Can. J. Math. Vol. **44** (4), 1992 pp. 843–855

DEDEKIND COMPLETENESS AND THE ALGEBRAIC COMPLEXITY OF o-MINIMAL STRUCTURES

ALAN MEKLER, MATATYAHU RUBIN AND CHARLES STEINHORN

ABSTRACT. An ordered structure is *o-minimal* if every definable subset is the union of finitely many points and open intervals. A theory is *o-minimal* if all its models are o-minimal. All theories considered will be o-minimal. A theory is said to be *n-ary* if every formula is equivalent to a Boolean combination of formulas in n free variables. (A 2-ary theory is called *binary*.) We prove that if a theory is not binary then it is not n-ary for any n. We also characterize the binary theories which have a Dedekind complete model and those whose underlying set order is dense. In [5], it is shown that if T is a binary theory, $\mathcal{M}$ is a Dedekind complete model of T, and I is an interval in $\mathcal{M}$, then for all cardinals κ there is a Dedekind complete elementary extension $\mathcal{N}$ of $\mathcal{M}$, so that $|I^{\mathcal{N}}| \geq \kappa$. In contrast, we show that if T is not binary and $\mathcal{M}$ is a Dedekind complete model of T, then there is an interval I in $\mathcal{M}$ so that if $\mathcal{N}$ is a Dedekind complete elementary extension of $\mathcal{M}$ then $I^{\mathcal{N}} = I^{\mathcal{M}}$.

0. **Introduction.** This paper continues the study of Dedekind complete o-minimal structures begun in [2] and [5]. We here attempt to give an explanation of why some o-minimal theories have a unique, up to isomorphism, Dedekind complete model and why others have Dedekind complete models of arbitrarily large power.

Before proceeding further, let us first set some notation and terminology. Throughout this paper all structures $\mathcal{M}$ are of the form $\mathcal{M} = (M, <, \ldots)$ where $(M, <)$ is a dense linear order without endpoints. Also, **T** always represents a complete first-order theory. An *open interval* in a linearly ordered structure $\mathcal{M}$ is a subset of $\mathcal{M}$ of the form (a, b) where $a \in \mathcal{M} \cup \{-\infty\}$ and $b \in \mathcal{M} \cup \{\infty\}$. We frequently use $I, J, \ldots$ to represent open intervals in structures, and if $I = (a, b) \subseteq \mathcal{M}$ and $\mathcal{N} \succ \mathcal{M}$, then we let $I^{\mathcal{N}}$ denote $\{x \in \mathcal{N} : \mathcal{N} \models a < x < b\}$. An *open box* in a structure $\mathcal{M}$ is a (definable) set of the form $B = I_1 \times \cdots \times I_n$ where $I_1, \ldots, I_n$ are open intervals in $\mathcal{M}$. If $f: A \longrightarrow M$, where $A \subseteq M^n$, then graph(f) denotes the graph of f. A linearly ordered structure is *o-minimal* if all definable sets in one variable are the union of finitely many points and open intervals in the structure. A theory **T** is o-minimal if all (equivalently, one) of its models is o-minimal. For basic facts about o-minimal structures and theories, we refer the reader to [1] and [4].

Now we return to our discussion. In [5], the following dichotomy is established:

The first author was partially supported by NSERC grant A8948; the third author was partially supported by NSF grants DMS-8606411 and DMS-8807505.

Received by the editors November 11, 1990.

AMS subject classification: Primary 03C07; secondary: 03C65.

THEOREM 0.1 [5]. *Let* $\mathbf{T}$ *be an o-minimal theory having a Dedekind complete model. Then* $\mathbf{T}$ *either has a unique (up to isomorphism) Dedekind complete model of order type* $(\mathbf{R}, <)$ *or has Dedekind complete models of arbitrarily large power.*

The second alternative can be regarded as pathology that cannot occur in mathematically interesting o-minimal structures. Indeed, if $\mathcal{R}$ is an o-minimal expansion of $(\mathbf{R}, +, <)$, then it is not difficult to check that up to isomorphism $\mathcal{R}$ is the unique Dedekind complete model of Th($\mathcal{R}$). In [5], the beginnings of a syntactic attempt to account for the dichotomy are made. There, the definition and theorem that follow are given.

DEFINITION 0.2. A theory $\mathbf{T}$ is *n-ary* if every formula is equivalent, relative to $\mathbf{T}$, to a Boolean combination of formulas in n free variables. A 2-ary theory is also called *binary*.

THEOREM 0.3 [5]. *Let* $\mathbf{T}$ *be a binary o-minimal theory, let* $\mathcal{M} \models \mathbf{T}$ *be Dedekind complete, and let* $I \subseteq \mathcal{M}$ *be an interval in* $\mathcal{M}$. *Then for all* κ, *there is a Dedekind complete* $\mathcal{N}_\kappa \succ \mathcal{M}$ *so that* $|I^{\mathcal{N}_\kappa}| \geq \kappa$.

In this paper, we complete the picture suggested by our remarks and Theorem 0.3 by proving:

THEOREM 0.4. *Let* $\mathbf{T}$ *be an o-minimal theory that is not binary and that has a Dedekind complete model* $\mathcal{M}$. *Then there is an interval* $I \subseteq \mathcal{M}$ *so that if* $\mathcal{N} \succ \mathcal{M}$ *is Dedekind complete, then* $I^{\mathcal{N}} = I^{\mathcal{M}}$.

Although this theorem is a local rather than a global result, it cannot be improved for trivial reasons. For example, let $\mathcal{M}$ be the o-minimal structure obtained by taking the ordered sum of $(\mathbf{R}, +, <)$, a point, and $(\mathbf{R}, <)$, with the induced structure. Then $\mathcal{M}$ has Dedekind complete elementary extensions of arbitrarily large power simply because it contains an (0-definable) interval whose theory is binary.

We also use the methods we develop to show that o-minimal structures whose theory is not binary behave in another way like $(\mathbf{R}, +, <)$. In particular, using the functions

$$f_k(x_1, \ldots, x_k) = x_1 + \cdots + x_k$$

it is easy to see that $(\mathbf{R}, +, <)$ does not have an n-ary theory for any n (see the proof of Theorem 0.5 for details). We prove the following general result.

THEOREM 0.5. *If* $\mathbf{T}$ *is o-minimal and not binary, then* $\mathbf{T}$ *is not n-ary for any n.*

In Section 1, we present the necessary preliminary material. Theorems 0.4 and 0.5 are proved in Section 2. We conclude the paper in Section 3 by characterizing binary o-minimal theories.

The research for this paper was done while the last two authors were visiting Simon Fraser University.

1. **Preliminaries.** We first recall the results that underly much of the study of o-minimal structures.

THEOREM 1.1 [4]. *Let $\mathcal{M} = (M, \ldots)$ be o-minimal and let $f\colon M \longrightarrow M$ be a definable function in $\mathcal{M}$. Then there are $-\infty = a_0 < a_1 < a_2 < \cdots < a_k < a_{k+1} = \infty$ in $\mathcal{M}$ definable from the parameters used to define f, such that for all $i = 0, \ldots, k$, the restriction of f to (a_i, a_{i+1}) is either constant or a monotone bijection onto an interval in $\mathcal{M}$.*

Let $b, a_1, \ldots, a_m \in \mathcal{M}$. We say that b is *algebraic* over $\{a_1, \ldots, a_m\}$ if there is some formula $\varphi(x, y_1, \ldots, y_m)$ so that $\mathcal{M} \models \varphi(b, a_1, \ldots, a_m)$ and $\varphi(x, a_1, \ldots, a_m)$ has only finitely many solutions in $\mathcal{M}$. Also, $\{a_1, \ldots, a_m\} \subset \mathcal{M}$ is said to be *independent* if there is no $i \leq m$ so that a_i is algebraic over $\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m\}$. Then we have the following consequence of Theorem 1.1.

COROLLARY 1.2. *Let $\mathcal{M}$ be o-minimal and let $b, c, a_1, \ldots, a_m \in \mathcal{M}$. If b is algebraic over $\{c, a_1, \ldots, a_m\}$ but not algebraic over $\{a_1, \ldots, a_m\}$, then c is algebraic over $\{b, a_1, \ldots, a_m\}$.*

We next review the definition of a cell given in [1].

DEFINITION 1.3. Let $\mathcal{M}$ be an ordered structure.

(i) Let $a, b \in \mathcal{M}$ with $a < b$. If $X = \{a\}$, then X is a cell and $\dim(X) = 0$. If $X = (a, b)$, then X is a cell and $\dim(X) = 1$.

(ii) Suppose that $Y \subset \mathcal{M}^n$ is a cell and that $\dim(Y) = m$. Also suppose that $f\colon Y \longrightarrow \mathcal{M}$ is definable and continuous, that $g, h\colon Y \longrightarrow \mathcal{M} \cup \{\pm\infty\}$ are definable and continuous, and that $g(\bar{b}) < h(\bar{b})$ for all $\bar{b} \in Y$. Then

 (a) $X_1 = \mathrm{graph}(f) = \{(\bar{b}, f(\bar{b})) : \bar{b} \in Y\}$ is a cell in $\mathcal{M}^{n+1}$ and $\dim(X_1) = m$;

 (b) $X_2 = (g, h)_Y = \{(\bar{b}, c) : \bar{b} \in Y \,\&\, g(\bar{b}) < c < h(\bar{b})\}$ is a cell in $\mathcal{M}^{n+1}$ and $\dim(X_2) = m + 1$.

It is apparent that cells are definable. We will need the following fundamental result from [1].

THEOREM 1.4 [1]. *Let $\mathcal{M}$ be o-minimal and let X be a definable subset of $\mathcal{M}^n$. Then X can be partitioned into the disjoint union of finitely many cells definable with the same parameters used to define X. Also, if $f\colon X \longrightarrow \mathcal{M}$ is definable, then X can be partitioned into the disjoint union of finitely many cells definable with the same parameters used to define f so that the restriction of f to each cell is continuous.*

The next definition from [5] is essential for the results we prove in §2.

DEFINITION 1.5. Let $a \in \mathcal{M}$. Then a is *left-attainable* if there is a definable function f in $\mathcal{M}$ and some $b \in \mathcal{M}$ such that $\langle f^n(b) : n \in \omega \rangle$ is defined (*i.e.*, for all n, $f^n(b)$ is in the domain of f) and is strictly increasing with supremum a. Similarly, a is *right-attainable* if there is a definable function f in $\mathcal{M}$ and some $b \in \mathcal{M}$ such that $\langle f^n(b) : n \in \omega \rangle$ is defined and is strictly decreasing with infimum a.

We say that ∞ is *attainable* if there is a definable function f in $\mathcal{M}$ and some $b \in \mathcal{M}$ such that $\langle f^n(b) : n \in \omega \rangle$ is defined and strictly increases to ∞. Likewise, $-\infty$ is *attainable* if there is a definable function f in $\mathcal{M}$ and some $b \in \mathcal{M}$ such that $\langle f^n(b) : n \in \omega \rangle$ is defined and strictly decreases to $-\infty$.

Before stating the next theorem, we must introduce some notation. Recall from [4] that a one-type over an o-minimal structure $\mathcal{M}$ is determined by the cut that it makes in the ordering of $\mathcal{M}$. Then, following [5], for an o-minimal structure $\mathcal{M}$ and $a \in \mathcal{M}$ we have the following types

$$\begin{aligned} p_a^+(x) &= \{x < b : b \in \mathcal{M} \,\&\, b > a\} \cup \{x > a\} \\ p_a^-(x) &= \{x > b : b \in \mathcal{M} \,\&\, b < a\} \cup \{x < a\} \\ p_\infty(x) &= \{x > b : b \in \mathcal{M}\} \\ p_{-\infty}(x) &= \{x < b : b \in \mathcal{M}\}. \end{aligned}$$

Now we can state a result linking the attainability of a point in a Dedekind complete o-minimal structure and the possibility of properly extending the structure to a larger Dedekind complete structure. It follows directly from Lemma 1.9 and Theorem 2.3 of [5].

THEOREM 1.6 [5]. *Let $\mathcal{M}$ be Dedekind complete and let $a \in \mathcal{M} \cup \{\pm\infty\}$ be attainable. Then no Dedekind complete extension of $\mathcal{M}$ realizes $p_a^{\pm}(x)$ (or $p_{\pm\infty}(x)$, as the case may be).*

Before stating and proving an easy characterization of an o-minimal theory being binary, we need one further fact. The statement below is stronger than the statement of Lemma 1.4 of [3], but the proofs of the two lemmas are the same.

LEMMA 1.7 [3]. *Let $\mathcal{M}$ be o-minimal and let $\{a_1, \ldots, a_n\} \subset \mathcal{M}$ be independent. Then every 0-definable set $X \subset \mathcal{M}^n$ containing $(a_1, \ldots, a_n)$ contains as a subset a 0-definable open cell C such that $(a_1, \ldots, a_n) \in C$.*

LEMMA 1.8. *An o-minimal theory $\mathbf{T}$ is binary if and only if for all $\mathcal{M} \models \mathbf{T}$ and all $\{a_1, \ldots, a_n\} \subset \mathcal{M}$, if the elements of $\{a_1, \ldots, a_n\}$ pairwise are independent, then $\{a_1, \ldots, a_n\}$ is independent.*

PROOF. We begin with the implication from left-to-right. Suppose that there is some $\mathcal{M} \models \mathbf{T}$ and $\{a_1, \ldots, a_n\} \subset \mathcal{M}$ the elements of which pairwise are independent, but so that $\{a_1, \ldots, a_n\}$ is not independent. We may suppose that n is minimal. By the minimality of n and Corollary 1.2, we see that a_n must be algebraic over $\{a_1, \ldots, a_{n-1}\}$. We may assume that this relationship is realized by some 0-definable function f so that $a_n = f(a_1, \ldots, a_{n-1})$. By the minimality of n and Lemma 1.7, it follows that the domain of f contains a 0-definable open cell, C, with $(a_1, \ldots a_{n-1})$ as a member. Without loss of generality we can assume that the domain of f is C. Since $\mathbf{T}$ is binary, we see that

$$\mathbf{T} \vdash y = f(x_1, \ldots, x_{n-1}) \longleftrightarrow \bigvee_{i \leq N} \varphi_i(\bar{x}, y)$$

https://doi.org/10.4153/CJM-1992-050-1 Published online by Cambridge University Press

where each $\varphi_i(\bar{x}, y)$ is a conjunction of formulas in two free variables. Fix i_0 so that $\mathcal{M} \models \varphi_{i_0}(a_1, \ldots, a_n)$ and write

$$\varphi_{i_0}(\bar{x}, y) \equiv \bigwedge_{j,k \leq n-1} \psi_{j,k}(x_j, x_k) \wedge \bigwedge_{j \leq n-1} \psi_j(x_j, y).$$

Now, by the fact that the elements of $\{a_1, \ldots, a_n\}$ pairwise are independent, for all j, k the subset of $\mathcal{M}$ defined by $\psi_{j,k}$ and the subset defined by ψ_j contain open sets each of which has $(a_1, \ldots, a_n)$ as an element. Hence, the subset of $\mathcal{M}^n$ defined by $\varphi_{i_0}(\bar{x}, y)$ contains an open set which has $(a_1, \ldots, a_n)$ as an element, which means that it is not possible that the disjunction asserted to define $y = f(x_1, \ldots, x_{n-1})$ actually could define a function. This completes the proof of the left-to-right implication.

Now we give the argument for the direction from right-to-left. By induction on n we show that formulas $\varphi(x_1, \ldots, x_n)$ are equivalent, relative to $\mathbf{T}$, to formulas in two free variables. We work in a fixed saturated $\mathcal{M} \models \mathbf{T}$. Let $\varphi(x_1, \ldots, x_{n+1})$ be given. By Theorem 1.4, we can partition the definable subset

$$X = \{(a_1, \ldots, a_n) \in \mathcal{M}^n : \mathcal{M} \models \exists y \varphi(a_1, \ldots, a_n, y)\}$$

into the pairwise disjoint 0-definable cells $C_1, \ldots, C_k$ so that on each cell C_i, the number and arrangement of points and open intervals in the sets

$$X_{\bar{a}} = \{b \in \mathcal{M} : \mathcal{M} \models \varphi(\bar{a}, b)\}$$

is uniform as $\bar{a} = (a_1, \ldots, a_n)$ ranges over C_i. We also insist that this uniformity include the provision that $X_{\bar{a}}$ be bounded or unbounded in the directions of both $\pm\infty$.

Now let us fix some such C_{i_0}. Let $f_1, \ldots, f_m \colon C_{i_0} \longrightarrow \mathcal{M}$ be the 0-definable functions so that for $\bar{a} \in C_{i_0}$, the values $f_1(\bar{a}), \ldots, f_m(\bar{a})$ uniformly give the isolated points and the boundary points of intervals in $X_{\bar{a}}$. Next let us fix some $j = 1, \ldots, m$. Since $f_j(\bar{a})$ depends on $\bar{a}$ for each $\bar{a} \in C_{i_0}$, it follows by the fact that dependence implies pairwise dependence, compactness, and, if necessary, a further application of Theorem 1.4, that we can assume that there is some $p \leq n$ and some 0-definable function $g_j \colon \mathcal{M} \longrightarrow \mathcal{M}$ so that

$$f_j(\bar{a}) = g_j(a_p) \text{ for all } \bar{a} = (a_1, \ldots, a_n) \in C_{i_0}.$$

Carrying out this argument for $f_1, \ldots, f_m$, and for $C_1, \ldots, C_k$, and applying induction hypothesis to the formulas defining $C_1, \ldots, C_k$, it is easy see now that $\varphi(x_1, \ldots, x_{n+1})$ is equivalent, relative to $\mathbf{T}$, to a Boolean combination of formulas in two free variables. This finishes the proof in the right-to-left direction. ∎

2. **0-minimal structures that are not binary.** We prove Theorems 0.4 and 0.5 in this section. We begin with the proof of Theorem 0.4. In outline, the proof proceeds in two major steps. The first step consists of the reduction from the hypothesis that the o-minimal theory $\mathbf{T}$ is not binary to the existence in the Dedekind complete $\mathcal{M} \models \mathbf{T}$ of intervals I and J and a definable function $f \colon I \times J \longrightarrow \mathcal{M}$ that is continuous and in each coordinate is uniformly increasing or decreasing. This is given in Lemma 2.1. Then, in a series of lemmas, the most basic of which being Lemma 2.3, we show that the existence of such a function leads to the conclusion of Theorem 0.4.

LEMMA 2.1. *Let* **T** *be an o-minimal theory that is not binary and let* $\mathcal{M} \models \mathbf{T}$. *Then, in* $\mathcal{M}$ *there exist intervals I and J and a definable function* $f: I \times J \longrightarrow \mathcal{M}$ *that is continuous and in each coordinate is uniformly increasing or decreasing.*

PROOF. By Lemma 1.8, there is some $\mathcal{M}' \models \mathbf{T}$ and $a_1, \ldots, a_n \in \mathcal{M}'$ so that $\{a_1, \ldots, a_n\}$ is pairwise independent but not independent. Let $n \geq 3$ be least so that such an $\mathcal{M}'$ and $a_1, \ldots, a_n$ exist. We may suppose that a_n is definable over $\{a_1, \ldots, a_{n-1}\}$ by some 0-definable function g, that is $a_n = g(a_1, \ldots, a_{n-1})$. Notice also by the minimality of n that any $n-1$ elements of $\{a_1, \ldots, a_n\}$ are independent. By Lemma 1.7, g is defined on some open $O \subseteq \mathcal{M}'^{n-1}$ containing $(a_1, \ldots, a_{n-1})$. By Theorem 1.4, we can partition O into finitely many 0-definable cells so that g is continuous on each cell, and that on each cell of dimension $n-1$, g is in each coordinate uniformly increasing, decreasing, or constant.

Let U be the cell in this decomposition of dimension $n-1$ containing $(a_1, \ldots, a_{n-1})$ (that U has dimension $n-1$ follows from Lemma 1.7, again). We now assert that g is in each coordinate uniformly increasing or decreasing. For if not, then we may suppose without any loss of generality that g is constant in the $(n-1)$-st coordinate throughout U, that is, for all $b_1, \ldots, b_{n-2} \in \mathcal{M}'$, we have that g is constant on the set

$$U \cap \{(b_1, \ldots, b_{n-2})\} \times \mathcal{M}'.$$

Using that U is 0-definable, it is evident that $a_n = g(a_1, \ldots, a_{n-1})$ must be dependent on $a_1, \ldots, a_{n-2}$. This, however, violates the minimality of n.

Now let $\psi(x_1, \ldots, x_{n-1}, y)$ be the formula so that

$$\mathcal{M}' \models \psi(b_1, \ldots, b_{n-1}, c) \Longleftrightarrow (b_1, \ldots, b_{n-1}) \in U \text{ and } c = g(b_1, \ldots, b_{n-1})$$

for all $b_1, \ldots, b_{n-1}, c \in \mathcal{M}'$. Let $\mathcal{M} \models \mathbf{T}$ and let

$$U' = \{(b_1, \ldots, b_{n-1}) : \mathcal{M} \models \exists y \psi(b_1, \ldots, b_{n-1}, y)\}.$$

It is clear that U' is an open cell of dimension $n-1$ in $\mathcal{M}$. We now fix $b_1, \ldots, b_{n-3} \in \mathcal{M}$, let

$$W = \{(a, b) : (b_1, \ldots, b_{n-3}, a, b) \in U'\},$$

and define $f: W \longrightarrow \mathcal{M}$ by

$$c = f(a, b) \Longleftrightarrow \mathcal{M} \models \psi(b_1, \ldots, b_{n-3}, a, b, c).$$

Since W is an open subset of $\mathcal{M}^2$, there are intervals $I, J \subseteq \mathcal{M}$ so that $I \times J \subseteq W$ and the restriction of f to $I \times J$ is as required in the conclusion of the Lemma. ∎

LEMMA 2.2. *Suppose that* $\mathcal{M}$ *is o-minimal and in* $\mathcal{M}$ *there exist intervals J and K and a definable function* $f: J \times K \longrightarrow \mathcal{M}$ *that is continuous and in each coordinate is uniformly increasing or decreasing. Then there are in* $\mathcal{M}$ *an interval I and a definable* *function $g: I^2 \longrightarrow \mathcal{M}$ that is continuous and in each coordinate is uniformly increasing or in each coordinate is uniformly decreasing.*

PROOF. We examine case-by-case the possibilities for f.

CASE (i): THE FUNCTION $f: J \times K \longrightarrow \mathcal{M}$ IS INCREASING IN BOTH COORDINATES. By shrinking the intervals if necessary, we may suppose that $J = [a_1, a_2]$ and that $K = [b_1, b_2]$. We now claim that there is some $I \subset J$ and definable $g: I^2 \longrightarrow \mathcal{M}$ that is continuous and in each coordinate is uniformly increasing.

Let $h_1: J \longrightarrow \mathcal{M}$ be given by $h_1(x) = f(x, b_1)$ and let $h_2: K \longrightarrow \mathcal{M}$ be given by $h_2(y) = f(a_1, y)$. Now choose $c \in h_1(J) \cap h_2(K)$ satisfying $c > f(a_1, b_1)$ and let

$$I = [a_1, h_1^{-1}(c)] \text{ and } I' = [b_1, h_2^{-1}(c)].$$

We next define $H: I \longrightarrow I'$ by

$$y = H(x) \text{ if and only if } h_1(x) = h_2(y)$$

It is a simple matter to check that H is an increasing order isomorphism between I and I'. Finally, we define $g: I^2 \longrightarrow \mathcal{M}$ by

$$g(x, y) = f\big(x, H(y)\big).$$

Again, it is clear that g is as required.

CASE (ii): THE FUNCTION $f: J \times K \longrightarrow \mathcal{M}$ IS INCREASING IN THE FIRST COORDINATE AND DECREASING IN THE SECOND COORDINATE. We assert that there exist intervals J' and K' and a function $h: J' \times K' \longrightarrow \mathcal{M}$ that is continuous and uniformly increasing in both coordinates, thereby reducing this case to Case (i). Let $J = [a_1, a_2]$.

It is not difficult to verify that the relation $R(y, z, x)$ given by

$$R(b, c, a) \text{ if and only if } f(a, b) = c$$

actually is a function $x = h(y, z)$ defined on the cell

$$D = \{(b, c) \in \mathcal{M}^2 : b \in J \,\&\, f(a_1, b) \leq c \leq f(a_2, b)\}.$$

We assert next that h is uniformly increasing in both coordinates. To prove this, we first let $(b_1, c), (b_2, c) \in D$ be such that $b_1 < b_2$, and let $e_1 = h((b_1, c)$ and $e_2 = h((b_2, c)$. If it were true that $e_1 \geq e_2$, then it would follow that

$$c = f(e_1, b_1) \geq f(e_2, b_1) > f(e_2, b_2) = c,$$

a contradiction. Hence, h is increasing in the first coordinate. The argument to show that g is increasing in the second coordinate is similar, and so we omit it. We now partition D into cells on which h is continuous. Since the dimension of D is two, it follows that there is a cell C in the partition of dimension two, and we just take $J' \times K' \subset C$. This completes the proof in Case (ii).

The possibility that f is decreasing in both coordinates is dealt with as in Case (i) and the possibility that f is decreasing in the first coordinate and increasing in the second is treated exactly as in Case (ii). Hence the lemma is proved. ∎

LEMMA 2.3. *Suppose that $\mathcal{M}$ is a Dedekind complete o-minimal structure and in $\mathcal{M}$ there exist an interval I and a definable function $f: I^2 \longrightarrow \mathcal{M}$ that is continuous and in each coordinate is uniformly increasing or in each coordinate is uniformly decreasing. Then there is in $\mathcal{M}$ an interval J each element of which is (uniformly) right-attainable or left-attainable.*

PROOF. Assume that $f: I^2 \longrightarrow \mathcal{M}$ is continuous and increasing in each coordinate. If f is decreasing in each coordinate, the argument is similar. By the continuity of f, its range is some interval J. Let $g: I \longrightarrow \mathcal{M}$ be defined by $g(a) = f(a,a)$ for $a \in I$. It follows that the range of g is J. It also is easily seen that g is a monotonically increasing bijection between I and J. Let $h: J \longrightarrow I$ be g^{-1}.

Let $a \in J$ and $J_a = J \cap (-\infty, a)$. We now claim that the definable function $F_a: J \longrightarrow J$ given by

$$F_a(x) = f\big(h(x), h(a)\big)$$

witnesses the left-attainability of a. (It can similarly be used to show the right-attainability of a.) Observe first that F_a is continuous and is increasing on J_a. Next, since g is increasing, its inverse h is also, and thus for $b \in J$ with $b < a$ we have

$$b = f\big(h(b), h(b)\big) < f\big(h(b), h(a)\big) = F_a(b)$$

and

$$F_a(b) = f\big(h(b), h(a)\big) < f\big(h(a), h(a)\big) = a.$$

From the second inequality, we see that the range of $F_a|_{J_a}$ is an interval contained in J_a. That is, for any $b \in J$ with $b < a$, we have $b < F_a(b) < F_a^2(b) < \cdots < a$. Since the entire sequence $\langle F_a^n(b) : n < \omega \rangle$ is bounded above by a and $\mathcal{M}$ is Dedekind complete, it follows that $\langle F_a^n(b) : n < \omega \rangle$ has a supremum in $\mathcal{M}$. The left-attainability of a now follows if we can show that $a = \sup_{n<\omega} \langle F_a^n(b) : n < \omega \rangle$. Indeed suppose that c is supremum and that $c < a$. In this case, it is a simple matter to check that $F_a(c) = c$, but this is impossible since $F_a(c) > c$ for $c \in J_a$. Hence $a = \sup_{n<\omega} \langle F_a^n(b) : n < \omega \rangle$, and so a is left-attainable, as required. ■

Now we complete the proof of Theorem 0.4.

PROOF OF THEOREM 0.4. Since **T** is not binary, By Lemmas 2.1, 2.2, and 2.3, there is an interval $J \subset \mathcal{M}$ so that every point in J is attainable. Since the only one-types over a Dedekind complete o-minimal structure are of the form $p_a^{\pm}$ or $p_{\pm\infty}$ it follows by Theorem 1.6 that if $\mathcal{N}$ is a Dedekind complete elementary extension of $\mathcal{M}$, then $J^{\mathcal{N}} = J^{\mathcal{M}}$. This completes the proof of the theorem. ■

For $i = 1, \ldots, m+1$, we denote by $\pi_i: \mathcal{M}^{m+1} \longrightarrow \mathcal{M}^m$ be the projection mapping given by

$$\pi_i(a_1, \ldots, a_{m+1}) = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m+1})$$

for $(a_1, \ldots, a_{m+1}) \in \mathcal{M}^{m+1}$. For the proof of Theorem 0.5, we require the following lemma.

LEMMA 2.4. *Let $\mathcal{M}$ be o-minimal and let $I_1, \ldots, I_m$ be open intervals in $\mathcal{M}$. Also, let*

$$f: I_1 \times \cdots \times I_m \longrightarrow \mathcal{M}$$

be definable, continuous, and uniformly increasing or decreasing in each coordinate. If $(a_1, \ldots, a_m, b) \in \text{graph}(f)$ and if $B \subset \mathcal{M}^{m+1}$ is an open box containing $(a_1, \ldots, a_m, b)$, then $\pi_i\big(B \cap \text{graph}(f)\big)$ has interior in m-space for all $i = 1, \ldots, m+1$.

PROOF. For $i = m+1$, the conclusion of the lemma is clear. For $i \neq m+1$, it is sufficient to prove that the dimension of $\pi_i\big(B \cap \text{graph}(f)\big)$ is m. For a contradiction, suppose that for some $i \leq m$ that the dimension of $A = \pi_i\big(B \cap \text{graph}(f)\big)$ is less than m. For ease of notation, let us fix $i = 1$. Observe next that the dimension of $B \cap \text{graph}(f)$ is m. If $\pi_1^{-1}(\bar{a})$ were finite for every $\bar{a} \in A$, then by the results of [1] there would be a uniform finite bound on the cardinality of all such $\pi_1^{-1}(\bar{a})$, and it easily would follow that the dimension of $B \cap \text{graph}(f)$ would be strictly less than m. Hence, $\pi_1^{-1}(\bar{a}_0)$ must be infinite for some $\bar{a}_0 = (a_2, \ldots, a_{m+1}) \in A$. It follows that $\pi_1^{-1}(\bar{a}_0)$ must contain a set of the form $J \times \{(a_2, \ldots, a_{m+1})\}$ for some interval J, and thus that f assumes the constant value a_{m+1} on the set $J \times \{(a_2, \ldots, a_m)\}$. But this is impossible since we have assumed that f is increasing or decreasing in each coordinate throughout $I_1 \times \cdots \times I_m$, and the lemma is proved. ∎

PROOF OF THEOREM 0.5. By Lemmas 2.1 and 2.2, we find in $\mathcal{M}$ an interval I and a definable function $g: I^2 \longrightarrow \mathcal{M}$ that is continuous and in each coordinate is uniformly increasing or in each coordinate is uniformly decreasing. Let the range of g be some interval J.

We first claim that there is a definable function $h: I^2 \longrightarrow I$ that is continuous, surjective, and either uniformly increasing in both coordinates or uniformly decreasing in each coordinate. To see this, let $H: I \longrightarrow J$ be the order isomorphism given by $H(a) = g(a, a)$ for $a \in I$. Observe that if g is increasing (decreasing) in each coordinate, then H and thus the order isomorphism $H^{-1}: J \longrightarrow I$ is increasing (decreasing). We now let $h = H^{-1} \circ g$. It is clear that h is as required.

Let m be given. To show that $\mathbf{T}$ is not m-ary, we claim that formula defining the graph of the definable function $f: I^m \longrightarrow I$ given by

$$f(a_1, \ldots, a_m) = h\Big(h\Big(\cdots h\big(h(a_1, a_2), a_3\big) \ldots, a_{m-1}\Big), a_m\Big)$$

is not equivalent, relative to $\mathbf{T}$, to a Boolean combination of formulas in m-variables. For a contradiction, suppose that

$$\mathbf{T} \vdash x_{m+1} = f(x_1, \ldots, x_m) \longleftrightarrow \bigvee_{j \leq p} \psi_j(x_1, \ldots, x_{m+1})$$

where each $\psi_j(x_1, \ldots, x_{m+1})$ is a conjunction of formulas in m variables (we are suppressing the parameters used to define f.). By Theorem 1.4, we can assume that there is some open box $B = I_1 \times \cdots \times I_m$ and some $j_0 \leq p$ so that

$$(*) \qquad \mathcal{M} \models c = f(\bar{b}) \longleftrightarrow \psi_{j_0}(\bar{b}, c) \text{ for all } \bar{b} \in B.$$

https://doi.org/10.4153/CJM-1992-050-1 Published online by Cambridge University Press

Next, we write

$$\psi_{j_0}(x_1,\ldots,x_{m+1}) \equiv \bigwedge_{k\le m+1} \varphi_k(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{m+1})$$

where for $k = 1,\ldots,m+1$, the formula φ_k contains only the m free variables $x_1,\ldots,x_{k-1}$, $x_{k+1},\ldots,x_{m+1}$, as shown.

By induction on $k \le m+1$, we now define open boxes $B_k \subset \mathcal{M}^{m+1}$ so that $\pi_{m+1}(B_k) \subset I_1 \times \cdots \times I_m$ and

$$\mathcal{M} \models \bigwedge_{i\le k} \varphi_i(\bar b) \text{ for all } \bar b \in B_k$$

where for each φ_i, if $\bar b = (b_1,\ldots,b_{m+1})$, we substitute b_q for the variable x_q. It then follows that

$$\mathcal{M} \models \psi_{j_0}(\bar b) \text{ for all } \bar b \in B_{m+1}.$$

By $(*)$ this would be impossible since the dimension of B_{m+1} is $m+1$, and the dimension of graph(f) is m.

Now, we carry out the construction of the boxes B_k for $k = 1,\ldots,m+1$. We let $B_1 = B \times \mathcal{M} = I_1 \times \cdots \times I_m \times \mathcal{M}$. It is clear that B_1 is as required. Assuming now that we have constructed $B_i = I_1^i \times \cdots \times I_{m+1}^i$, we show how to construct B_{i+1}. By Lemma 2.4, it follows that $\pi_{i+1}\big(B_i \cap \text{graph}(f)\big)$ has interior in m-space. Hence, there is an open box $J_1 \times \cdots \times J_i \times J_{i+2} \times \cdots \times J_{m+1}$ which is contained in $\pi_{i+1}\big(B_i \cap \text{graph}(f)\big)$. Let $\bar b \in B_i \cap \text{graph}(f)$ be a point so that $\pi_{i+1}(\bar b)$ is contained in $J_1 \times \cdots \times J_i \times J_{i+2} \times \cdots \times J_{m+1}$. We then see that

$$U = J_1 \times \cdots \times J_i \times I_{i+1}^i \times J_{i+2} \times \cdots \times J_{m+1} \cap B_i$$

contains $\bar b$ and so is a nonempty open set whose intersection with graph(f) is nonempty. We thus can find an open box $B_{i+1} \subset U$ so that $\bar b \in B_{i+1}$. We are done if

$$\mathcal{M} \models \varphi_{i+1}(\bar c) \text{ for all } \bar c \in B_{i+1}.$$

But this is clear since

$$\mathcal{M} \models \varphi_{i+1}(c_1,\ldots,c_i,c_{i+2},\ldots,c_{m+1})$$

for all $(c_1,\ldots,c_i,c_{i+2},\ldots,c_{m+1}) \in \pi_{i+1}(B_{i+1})$ by $(*)$ and the construction. ∎

3. Characterization of o-minimal binary theories.

Here we demonstrate that all o-minimal binary theories with a Dedekind complete model must have a particularly simple form. We say that two partial unary functions f and g defined in some structure $\mathcal{M}$ *cross* at a point a if $f(a) = g(a)$ and for all open intervals I in $\mathcal{M}$ containing a there is $b \in I$ so that $f(b) \ne g(b)$.

DEFINITION 3.1. A structure $\mathcal{M} = (M,<,f)_{f\in\mathcal{F}}$ is a *canonical binary structure* if

(i) $(M,<) \cong (\mathbf{R},<)$,

(ii) each $f \in \mathcal{F}$ is a partial unary function whose domain is an interval in $\mathcal{M}$ and which is strictly monotonic and continuous, the identity function is a member $\mathcal{F}$,

(iii) $\mathcal{F}$ is closed under inverses and composition,

(iv) all distinct $f, g \in \mathcal{F}$ cross at only finitely many points.

Now we can state our characterization of o-minimal binary structures and theories.

THEOREM 3.2. *(a.) Any canonical binary structure is o-minimal.*

(b) Let **T** *be a binary o-minimal theory having a Dedekind complete model. Then,* **T** *has a Dedekind complete model* $\mathcal{M} = (M, <, \ldots)$ *so that* $(M, <) \cong (\mathbf{R}, <)$*, and if we let* $\mathcal{F}^*$ *consist of all definable partial functions in* $\mathcal{M}$ *satisfying clause (ii) in Definition 3.1, then* $\mathcal{M}^* = (M, <, f)_{f \in \mathcal{F}^*}$ *is a canonical binary structure and every definable relation in* $\mathcal{M}$ *is definable in* $\mathcal{M}^*$*.*

PROOF OF (a). Let $\mathcal{M} = (\mathbf{R}, <, f)_{f \in \mathcal{F}}$ be a canonical binary structure. We show that

$$\mathcal{M}' = (\mathbf{R}, <, f, r)_{f \in \mathcal{F}, r \in \mathbf{R}}$$

admits elimination of quantifiers, from which it is evident that $\mathcal{M}$ is o-minimal.

To show that $\mathcal{M}'$ admits elimination of quantifiers, it is sufficient to consider a formula $\exists x \varphi(x, \bar{y})$ where $\varphi(x, \bar{y})$ is a conjunction of atomic formulas of the form

$$f(x) = r,\ f(x) < r,\ f(x) > r,$$
$$f(x) = g(x),\ f(x) < g(x),$$
$$f(x) = g(y_i),\ f(x) < g(y_i),\ f(x) > g(y_i),$$

where $f, g \in \mathcal{F}$. Here, we consider $x > r$ to have the form $f(x) > r$ where $f(x) = x$. Also, we pull out from inside the scope of the quantifier any formulas involving just $\bar{y}$, and we dispose of negated atomic formulas by writing them as disjunctions and distributing the existential quantifier over the disjunction.

We now claim that we actually may take $\varphi(x, \bar{y})$ to be a conjunction of atomic formulas of the form

$$(*) \qquad x = r,\ x < r,\ x > r,$$
$$x = g(y_i),\ x < g(y_i),\ x > g(y_i).$$

Assuming this for the moment, we show how to complete the proof of quantifier elimination. Let us denote the collection of terms r and $g(y)$ occurring in formulas in $(*)$ of the form $x > r$ or $x > g(y)$ by $\mathcal{T}_<$. We similarly define the sets of terms $\mathcal{T}_=$ and $\mathcal{T}_>$. Then, it is easy to see that

$$\mathrm{Th}(\mathcal{M}') \models \exists x \varphi(x, \bar{y}) \longleftrightarrow \bigwedge_{\tau_1, \tau_2 \in \mathcal{T}_=} \tau_1 = \tau_2 \wedge \bigwedge_{\substack{\sigma \in \mathcal{T}_< \\ \tau \in \mathcal{T}_= \cup \mathcal{T}_>}} \sigma < \tau \wedge \bigwedge_{\substack{\tau \in \mathcal{T}_= \\ \upsilon \in \mathcal{T}_>}} \tau < \upsilon$$

as required.

So it remains to demonstrate that the conjuncts in $(*)$ are sufficient. We begin by showing how to do away with a conjunct of the form $f(x) < r$. By clause (ii) of Definition 3.1, we may assume without loss of generality that the domain of f is (r_1, r_2), the range of f is (s_1, s_2), and that f is continuous and monotone, say monotonically decreasing. Here r_1 and s_1 can be a real number or $-\infty$ and r_2 and s_2 can be a real number or ∞. If $r \leq s_1$, then the conjunction in which $f(x) < r$ appears is impossible to satisfy, and so trivial to deal with, so we may suppose that there is some value c satisfying $f(c) = r$. In this case we can replace $f(x) < r$ by

$$c < x \wedge x < r_2.$$

A conjunct of the form $f(x) = r$ or $f(x) > r$ are dealt with similarly.

Next, we consider a conjunct of the form $f(x) < g(x)$. If there are no solutions of the inequality then the conjunct can be replaced by $x \neq x$. If there are solutions then the hypothesis that the functions cross only finitely many times says that there are open intervals $(b_1, c_1), \ldots, (b_n, c_n)$ so that for all a, $f(a) < g(a)$ if and only if there is some i such that $a \in (b_i, c_i)$. In which case the conjunct can be replaced by the disjunction of the clauses $b_i < x \wedge x < c_i$ $(i = 1, \ldots, n)$. Other possibilities, as well as the conjuncts $f(x) = g(x)$ and $f(x) > g(x)$, are dealt with similarly.

Lastly, we consider a conjunct of the form $f(x) < g(y)$. We are confronted by several possibilities. By clause (ii) of Definition 3.1, we may assume that the domain of f is (r_1, r_2), the range of f is (s_1, s_2), the domain of g is (t_1, t_2), the range of g is (u_1, u_2), and that, say, f is monotonically decreasing and g is monotonically increasing. Also, r_i, s_i, t_i, and u_i for $i = 1, 2$ can be real numbers or $\pm\infty$, appropriately. Suppose also that there is some c between t_1 and t_2 for which $g(c) = s_2$. Then, it is easy to see that we can replace $f(x) < g(y)$ by

$$(r_1 < x \wedge x < r_2) \wedge \Big[y > c \vee \big(s_1 < g(y) \wedge g(y) < s_2 \wedge x < f^{-1}(g(y))\big)\Big].$$

Since $\mathcal{F}$ is closed under inverses and composition by (iii) of Definition 3.1, it is clear that the formula above is as desired. Other cases, and the conjuncts $f(x) = g(y)$ and $f(x) > g(y)$ are dealt with likewise. So the proof of (a) is complete. ∎

PROOF OF (b). The existence of $\mathcal{M}$ is given by Theorem 1.11 of [5]. Clearly, $\mathcal{M}^*$ satisfies clause (iii) in Definition 3.1. It also satisfies clause (iv) since $\mathcal{M}$ is o-minimal. Hence, $\mathcal{M}^*$ is a canonical binary structure. Using cell-decomposition (Theorem 1.4) and the fact that **T** is binary, it is a simple matter to show that every definable relation in $\mathcal{M}$ is definable in $\mathcal{M}^*$. ∎

Theorem 3.2 can be extended to binary theories whose underlying order is dense without endpoints at the price of introducing more complicated definitions. The reader can probably supply the extension but we will indicate how it goes for convenience. We say that two partial unary functions f and g defined in some structure $\mathcal{M}$ *cross between* a and b if either $g(a) < f(a)$ and $g(b) > f(b)$ or $g(a) > f(a)$ and $g(b) < f(b)$.

DEFINITION 3.3. A structure $\mathcal{M} = (M, <, f)_{f \in \mathcal{F}}$ is a *canonical dense binary structure* if

(i) $(M, <)$ is a dense linear order without endpoints,

(ii) each $f \in \mathcal{F}$ is a partial unary function and which is strictly monotonic and continuous bijection from one interval in $\mathcal{M}$ to another, the identity function is a member of $\mathcal{F}$,

(iii) $\mathcal{F}$ is closed under inverses and composition,

https://doi.org/10.4153/CJM-1992-050-1 Published online by Cambridge University Press

(iv) all distinct $f, g \in \mathcal{F}$ cross at only finitely many points,

(v) for all $a < b$ if f, g cross between a and b and the interval (a, b) is contained in the domain of both f and g then there is some c so that $a < c < b$ and f, g cross at c.

THEOREM 3.4. *(a.) Any canonical dense binary structure is o-minimal.*

(b) Let $\mathbf{T}$ *be a binary o-minimal theory having a model* $\mathcal{M}$ *whose underlying order is dense without endpoints. If we let* $\mathcal{F}^*$ *consist of all definable partial functions in* $\mathcal{M}$ *satisfying clause (ii) in Definition 3.1, then* $\mathcal{M}^* = (M, <, f)_{f \in \mathcal{F}^*}$ *is a canonical dense binary structure and every definable relation in* $\mathcal{M}$ *is definable in* $\mathcal{M}^*$.

PROOF. The proof is essentially the same as the proof of Theorem 3.2. ■

REFERENCES

1. J. Knight, A. Pillay and C. Steinhorn, *Definable sets in ordered structures II*, Trans. Amer. Math. Soc. **295**(1986), 593–605.
2. D. Marker, *Omitting types in o-minimal theories*, Journal of Symbolic Logic **51**(1986), 63–74.
3. A. Pillay, *Some remarks on definable equivalence relations in o-minimal structures*, Journal of Symbolic Logic **51**(1986), 709–714.
4. A. Pillay and C. Steinhorn, *Definable sets in ordered structures I*, Trans. Amer. Math. Soc. **295**(1986), 565–592.
5. A. Pillay and C. Steinhorn, *On Dedekind complete o-minimal structures*, Journal of Symbolic Logic **52** (1987), 156–164.

Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B.C. V5A 1S6

Department of Mathematics
Ben Gurion University of the Negev
Beer Sheva, Israel

Department of Mathematics
Vassar College
Poughkeepsie, New York
USA 12601