 = \xi_0$, for $\xi_0 = -10, -4, 0, 5, 10$. All the red dashed parabolic curves coincide with the dynamics of the plasma wave. Therefore, we conclude that the whole plasma wave propagates with acceleration β given by (3.3).

3.2. *Case for velocity v parallel to the acceleration*

In this case, the velocity v produces only a modification of the initial behaviour of the solution, accelerating along the same direction. As acceleration (3.3) is positive, we graphically display this solution with $v > 0$. In figure 2, we show a numerical solution

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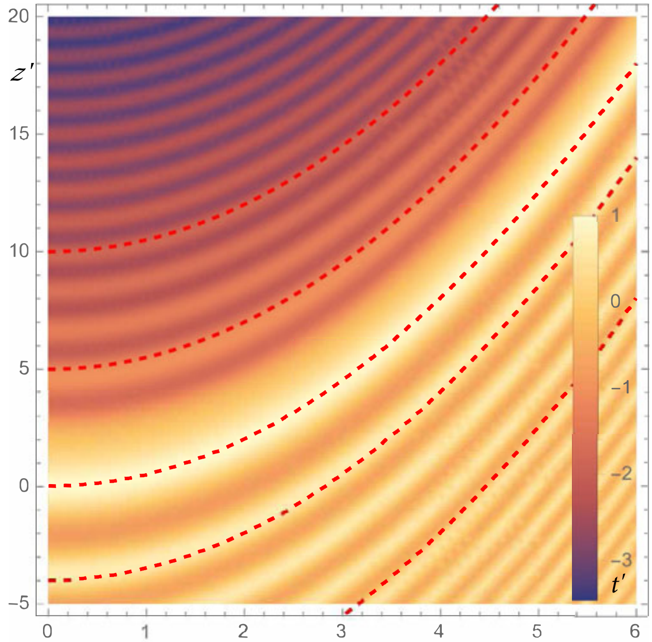

FIGURE 1. Density plot for $f(\xi)$, with $f(\xi=0)=1$ and $d_\xi f(\xi=0)=0$, in terms of time t' and distance z', for $v=0$. Red dashed lines correspond to parabolic trajectories $z'-t'^2/2=\xi_0$, with $\xi_0=-10,-4,0,5,10$.

of (3.5) for $f(\xi=0)=1$, $d_\xi f(\xi=0)=0$, in terms of normalized time $t'=Qt/\sqrt{2}$, normalized distance $z'=\sqrt{Q/2P}\,z$ and normalized velocity $v'=v/\sqrt{PQ}=2$. Any part of the solution follows accelerated curved parabolic trajectories in the $t-z$ space. We display those curved trajectories as dashed lines for $\xi=z'-v't'-t'^2/2=\xi_0$, with $\xi_0=-10,-4,0,5,10$.

3.3. *Case for velocity v antiparallel to the acceleration*

This case is more interesting as it implies that the acceleration of the wave dynamics can change its direction of propagation. To exemplify this case we consider a negative initial velocity $v<0$. In figure 3, we show what occurs for the specific case of normalized velocity equal to $v'=v/\sqrt{PQ}=-2$. In this figure, we display the numerical behaviour of f in a density plot in terms of the same normalized variables t' and z' as above. Because of its accelerating nature, the whole wavepacket changes its initial direction of propagation. This occurs for any part of the wave, as is shown by the parabolic curves in $t-z$ space (in red dashed lines), for $\xi=z'-v't'-t'^2/2=\xi_0$, with $\xi_0=-4,0,5,10,15$. The acceleration, therefore, allows this nonlinear plasma solution to reverse its propagation direction when the initial velocity has the opposite direction to the direction of acceleration. It is straightforward to calculate that this change in direction takes a time equal to

$$\Delta t=-\frac{v}{\beta}>0. \tag{3.6}$$

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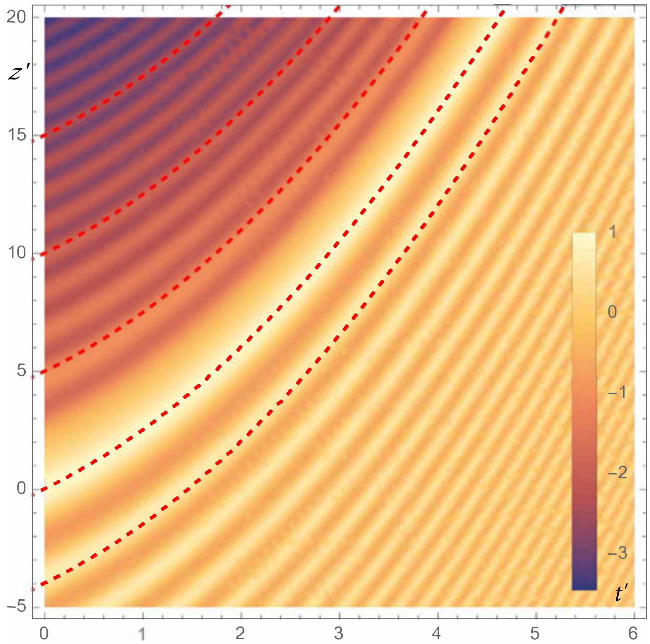

FIGURE 2. Density plot for $f(\xi)$, with $f(\xi = 0) = 1$ and $d_\xi f(\xi = 0) = 0$, in terms of time t', distance z' and normalized positive velocity $v' = 2$. Red dashed lines correspond to parabolic trajectories $z' - v't' - t'^2/2 = \xi_0$, with $\xi_0 = -4, 0, 5, 10, 15$.

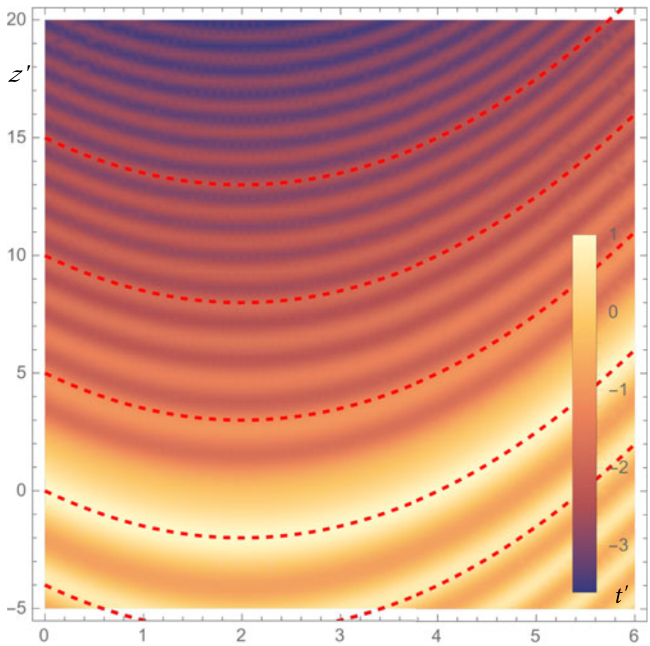

FIGURE 3. Density plot for $f(\xi)$, with $f(\xi = 0) = 1$ and $d_\xi f(\xi = 0) = 0$, in terms of time t', distance z' and normalized negative velocity $v' = -2$. Red dashed lines correspond to parabolic trajectories $z' - v't' - t'^2/2 = \xi_0$, with $\xi_0 = -4, 0, 5, 10, 15$.

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3.4. *Asymptotic behaviour*

On the other hand, although a numerical study of the whole solution is possible, the analytical properties of the solution for f can be found in the case when $\xi \rightarrow -\infty$. In this limit, the solution of the modified second Painlevé equation (3.5) behaves as (Clarkson 2003)

$$f(\xi) \approx \kappa\, \mathrm{Ai}(\xi), \tag{3.7}$$

where Ai is the Airy function, and κ is an arbitrary constant. As the accelerating propagating properties of this electromagnetic plasma wave are present for any value of ξ, this solution pertains to the same family of other accelerating Airy solutions already found in optics and plasmas (Baumgartl, Mazilu & Dholakia 2008; Abdollahpour *et al.* 2010; Chong *et al.* 2010; Chávez-Cerda *et al.* 2011; Jiang, Huang & Lu 2012; Kaminer *et al.* 2012; Mahalov & Suslov 2012; Panagiotopoulos *et al.* 2013; Minovich *et al.* 2014; Li *et al.* 2016; Wiersma *et al.* 2016; Esat Kondakci & Abouraddy 2018; Efremidis *et al.* 2019; Bouchet *et al.* 2022; Winkler *et al.* 2023).

4. Final remark

We have presented a new nonlinear plasma solution with accelerating properties. As the initial velocity of the argument (3.2) is arbitrary, a whole set of a new kind of different propagations can be obtained. This is achieved in a relativistic plasma regime, depending on how magnetized the plasma is, producing wavepackets with acceleration (3.3).

As a nonlinear dynamics of electron–positron plasmas can be found in pulsar magnetospheres (Chian & Kennel 1983; Beskin, Gurevich & Istamin 1993), relativistic jets (Iwamoto & Takahara 2002), the early universe (Lesch & Bisk 1998) or supernovae (Hardy & Thoma 2000), these electromagnetic propagation solutions can produce new mechanisms for plasma acceleration in those regimes. In addition, as the nonlinear Schrödinger equation also appears in electrostatic plasma propagation (see for instance Kourakis & Shukla 2004; Misra & Shukla 2011; Rajabi & Mohammadnejad 2023), the above solution also predicts nonlinear acceleration for such phenomena. In general, it is expected that the propagation of the wavepacket experiences an acceleration along the direction of propagation, even having the possibility to reverse such a direction. Those implications are left for future studies.

Finally, the Painlevé equation has been explored in different realms of plasma physics (Khater *et al.* 1997; Khater, Callebaut & Ibrahim 1998; Ibrahim 2003; Rogers & Clarkson 2018; Kumar, Mohan & Kumar 2022). Therefore, this work contributes to showing that the Painlevé equation is also a straightforward consequence of accelerating solutions in relativistic nonlinear plasmas.

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Declaration of interests

The author reports no conflict of interest.

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