A PRESENTATION OF THE MATHIEU GROUP M_{12}

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In the course of work on factor groups of the modular group, A. O. L. Atkin (private communication) obtained the permutations

$$\begin{aligned} S&: (0)(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11), \\ T_1&: (0\ 1)(2\ 11)(3\ 4)(5\ 10)(6\ 7)(8\ 9), \\ T_2&: (0\ 1)(2\ 11)(4\ 10)(6\ 9)(3)(5)(7)(8), \end{aligned}$$

which satisfy the relations

$$(1) \qquad S^{11} = T^2 = (ST)^3 = (S^{-1}TST)^6 = (S^{-2}TS^2T)^5 = E$$

for $T = T_1, T_2$. It was evident that neither pair S, T generates either A_{12} or $LF(2, 11)$, so he suggested that they probably generate the Mathieu group M_{12}.

To see that they do so, we can check that each pair S, T is transitive on the 132 hexads of the Steiner system $S(5, 6, 12)$ comprising the pairs of complementary hexads

0	1	2	3	4	6,	5	7	8	9	10	11,
0	1	2	3	7	10,	4	5	6	8	9	11,
0	1	2	3	8	9,	4	5	6	7	10	11,
0	1	2	4	5	8,	3	6	7	9	10	11,
0	1	2	4	7	9,	3	5	6	8	10	11,
0	1	2	6	8	10,	3	4	5	7	9	11,

and their transforms under the cyclic permutation

$$S: (0)(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11).$$

Thus each of the pairs S, T generates M_{12} which is the group of automorphisms of this Steiner system.

Having found these generators for M_{12}, we may enquire

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what further relations they satisfy, and whether these lead to a concise presentation for M_{12}. We note first that although S, T_1 and S, T_2 satisfy the same relations (1), they are not automorphs, as they satisfy the different relations

$$(S^{-3}T_1S^3T_1)^5 = E, \qquad (S^{-3}T_2S^3T_2)^6 = E.$$

In fact, in terms of either pair of generators S, T, the outer automorphism of M_{12} may be taken as fixing T and exchanging ST with its inverse. So in looking for presentations of M_{12} we cannot consider both pairs S, T together.

The following method was adopted to obtain a specific presentation. Pairs of elements were examined, looking for a pair which (a) generates a subgroup with a known small set of defining relations, and (b) admits enumeration of the cosets of the subgroup using only a small set of further relations. The following is the most concise presentation found.

We write $U = T_1S^2T_1S^{-1}$; then T_1, U generate the group $PGL(2, 3^2)$ of order 720 defined by the relations

$$\text{(2)} \qquad U^{10} = T_1^{\,2} = (UT_1)^3 = (U^{-1}T_1UT_1)^4 = (U^{-4}T_1U^4T_1)^2 = E,$$

and its 132 cosets in M_{12} can be enumerated using only the relations

$$\text{(3)} \qquad S^{11} = T_1^{\,2} = (ST_1)^3 = (S^{-1}T_1ST_1)^6 = (S^3T_1S^6T_1)^3 = E.$$

Since $T_1^{\,2} = E$ is common to (2) and (3), and UT_1 is a conjugate of ST_1, there are only eight distinct relations in this presentation. In fact either of the last two relations of (2) may be omitted also, as we can see thus. If either of these relations is omitted, it can be shown that the remaining four relations (2) define groups of order 2160 = 3.720.* So together with the relations (3) they define either M_{12} or a group three times greater with M_{12} as a factor group. If they defined a group three times greater, it would have to have a representation by transitive permutations

* These groups are not isomorphic. That with the last relation omitted has no subgroup of index 3, while that with the other relation omitted has a subgroup of index 3 generated by U^2, T_1, which turns out to be $PGL(2, 3^2)$ again.

S': (0) (1 2 3 4 5 6 7 8 9 10 11) (0') (1' 2' 3' 4' 5' 6' 7' 8' 9' 10' 11') (0'') (1'' 2'' 3'' 4'' 5'' 6'' 7'' 8'' 9'' 10'' 11''),

T' including (0 1) (0' 1') (0'' 1''),

reducing by identification of corresponding numbers to S, T_1 for M_{12}. But it is not difficult to show that there are no such permutations S', T' compatible with the relations (3).

We have thus reduced the presentation to the following seven relations, in which U has been expressed in terms of S, T_1, and some later relations have been simplified by use of the relation $(ST_1)^3 = E$:

$$S^{11} = T_1^2 = (ST_1)^3 = (S^3T_1)^6 = (S^3T_1S^6T_1)^3$$
$$= (S^4T_1)^{10} = (S^2T_1S^{-2}T_1S^3T_1)^4 = E.$$

It is not known whether this set is irreducible. The presentations of Moser [1, 3] and Garbe and Mennicke [2] comprise more relations, as they are based on presentations of M_{11} extended by adjunction of a further generator, and no known presentation of M_{11} is as concise as (2). The work of Atkin shows that M_{11} is not a factor group of the modular group, so it cannot be generated by a pair of elements of periods 2 and 3. He has also shown that, up to automorphisms, the generators T_1, ST_1 and T_2, ST_2 are the only possibilities for M_{12}, so any such generators for M_{12} satisfy the relations (1).

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