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ON THE DIOPHANTINE EQUATION $x^m = y^{n_1} + y^{n_2} + \cdots + y^{n_k}$

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Abstract. We find all solutions of $x^m = y^{n_1} \pm y^{n_2} \pm y^{n_3} \pm y^{n_4}$ in integers m, n_1, n_2, n_3, n_4 for all relatively prime integers x, y below 100.

2000 *Mathematics Subject Classification.* 11A05, 11A25.

1. Introduction. In 1970, Senge and Strauss [**5**] proved that the sums of all digits of n in bases a and b cannot be both small if and only if $\log a/\log b$ is irrational. Denote the sum of all digits in the canonical expansion of n in base a by $N_a(n)$. They proved that there are only finitely many integers n with both of $N_a(n)$, $N_b(n)$ below a fixed bound. Their proof is not effective, depending upon Mahler's generalization of Thue-Siegel-Roth theorem.

In 1980, Stewart [**6**] proved this result in an effective way, using Baker theory of linear forms in logarithms. Thus, in principle, we can effectively determine all integers n satisfying $N_a(n), N_b(n) \leq k$ for any given a, b, k such that $\log a/\log b$ is irrational.

Tijdeman and Wang [**7**], Wang [**8**] and Deze and Tijdeman [**2**] determined all integers n with $N_p(n) + N_q(n) \leq 4$ for primes $p, q < 200$. Indeed, they determined all solutions of the equation $x \pm y \pm z \pm w = 0$ in integers $x, y, z, w \in S(p, q)$ for two primes $p, q < 200$, where $S(p, q)$ denotes the set of integers composed only of p, q.

In this paper, we determine all prime powers p^e with $N_q(p^e) \leq 4$ for small primes p, q. Indeed, we shall find all the solutions of the slightly generalized equation

$$x^e = \epsilon_1 y^{f_1} + \epsilon_2 y^{f_2} + \epsilon_3 y^{f_3} + \epsilon_4 y^{f_4}, \tag{1}$$

in integers e, $\epsilon_i \in \{-1, 0, 1\}$ and $f_1 \geq f_2 \geq f_3 \geq f_4 \geq 0$, for relatively prime integers $2 \leq x, y \leq 100$. The case $(x, y) > 1$ shall be discussed in the forthcoming papers.

Since the case $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 0$ is dealt in [**2**], we deal only the case no subsum of the right of (1) vanishes (in particular, none of ϵ_i's is equal to zero).

We begin by noting that we must have $f_4 = 0$ since $(x, y) = 1$. Moreover, we find that ϵ_1 must be positive. Assume that $\epsilon_1 = -1$ and $f_1 > f_2$. Then, we must have $x^e \leq -y^{f_1} + y^{f_1-1} + y^{f_1-1} + 1 \leq 1 - y^{f_1-1}(y-2) \leq 1$, which is a contradiction. If $\epsilon_1 = -1$ and $f_1 = f_2$, then we must have $\epsilon_2 = -1$.

Moreover, we may assume that x, y are not perfect powers. Otherwise we can replace e by el if $x = x_0^l$ for some $x_0, f \geq 2$ and we can replace f_i's by $f_i g$ if $y = y_0^g$ for some $y_0, g \geq 2$.

THEOREM 1.1. *Let* $2 \leq x, y \leq 100$ *be relatively prime integers below* 100 *which are not perfect powers. Any solution of (1) satisfies one of the following conditions:*

(1) *The solution appears in Table 1.*

Table 1. Perfect powers representable as $y^{f_1} \pm y^{f_2} \pm y^{f_3} \pm y^{f_4}$

x^e	One of representations	x^e	One of representations
2^4	$3^2+3^2-3^0-3^0$	15^2	$2^7+2^7-2^5+2^0$
2^4	$3^2+3^2-3^1+3^0$	17^2	$2^8+2^5+2^0$
2^5	$3^3+3^1+3^1-3^0$	17^2	$6^3+6^2+6^2+6^0$
2^5	$3^3+3^2-3^1-3^0$	17^2	$12^2+12^2+12^0$
2^8	$3^5+3^2+3^1+3^0$	18^2	$13^2+13^2-13^1-13^0$
2^9	$3^5+3^5+3^3-3^0$	20^2	$7^3+7^2+7^1+7^0$
2^9	$3^6-3^5+3^3-3^0$	20^3	$63^2+63^2+63^1-63^0$
2^5	$5^2+5^1+5^0+5^0$	21^2	$2^9-2^6-2^3+2^0$
2^7	$5^3+5^1-5^0-5^0$	22^2	$3^5+3^5-3^1+3^0$
2^8	$5^3+5^3+5^1+5^0$	23^2	$2^8+2^8+2^4+2^0$
2^{12}	$45^2+45^2+45^1+45^0$	23^2	$2^9+2^3+2^3+2^0$
2^{13}	$91^2-91^1+91^0+91^0$	23^2	$2^9+2^4+2^0$
3^3	$2^5-2^1-2^1-2^0$	23^3	$78^2+78^2-78^0$
3^3	$2^5-2^2-2^0$	24^2	$17^2+17^2-17^0-17^0$
3^3	$2^4+2^3+2^2-2^0$	26^2	$3^6-3^4+3^3+3^0$
3^3	$2^4+2^4-2^2-2^0$	26^3	$3^9-3^7+3^4-3^0$
3^4	$2^6+2^4+2^0$	26^2	$15^2+15^2+15^2+15^0$
3^5	$2^8-2^4+2^2-2^0$	28^2	$3^6+3^4-3^3+3^0$
3^3	$5^2+5^0+5^0$	28^3	$3^9+3^7+3^4+3^0$
3^5	$11^2+11^2+11^0$	31^2	$2^9+2^9-2^6+2^0$
5^2	$2^3+2^3+2^3+2^0$	31^2	$2^{10}-2^6+2^0$
5^2	$2^4+2^3+2^0$	33^2	$2^9+2^9+2^6+2^0$
5^2	$2^5-2^3+2^0$	33^2	$2^{10}+2^6+2^0$
5^3	$2^6+2^6-2^1-2^0$	33^2	$10^3+10^2-10^1-10^0$
5^3	$2^6+2^6-2^2+2^0$	39^2	$2^{10}+2^9-2^4+2^0$
5^3	$2^7-2^1-2^0$	39^2	$2^{11}-2^9-2^4+2^0$
5^3	$2^7-2^2+2^0$	41^2	$29^2+29^2-29^0$
5^4	$2^9+2^7-2^4+2^0$	44^2	$3^7-3^5-3^2+3^0$
5^2	$3^3+3^0+3^0$	45^2	$2^{11}-2^4-2^3+2^0$
5^2	$3^3-3^1+3^0$	46^2	$3^7-3^4+3^2+3^0$
7^2	$2^4+2^4+2^4+2^0$	47^2	$2^{11}+2^7+2^5+2^0$
7^2	$2^5+2^3+2^3+2^0$	47^2	$13^3+13^1-13^0$
7^2	$2^5+2^4+2^0$	49^2	$2^6-2^4+2^0$
7^2	$2^6-2^4+2^0$	49^2	$2^5+2^5-2^4+2^0$
7^2	$5^2+5^2-5^0$	53^2	$6^4+6^4+6^3+6^0$
7^3	$18^2+18^1+18^0$	56^2	$5^5+5^1+5^1+5^0$
7^3	$19^2-19^1+19^0$	56^2	$15^3-15^2-15^1+15^0$
10^2	$3^4+3^3-3^2+3^0$	57^2	$5^5+5^3-5^0$
10^3	$3^6+3^5+3^3+3^0$	58^2	$41^2+41^2+41^0+41^0$
10^2	$7^2+7^2+7^0+7^0$	63^2	$2^{11}+2^{11}-2^7+2^0$
11^2	$2^6+2^6-2^3+2^0$	63^2	$2^{12}-2^7+2^0$
11^2	$2^7-2^2-2^1-2^0$	65^2	$2^{11}+2^{11}+2^7+2^0$
11^2	$2^7-2^2-2^2+2^0$	65^2	$2^{12}+2^7+2^0$
11^2	$2^7-2^3+2^0$	72^2	$17^3+17^2-17^1-17^0$
11^2	$5^3-5^1+5^0$	80^2	$3^8-3^5+3^4+3^0$
11^3	$6^4+6^2-6^0$	80^3	$3^{12}-3^9+3^5-3^0$
11^3	$37^2-37^1-37^0$	82^2	$3^8+3^5-3^4+3^0$
12^2	$5^3+5^2-5^1-5^0$	82^3	$3^{12}+3^9+3^5+3^0$
13^2	$2^7+2^5+2^3+2^0$	89^2	$2^{13}-2^8-2^4+2^0$
13^3	$3^7+3^2+3^0$	97^2	$56^2+56^2+56^2+56^0$
15^2	$2^8-2^5+2^0$	99^2	$70^2+70^2+70^0$

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(2) *There exists a pair of integers* w, g *such that* $e = 2g$, $(x, y) = (w, w^g \pm 1)$, $(f_1, f_2, f_3) = (2, 1, 1)$ *and* $\epsilon_2 = \epsilon_3 = \mp 1$, $\epsilon_4 = 1$.

(3) $e \leq 1$ *or* $f_1 \leq 1$.

(4) *The solution arises from the relations* $2^3 = 3^2 - 3^0$ *and* $3^2 = 2^3 + 2^0$.

Our method is similar to the one of Deze, Tijdeman and Wang in its essence. Deze, Tijdeman and Wang used de Weger's lower bound for differences between two prime powers [**9**], which follows from lower bounds for linear forms in two logarithms and classical theory of continued fractions.

However, we need to deal linear forms of three logarithms of the form $e \log x - f \log y - \log A$. In order to improve lower bounds for such linear forms, we use a variant of Davenport's method [**1**].

Our table suggests that an integer solution of (1) with $\min\{e, f_1\} \geq 3$ can be derived from one of the identities (a) $2^3 = 3^f - 3^f + 3^2 - 3^0$ and (b) $(3^g \pm 1)^3 = 3^{3g} \pm 3^{2g+1} + 3^{g+1} \pm 3^0$, except from finitely many ones.

2. Results on linear forms of logarithms. First, we introduce some notations. We denote by $\|\alpha\|$ the smallest distance between α and the integers and denote $v_m(n)$ by the largest integer v such that $m^v \mid n$.

We use Matveev's lower bound for linear forms in logarithms, which is the best known result applicable for an arbitrary one.

THEOREM 2.1 (Matveev's theorem). *Let* $a_1, a_2, \ldots, a_n$ *be non-zero integers such that* $\log a_1, \ldots, \log a_n$ *are not all zero. Let* $A_1, \ldots, A_n$ *be real numbers such that* $A_j \geq \max\{0.16, \log a_j\}$ *for each* j.

Put

$$\Lambda = b_1 \log a_1 + \cdots + b_n \log a_n \tag{2}$$

and

$$\begin{aligned} B &= \max\{1, |b_1|\, A_1/A_n, |b_2|\, A_2/A_n, \ldots, |b_n|\}, \\ \Omega &= A_1 A_2, \ldots, A_n, \\ C(n) &= \frac{16}{n!} e^n (2n+3)(n+2)(4(n+1))^{n+1} \left(\frac{1}{2} en\right) (4.4n + 5.5 \log n + 7). \end{aligned} \tag{3}$$

Then we have

$$\log |\Lambda| > -C(n)(\log 3 - \log 2 + 1 + \log B) \max\left\{1, \frac{n}{6}\right\} \Omega. \tag{4}$$

Although Matveev's theorem gives an effective bound for solutions of our equation, this bound is fairly large. We wish to obtain a stronger lower bounds to $|\Lambda|$ for coefficients below an upper bound given by Matveev's theorem. Davenport's lemma is useful for this purpose (see [**1**]). We use the following formulation.

THEOREM 2.2 (Davenport's lemma). *Let* $\beta, \theta_1, \theta_2$ *be real numbers such that* $\theta = -\theta_1/\theta_2$ *is irrational. Let* x_1, x_2, X_0 *be integers and* $X = \max\{x_1, x_2\}$. *Put* $\Lambda = \beta + x_1\theta_1 + x_2\theta_2$ *and* $\psi = \beta/\theta_2$.

If a convergent p/q *of* θ *with* $q > X_0$ *satisfies the inequality* $\|q\psi\| > 2X_0/q$, *then* $|\Lambda| > X_0\, |\theta_2|\, /q^2$.

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Proof. Using the fact that $q(x_1\theta - x_2) - x_1(q\theta - p) = px_1 - qx_2$ is an integer, we obtain

$$\|q\psi\| = \|q\Lambda/\theta_2 + q(x_1\theta - x_2)\| = \|q\Lambda/\theta_2 + x_1(q\theta - p)\|. \tag{5}$$

We can easily see that this is at most

$$q|\Lambda/\theta_2| + |x_1|/q \le q|\Lambda/\theta_2| + X_0/q. \tag{6}$$

□

From Theorems 94 and 95 in Nagell [**4**], we obtain the following lemma.

LEMMA 2.3. *For any odd prime q dividing y, we have*

$$v_y(x^e - 1) \le (v_q(e) + v_q(x^{q-1} - 1))/v_q(y) \tag{7}$$

and

$$v_y(x^e + 1) \le (v_q(e) + v_q(x^{q-1} - 1))/v_q(y). \tag{8}$$

Moreover, we have

$$v_y(x^e - 1) \le (v_2(e) + v_2(x^2 - 1) - 1)/v_2(y) \tag{9}$$

and

$$v_y(x^e + 1) \le (v_2(x^2 - 1))/v_2(y). \tag{10}$$

Let $a(x, y) = \min v_q(x^{2(q-1)} - 1)/v_q(y)$ over primes q dividing y. We can confirm by computation that $v_q(x^{2(q-1)} - 1) \le 6$ for any prime $q < 100$ and integer $x \le 100$. This immediately gives the following result.

COROLLARY 2.4. *If $x, y < 100$, then we have*

$$v_y(x^e \pm 1) \le a(x, y) + \log e/\log y \tag{11}$$

and

$$v_y(x^e \pm 1) \le 6 + \log e/\log y. \tag{12}$$

3. Proof of main theorem. Throughout this section, we let x, y be relatively prime integers less than 100 which are not perfect powers and put $A = y^{f_1-f_2} + \epsilon_2$, $\Lambda_1 = e\log x - f_1\log y$, $\Lambda_2 = e\log x - f_2\log y - \log A$.

We shall give the first upper bound using Matveev's theorem.

LEMMA 3.1. *If (e, f_1, f_2, f_3) is a solution of* (1)*, then $e < 3 \times 10^{21}$ and $f_1 < 3 \times 10^{21}\log x/\log y$.*

Proof. Assume that $(e, f_1, f_2, f_3, \epsilon_2, \epsilon_3, \epsilon_4)(\epsilon_i \in \{-1, 0, 1\}, f_i \ge 0)$ is a solution of (1) such that the right-hand side of (1) has no vanishing subsum.

First we shall obtain an upper bound for A. By (1), we have $y^{f_2} \ge |x^e - y^{f_1}|/2 - 1$. Since $|x^e - y^{f_1}| > y^{f_1}|\Lambda_1|/2$, we have $y^{f_2} > y^{f_1}|\Lambda_1|/5$. This gives

$$|A| \le y^{f_1-f_2} + 1 \le 1 + 5/|\Lambda_1| \le \exp(2)/|\Lambda_1|. \tag{13}$$

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Now we apply Theorem 2.1 to estimate Λ_1. We have

$$\log|A| \leq 2 + C(2)(\log 3 - \log 2 + 1 + \log B_1)\log x \log y, \tag{14}$$

where $B_1 = \max\{e \log x / \log y, f_1\}$ if $x < y$ and $B_1 = \max\{e, f_1 \log y / \log x\}$ if $x > y$.

Since the right-hand side of (1) has no vanishing subsum, we have $y^{f_1} \leq 8x^e$. Therefore we obtain

$$\log|A| \leq 2 + C(2)(\log 3 - \log 2 + 1 + \log(e+3))\log x \log y. \tag{15}$$

Put $A' = \exp(2 + C(2)(\log 3 - \log 2 + 1 + \log(e+3))\log x \log y)$, so that the inequality $|A| \leq A'$ holds.

Second we shall obtain a lower bound for f_3. We can easily see that

$$y^{f_3} \geq x^e - y^{f_1} - y^{f_2} - 1 \geq x^e - Ay^{f_2} - 1 \geq x^e\,|\Lambda_2|\,/3. \tag{16}$$

Applying Theorem 2.1 again, we have

$$\log|\Lambda_2| \geq W \log x \log y \log A', \tag{17}$$

where

$$W = C(3)(\log 3 - \log 2 + 1 + \log B_2), \tag{18}$$

$$B_2 = \max\{e \log x / \log A', f_2 \log y / \log A', 1\}. \tag{19}$$

Combining (16) and (17), we have

$$f_3 \geq \frac{e \log x - \log 3}{\log y} - C(3)(\log 3 - \log 2 + 1 + \log B_2)\log x \log A'. \tag{20}$$

On the other hand, Corollary 2.4 gives

$$f_3 = v_y(x^e \pm 1) \leq 6 + \log e / \log y. \tag{21}$$

It follows from (16), (17) and (21) that

$$6 \log y + \log 3 + \log e + W(\log x)(\log y)(\log A') - e \log x \geq 0. \tag{22}$$

For each pair of relatively prime integers x, y below 100, we see that the left-hand side of (22) is negative for $e = 2.9 \times 10^{21}$ and decreasing for $e \geq 2.9 \times 10^{21}$. This proves the lemma. □

LEMMA 3.2. *If* (e, f_1, f_2, f_3) *is a solution of* (1), *then we have* $|\Lambda_1| \geq 10^{-27}$ *and* $y^{f_1 - f_2} < 5 \times 10^{27}$.

Proof. Let p_n/q_n be the nth convergent of $\log x / \log y$. We set n to be the smallest index for which $q_n > 3 \times 10^{21}$. We confirmed by computation that $|q_n \log x - p_n \log y| \geq 10^{-27}$ for all relatively prime integers x, y below 100. By Lemma 3.1, we have $e < 3 \times 10^{21} \leq q_n$. Hence, by Theorem 184 of [**3**], we have $|\Lambda_1| = |e \log x - f_1 \log y| \geq |q_n \log x - p_n \log y|$. Since we have $y^{f_2} > y^{f_1}\,|\Lambda_1|\,/5$, we have $y^{f_1 - f_2} < 5 \times 10^{27}$. □

LEMMA 3.3. *If* (e, f_1, f_2, f_3) *is a solution of* (1), *then we have* $x^e \geq 3 \times 10^{35} y^{f_3}$.

Proof. We distinguish two cases according to whether $A = x^g$ for some integer $g \geq 0$ or not.

Assume that $A = x^g$ for some integer $g \geq 0$. Then we have $x^e = Ay^{f_2} \pm y^{f_3} \pm 1$. Hence $y^{f_3} \geq \left|x^e - Ay^{f_2}\right| - 1 \geq x^g \left|x^{e-g} - y^{f_2}\right| - 1$. Similarly, with Lemma 3.2, we obtain that $|(e-g)\log x - f_2 \log y| > 10^{-27}$ and therefore $y^{f_3} \geq 10^{-28}x^e$.

Assume that A is not of the form x^g with g a non-negative integer. By definition, A is an integer of the form $y^f \pm 1$, where f is a non-negative integer satisfying $y^f < 5 \times 10^{27}$, but not of the form x^g with g a non-negative integer. For all of such integers, we apply Davenport's method to obtain a lower bound for $|\Lambda_2|$ with $e < 3 \times 10^{21}$. Put $\beta = \log A$, $\theta_1 = \log y$, $\theta_2 = \log x$, $X_0 = 3 \times 10^{21} \log x / \log y$ and apply Theorem 2.2 for all x, y, $A = y^f \pm 1$ with $(x, y) = 1$, $2 \leq x, y < 100$ and $f < \log(5 \times 10^{27})/\log y$.

Computation shows that we can apply Theorem 2.2 with an appropriate convergent p/q of $\theta = -\theta_1/\theta_2$ and we have the inequality $|\Lambda_2| \geq 10^{-35}$ for all x, y, A satisfying this condition. It follows from (16) that $y^{f_3} \geq x^e |\Lambda_2|/3 \geq 10^{-35}x^e/3$. □

Now we prove Theorem 1.1. By Lemma 3.3, we have

$$x^e \leq 3 \times 10^{35} e y^{a(x,y)} \tag{23}$$

by Corollary 2.4.

Computation shows that $a(x, y) \leq 6$ for $x < 100$, $y = 2$, $a(x, y) \leq 4$ for $x < 100$, $y = 3$, $a(x, y) \leq 3$ for $x < 100$, $5 \leq y \leq 23$ and $a(x, y) \leq 2$ for $x < 100$, $24 \leq y < 100$. Hence (23) gives that $e \leq 138$.

Computer search over all integers of the form x^e with $x < 100$, $e \leq 138$ revealed that for any relatively prime integers $x, y < 100$, all solutions of (1) with $e, f_1 \geq 2$ are derived from the identities given in the statement of the theorem except those given in Table 1. This completes the proof.

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