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AREA INTEGRALS AND BOUNDARY BEHAVIOUR OF HARMONIC FUNCTIONS

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Abstract This paper introduces a family of area-type integrals over cones. These are used to investigate non-tangential boundary behaviour of harmonic functions on a half-space, extending results of Stein and Brossard.

Keywords: harmonic function; boundary behaviour; area integral

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1. Introduction

Let $D = \{X = (X', x_n) : X' \in \mathbb{R}^{n-1} \text{ and } x_n > 0\}$, where $n \geqslant 2$, and let u be a harmonic function on D. For fixed $\beta > 0$ and $t > 0$ we define the cone

$$\Gamma(Z'; \beta, t) = \{(X', x_n) \in \mathbb{R}^{n-1} \times (0, t) : |X' - Z'| < \beta x_n\} \quad (Z' \in \mathbb{R}^{n-1}),$$

where $|\cdot|$ denotes the Euclidean norm. Also, let λ_n denote the Lebesgue measure on $\mathbb{R}^n$. A well-known result of Stein [**9**] asserts that if the 'area integral'

$$\left(\int_{\Gamma(Z';\beta,t)} x_n^{2-n} |\nabla u(X)|^2 \, \mathrm{d}\lambda_n(X)\right)^{1/2} \tag{1.1}$$

is finite for every Z' in a Borel set $E' \subset \mathbb{R}^{n-1}$, then u has a finite non-tangential limit $u_{\mathrm{nt}}(Z')$ at $(Z', 0)$ for λ_{n-1}-almost every point $Z' \in E'$. More recently, Brossard [**3**] used a variant of the integral in (1.1) in connection with zero boundary limits of positive harmonic functions. More precisely, he showed that if $u > 0$ (and so $u_{\mathrm{nt}}(Z')$ exists almost everywhere), then the equivalence

$$\int_{\Gamma(Z';\beta,t)} x_n^{2-n} |\nabla u(X)|^2 (u(X))^{-2} \, \mathrm{d}\lambda_n(X) = +\infty \iff u_{\mathrm{nt}}(Z') = 0 \tag{1.2}$$

holds for almost all $Z' \in \mathbb{R}^{n-1}$. (We say that two assertions are equivalent almost everywhere if, outside some set of measure zero, they are simultaneously either true or false.)

The purpose of this paper is to introduce a family of area-type integrals which can be used to extend the results of Stein and Brossard and to yield some additional theorems concerning the boundary behaviour of harmonic functions.

Given a decreasing function $f : (0, +\infty) \to [0, +\infty)$ and a positive harmonic function h on D we define

$$A(u, h, f)(Z') = \int_{\Gamma(Z';\beta,t)} x_n^{2-n} h(X) \left| \nabla\left(\frac{u}{h}\right)(X) \right|^2 f\left(\frac{|u(X)|}{h(X)}\right) \mathrm{d}\lambda_n(X) \quad (Z' \in \mathbb{R}^{n-1}),$$

where $f(0)$ is interpreted as $\lim_{s\to 0+} f(s)$. The expression in (1.1) is $(A(u, 1, 1))^{1/2}$ and the integral in (1.2) is $A(u, 1, t \mapsto t^{-2})$.

Theorem 1.1. *Let u be a positive harmonic function on D and suppose that $f : (0, +\infty) \to [0, +\infty)$ is a decreasing function such that $f(s) \to +\infty$ as $s \to 0+$ and $\int_0^1 sf(s)\,\mathrm{d}s = +\infty$. Then*

$$A(u, 1, f)(Z') = +\infty \iff u_{\mathrm{nt}}(Z') = 0$$

for λ_{n-1}-almost every $Z' \in \mathbb{R}^{n-1}$.

In addition to the case $f(s) = s^{-2}$ (Brossard's result), possible choices of f in Theorem 1.1 include $s^{-2}[1 + \log^+(s^{-1})]^{-a}$ when $0 < a \leqslant 1$. The sharpness of the integral condition on f is easily seen by considering the harmonic function $u(X) = x_n$. We will prove Theorem 1.1 by analytic arguments, in contrast to the probabilistic approach in [**3**].

Theorem 1.2. *Let u and h be harmonic functions on D, with $h > 0$, and let f be the characteristic function valued 1 on $[0, 1)$ and 0 on $[1, +\infty)$. Then u has a finite non-tangential limit at $(Z', 0)$ for λ_{n-1}-almost every $Z' \in \mathbb{R}^{n-1}$ such that $A(u, h, f)(Z') < +\infty$.*

We note that, in the particular case where $h = 1$, Theorem 1.2 relaxes the area integral hypothesis in the result of Stein mentioned earlier. This particular case can also be deduced from a result of Brossard on the density of the area integral (Theorem 2 of [**4**]).

Since our next result holds in a very general context, we need to introduce some notation concerning the Martin representation. (We refer to Chapter 8 of [**1**] for an introduction to this notion, which generalizes the well-known Poisson integral representation for a ball.) Let Ω be a connected open set with Green function $G(\cdot, \cdot)$, and let $X_0 \in \Omega$ be our reference point. The Martin boundary (respectively, minimal Martin boundary) will be denoted by Δ (respectively, Δ_1), and the Martin kernel by $M(X, Y)$. Thus

$$M(X, Y) = \begin{cases} \dfrac{G(X, Y)}{G(X_0, Y)} & (X \in \Omega,\ Y \in \Omega \setminus \{X_0\}), \\[2ex] \lim\limits_{Z\to Y} \dfrac{G(X, Z)}{G(X_0, Z)} & (X \in \Omega,\ Y \in \Delta), \end{cases}$$

and, for each positive harmonic function h on Ω, there is a unique measure μ_h on Δ such that $\mu_h(\Delta \setminus \Delta_1) = 0$ and

$$h(X) = \int_{\Delta} M(X, Y)\,\mathrm{d}\mu_h(Y) \quad (X \in \Omega).$$

Theorem 1.3. *Let u and h be harmonic functions on Ω, where $h > 0$, and let $X_1 \in \Omega$. The following are equivalent:*

(a) *there is a decreasing function $\psi : [0, +\infty) \to (0, +\infty)$ such that $\int_0^{+\infty} \psi(t)\,\mathrm{d}t = +\infty$ and*

$$\int_\Omega G(X_1, X)h(X)\left|\nabla\left(\frac{u}{h}\right)(X)\right|^2 \psi\left(\frac{|u(X)|}{h(X)}\right) \mathrm{d}\lambda_n(X) < +\infty; \tag{1.3}$$

(b) *there is a function $g \in L^1(\mu_h)$ such that $u(X) = \int M(X, Y)g(Y)\,\mathrm{d}\mu_h(Y)$.*

Corollary 1.4. *Let u be a harmonic function on $\Omega \setminus F$, where F is a relatively closed polar subset of Ω. If there is a decreasing function $\psi : [0, +\infty) \to (0, +\infty)$ such that $\int_0^{+\infty} \psi(t)\,\mathrm{d}t = +\infty$ and*

$$\int_\Omega |\nabla u|^2 \psi(|u|)\,\mathrm{d}\lambda_n < +\infty, \tag{1.4}$$

then u has a (unique) harmonic extension to Ω.

This corollary relaxes the well-known result that polar sets are removable for harmonic functions with finite Dirichlet integral (condition (1.4) with $\psi = 1$). Other suitable choices for $\psi(t)$ are $(1 + t)^{-1}$ and $[(1 + t)\log(2 + t)]^{-1}$. The sharpness of the result is seen if we consider the case where Ω is the unit ball, $F = \{0\}$, and $u(X) = \log|X|$ when $n = 2$ or $u(X) = |X|^{2-n}$ when $n \geqslant 3$.

In our next application of Theorem 1.3, the open set Ω is a Lipschitz domain. In this case the Martin compactification of Ω is homeomorphic to its Euclidean closure and all points of the Martin boundary are minimal (see [**7**], or Theorem 8.8.4 of [**1**]), so we can identify Δ (and Δ_1) with the Euclidean boundary $\partial\Omega$.

Corollary 1.5. *Let u and h be harmonic functions on a Lipschitz domain Ω, where $h > 0$, and let E be a subset of $\partial\Omega$ such that $\mu_h(E) = 0$. If $X_1 \in \Omega$ and there is a decreasing function $\psi : [0, +\infty) \to (0, +\infty)$ with $\int_0^\infty \psi(t)\,\mathrm{d}t = +\infty$ such that*

$$\int_\Omega G(X_1, X)h(X)\left|\nabla\left(\frac{u}{h}\right)(X)\right|^2 \psi\left(\frac{|u(X)|}{h(X)}\right) \mathrm{d}\lambda_n(X) < +\infty, \tag{1.5}$$

then u has non-tangential limit 0 at σ-almost every point of E, where σ denotes the surface area measure on $\partial\Omega$.

Our final corollary bears a superficial resemblance to Theorem 1.1. However, unlike that result, it applies to harmonic functions u of variable sign.

Corollary 1.6. *Let u be a harmonic function on D and let $\psi : [0, +\infty) \to (0, +\infty)$ be a decreasing function such that $\int_0^{+\infty} \psi(t)\,\mathrm{d}t = +\infty$. Then $u_{\mathrm{nt}}(Z') = 0$ for λ_{n-1}-almost every $Z' \in \mathbb{R}^{n-1}$ such that $A(u, x_n, \psi)(Z') < +\infty$.*

We will prove Theorems 1.1 and 1.2 in § 3, and Theorem 1.3 and its corollaries in § 4, following some preparatory observations in the next section.

2. Preparatory material

Theorem 2.1. *Let E' be a measurable subset of $\mathbb{R}^{n-1}$ and let u be a harmonic function on D. The following are equivalent:*

(a) *u has a finite non-tangential limit at $(Z', 0)$ for λ_{n-1}-almost every $Z' \in E'$;*

(b) *u is bounded in a cone of vertex Z' for λ_{n-1}-almost every $Z' \in E'$;*

(c) *$A(u, 1, 1)(Z') < +\infty$ for λ_{n-1}-almost every $Z' \in E'$.*

The equivalence of (a) and (b) in Theorem 2.1 was shown by Calderón [**5**]. We have already mentioned that Stein proved that (c) implies (a). In the same paper [**9**] he showed that (b) implies (c), and also proved the following lemma.

Lemma 2.2. *Let $\Phi : D \to [0, +\infty)$ be a measurable function, let E' be a bounded Borel subset of $\mathbb{R}^{n-1}$ such that*

$$\int_{\Gamma(Z';\beta,t)} x_n^{2-n} \Phi(X) \, \mathrm{d}\lambda_n(X) < +\infty \quad (Z' \in E'),$$

and let $\varepsilon > 0$, $\alpha < \beta$ and $r < t$. Then there is a compact subset F' of E' satisfying $\lambda_{n-1}(F') > \lambda_{n-1}(E') - \varepsilon$ and such that

$$\int_U x_n \Phi(X) \, \mathrm{d}\lambda_n(X) < +\infty,$$

where $U = \bigcup_{Z' \in F'} \Gamma(Z'; \alpha, r)$.

The proof of the following elementary lemma is left to the reader.

Lemma 2.3. *Let $\sum a_m$ be a convergent series of positive real numbers. Then there is an unbounded, strictly increasing sequence (k_m) of positive numbers such that $\sum_m k_m a_m < +\infty$, and $k_{m+2} - k_{m+1} \leqslant k_{m+1} - k_m$ for all m.*

3. Proofs of Theorems 1.1 and 1.2

3.1. Proof of Theorem 1.1

Suppose that $f : (0, +\infty) \to [0, +\infty)$ is a decreasing function such that $f(s) \to +\infty$ as $s \to 0+$ and $\int_0^1 s f(s) \, \mathrm{d}s = +\infty$.

First we make the easy observation that $u_{\mathrm{nt}} = 0$ almost everywhere (λ_{n-1}) on the set of points Z', where $A(u, 1, f)(Z') = +\infty$. To see this, we note that, since $u > 0$, the function u_{nt} exists and is finite at λ_{n-1}-almost every point $Z' \in \mathbb{R}^{n-1}$. Hence, by Theorem 2.1,

$$A(u, 1, 1)(Z') < +\infty \quad \text{for } \lambda_{n-1}\text{-almost every } Z' \in \mathbb{R}^{n-1}.$$

The conclusion follows since, if $u_{\mathrm{nt}}(Z') > 0$, then $f(u)$ is bounded (by $c_{Z'}$, say) on the cone $\Gamma(Z'; \beta, t)$, and so $A(u, 1, f)(Z') \leqslant c_{Z'} A(u, 1, 1)(Z')$.

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Conversely, we will show that $u_{\mathrm{nt}}(Z') > 0$ for λ_{n-1}-almost every $Z' \in E'_k$, where

$$E'_k = \{Z' : A(u,1,f)(Z') < +\infty \text{ and } |Z'| \leqslant k\}.$$

Let $\varepsilon > 0$, let $\alpha < \beta$ and $r < t$. We apply Lemma 2.2 with $\Phi = |\nabla u|^2 f \circ u$ to see that there is a compact subset F' of E'_k satisfying $\lambda_{n-1}(F') > \lambda_{n-1}(E'_k) - \varepsilon$ and such that

$$\int_U x_n |\nabla u(X)|^2 f(u(X))\,\mathrm{d}\lambda_n(X) < +\infty, \tag{3.1}$$

where $U = \bigcup_{Z' \in F'} \Gamma(Z'; \alpha, r)$.

The function f is not necessarily continuous so we define $f_1 : (0,+\infty) \to [0,+\infty)$ to be that function whose graph consists of the line segments joining the points

$$\{(2^{-k}, f(2^{1-k})) : k \in \mathbb{N}\} \cup \{(s,0) : s \geqslant 1\}.$$

Then f_1 is a decreasing continuous function, $f_1 \leqslant f$ and $\int_0^1 s f_1(s) = +\infty$. We define $\Psi : (0,+\infty) \to \mathbb{R}$ by $\Psi(x) = \int_1^x \int_1^y f_1(s)\,\mathrm{d}s\,\mathrm{d}y$. Then Ψ is a positive C^2 convex function that satisfies $\Psi(s) \to +\infty$ as $s \to 0+$.

Let χ_U denote the characteristic function of U and let $a_n = \sigma_n \max\{1, n-2\}$, where σ_n is the surface area of the unit sphere in $\mathbb{R}^n$. Using standard estimates of the Green function for D, it is clear from (3.1) that we can form the potential v on D of the measure

$$\frac{1}{a_n} \chi_U(X) |\nabla u(X)|^2 f_1(u(X))\,\mathrm{d}\lambda_n(X).$$

It follows that

$$\Delta v = -|\nabla u|^2 f_1(u) = -|\nabla u|^2 \Psi''(u) = -\triangle \Psi(u) \quad \text{on } U.$$

Hence $v + \Psi \circ u$ is a positive harmonic function on U and so, by Théorème 10 of [**2**], has a finite non-tangential limit at $(Z', 0)$ for λ_{n-1}-almost every point $Z' \in F'$. It follows that $\Psi \circ u$ is non-tangentially bounded at such points. However, $\Psi(u) \to +\infty$ as $u \to 0+$. Hence $u_{\mathrm{nt}}(Z') > 0$ for λ_{n-1}-almost every $Z' \in F'$. Since ε and k are arbitrary, it follows that $u_{\mathrm{nt}}(Z') > 0$ for λ_{n-1}-almost every $Z' \in \mathbb{R}^{n-1}$, where $A(u,1,f)(Z') < +\infty$.

3.2. Proof of Theorem 1.2

Let u, h and f be as in the statement of the theorem, let

$$E'_k = \{Z' : A(u,h,f)(Z') < +\infty \text{ and } |Z'| \leqslant k\},$$

and let $\alpha < \beta$, $r < t$ and $\varepsilon > 0$. If we define $V = \{X \in D : |u(X)| < h(X)\}$, then we can apply Lemma 2.2 with $\Phi = h|\nabla(u/h)|^2 f(|u|/h)$ to see that there is a compact subset F' of E'_k satisfying $\lambda_{n-1}(F') > \lambda_{n-1}(E'_k) - \varepsilon$ and such that

$$\int_{U \cap V} x_n h(X) \left| \nabla \left(\frac{u}{h} \right)(X) \right|^2 \mathrm{d}\lambda_n(X) < +\infty, \tag{3.2}$$

where $U = \bigcup_{Z' \in F'} \Gamma(Z'; \alpha, r)$.

Suppose that $\Psi : \mathbb{R} \to (0, +\infty)$ is a C^2 convex function such that $\Psi(t) = |t|$ on $\mathbb{R} \setminus (-1, 1)$. Then $\Psi''(u/h) = 0$ on $D \setminus V$. Hence, in view of (3.2), we can form the potential v on D of the measure

$$\frac{1}{a_n}\chi_U(X)h(X)\left|\nabla\left(\frac{u}{h}\right)(X)\right|^2\Psi''\left(\frac{u(X)}{h(X)}\right)\mathrm{d}\lambda_n(X),$$

and

$$\Delta v = -h\left|\nabla\left(\frac{u}{h}\right)\right|^2\Psi''\left(\frac{u}{h}\right) = -\Delta\left(h\Psi\left(\frac{u}{h}\right)\right) \quad \text{on } U.$$

The function $H = v + h\Psi(u/h)$ is positive and harmonic on U. Since

$$|u| = h\left|\frac{u}{h}\right| \leqslant h\Psi\left(\frac{u}{h}\right) \leqslant v + h\Psi\left(\frac{u}{h}\right) = H \quad \text{on } U,$$

we can apply Théorème 10 of [**2**] to the positive harmonic functions H and $H - u$ to see that each, and thus u, has a finite non-tangential limit at $(Z', 0)$ for λ_{n-1}-almost every point Z' of F'. Since ε and k are arbitrary, the result follows.

4. Proof of Theorem 1.3 and corollaries

4.1. Proof of Theorem 1.3

The proof of Theorem 1.3 relies, in part, on ideas from Parreau [**8**] (see also Theorem 9.4.8 of [**1**]).

Suppose that condition (a) of the theorem holds. Then we can form the potential v on Ω of the measure

$$\frac{1}{a_n}h\left|\nabla\left(\frac{u}{h}\right)\right|^2\psi\left(\frac{|u|}{h}\right)\mathrm{d}(\lambda_n|_\Omega).$$

Define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(x) = \int_0^x \int_0^y \psi(t)\,\mathrm{d}t\,\mathrm{d}y$. Then ϕ is a convex increasing function, $x^{-1}\phi(x) \to +\infty$ as $x \to +\infty$ and $\phi'' = \psi$, and we have

$$\Delta v = -h\left|\nabla\left(\frac{u}{h}\right)\right|^2\psi\left(\frac{|u|}{h}\right) = -\Delta\left(h\phi\left(\frac{|u|}{h}\right)\right) \quad \text{on } \Omega.$$

Let $w = v + h\phi(|u|/h)$. Then w is a positive harmonic function on Ω which majorizes $h\phi(|u|/h)$ and thus $|u| \leqslant h\phi^{-1}(w/h)$. But $h\phi^{-1}(w/h)$ is a superharmonic function on Ω (see Theorem 3.4.3 of [**1**]), so it follows that the subharmonic function $|u|$ has a harmonic majorant H such that $H \leqslant h\phi^{-1}(w/h)$. Whence $h\phi(H/h) \leqslant w$. Let $m \in \mathbb{N}$ and define $b_m = \sup\{t/\phi(t) : t \geqslant m\}$. Then

$$t \leqslant b_m\phi(t) + m \quad (t > 0,\ m \in \mathbb{N}),$$

and

$$H \leqslant b_m h\phi\left(\frac{H}{h}\right) + mh \leqslant b_m w + mh. \tag{4.1}$$

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Let ν_H denote the singular part of μ_H with respect to μ_h. We can choose sets A and B such that $\nu_H(B) = 0 = \mu_h(A)$ and $A \cup B = \Delta_1$, and then observe that

$$\nu_H(\Delta_1) = \nu_H(A) = \mu_H(A) \leqslant b_m \mu_w(A) \quad \text{for all } m.$$

Since $b_m \to 0$ as $m \to +\infty$, we have $\nu_H(\Delta_1) = 0$. Similarly, $\nu_{H-u}(\Delta_1) = 0$, since, from (4.1),

$$0 \leqslant H - u \leqslant 2H \leqslant 2b_m w + 2mh.$$

If we define $\mu_u = \mu_H - \mu_{(H-u)}$, then μ_u is absolutely continuous with respect to μ_h and so, by the Radon–Nikodým Theorem, condition (b) holds.

Conversely, suppose that (b) holds. Our first step is to show that there is an increasing convex C^2 function $\phi : [0, +\infty) \to [0, +\infty)$ such that ϕ'' is decreasing, $t^{-1}\phi(t) \to +\infty$ as $t \to +\infty$ and $\phi(|g|)$ is μ_h-integrable. (If g is bounded, then $\phi(t) = t^2$ will suffice.) Let

$$A_m = \{Y \in \Delta : m - 1 \leqslant |g(Y)| < m\}, \quad m \in \mathbb{N}.$$

Then $\sum_m m\mu_h(A_m) < +\infty$ and it follows from Lemma 2.3 that there is an unbounded strictly increasing sequence (k_m) of positive numbers such that

$$k_{m+2} - k_{m+1} \leqslant k_{m+1} - k_m \leqslant k_1 \quad \text{for all } m,$$

and $\sum_m m k_m \mu_h(A_m) < +\infty$. We define $\psi : [0, +\infty) \to (0, +\infty)$ to be that function whose graph comprises the line segments joining the points $\{(0, k_1)\} \cup \{(m, k_{m+1} - k_m) : m \in \mathbb{N}\}$. Then ψ is a decreasing function and

$$\int_0^s \psi(t)\,\mathrm{d}t \geqslant k_{[s]} - k_1 \to +\infty \quad \text{as } s \to +\infty.$$

Now we define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(t) = \int_0^t \int_0^y \psi(s)\,\mathrm{d}s\,\mathrm{d}y$ and observe that

$$0 \leqslant \phi(t) \leqslant \sum_{i=1}^{m} k_i \quad (t \leqslant m,\ m \geqslant 1).$$

Clearly, ϕ is an increasing convex function such that $\phi'' = \psi$, and $t^{-1}\phi(t) \to +\infty$ as $t \to +\infty$. Further, $\phi(|g|)$ is μ_h-integrable since

$$\int_A \phi(|g|)\,\mathrm{d}\mu_h \leqslant \sum_m \phi(m)\mu_h(A_m) \leqslant \sum_m m k_m \mu_h(A_m) < +\infty.$$

Given $X \in \Omega$, we apply Jensen's inequality with the unit measure $(M(X, \cdot)/h(X))\,\mathrm{d}\mu_h$ to obtain

$$\begin{aligned}\phi\biggl(\frac{|u(X)|}{h(X)}\biggr) &\leqslant \phi\biggl(\frac{\int M(X,Y)|g(Y)|\,\mathrm{d}\mu_h(Y)}{h(X)}\biggr) \\ &\leqslant \int \frac{M(X,Y)}{h(X)} \phi(|g(Y)|)\,\mathrm{d}\mu_h(Y).\end{aligned}$$

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Hence, the subharmonic function $h\phi(|u|/h)$ has a harmonic majorant H_1 given by

$$H_1 = \int M(\cdot, Y)\phi(|g(Y)|)\,\mathrm{d}\mu_h(Y).$$

The non-negative superharmonic function $H_1 - h\phi(|u|/h)$ has Riesz decomposition $v + H_2$, where v is a potential on Ω and H_2 is harmonic there. Hence

$$\Delta v = -\Delta\biggl(h\phi\biggl(\frac{|u|}{h}\biggr)\biggr) = -h\biggl|\nabla\biggl(\frac{u}{h}\biggr)\biggr|^2 \phi''\biggl(\frac{|u|}{h}\biggr) = -h\biggl|\nabla\biggl(\frac{u}{h}\biggr)\biggr|^2 \psi\biggl(\frac{|u|}{h}\biggr)$$

and (a) follows. This completes the proof of Theorem 1.3.

4.2. Proof of Corollary 1.4

Let u be a harmonic function on $\Omega \setminus F$, where F is a relatively closed polar subset of Ω. We choose the reference point X_0 for the Martin compactification of Ω to belong to $\Omega \setminus F$. Then, for each $Y \in \Delta_1 \cup F$, the Martin kernel of $\Omega \setminus F$ is equal to the Martin kernel $M(X, Y)$ of Ω restricted to $\Omega \setminus F$, and the minimal Martin boundary of $\Omega \setminus F$ is $\Delta_1 \cup F$ (see Theorem 9.5.1 (i) of [**1**]). Now, suppose that there is a function ψ as described in Corollary 1.4. It follows that condition (a) of Theorem 1.3, with $h = 1$, holds. So, by condition (b) of Theorem 1.3, there is a μ_1-integrable function g on $\Delta_1 \cup F$ such that

$$u(X) = \int_{\Delta_1 \cup F} M(X, Y)g(Y)\,\mathrm{d}\mu_1(Y) \quad (X \in \Omega \setminus F).$$

Since $\mu_1(F) = 0$ the function u has a harmonic extension to Ω which is clearly unique.

4.3. Proof of Corollary 1.5

Now let u, h, E and ψ be as in Corollary 1.5, and suppose that (1.5) holds. Then condition (a), and hence (b), of Theorem 1.3 holds. Since we can identify both Δ and Δ_1 with the Euclidean boundary $\partial\Omega$, there is a μ_h-integrable function g on $\partial\Omega$ such that

$$u(X) = \int_{\partial\Omega} M(X, Y)g(Y)\,\mathrm{d}\mu_h(Y) \quad (X \in \Omega).$$

It is sufficient to consider the case where $u \geqslant 0$, since u is the difference of the two non-negative harmonic functions

$$\int_{\partial\Omega} M(\cdot, Y)g^+(Y)\,\mathrm{d}\mu_h(Y) \quad \text{and} \quad \int_{\partial\Omega} M(\cdot, Y)g^-(Y)\,\mathrm{d}\mu_h(Y).$$

Now $\mu_u(E) = \int_E g\,\mathrm{d}\mu_h = 0$, so u has minimal fine limit 0 at μ_1-almost every point of E (see Corollary 9.4.2 of [**1**]). Since Ω is Lipschitz, Theorem 5.5 of [**7**] asserts that u has non-tangential limit 0 at every point of E where it has a minimal fine limit 0. Further, by Theorem 3 of [**6**], the measures μ_1 and σ on $\partial\Omega$ are mutually absolutely continuous. The result follows.

4.4. Proof of Corollary 1.6

It remains to establish Corollary 1.6. Let u and ψ be as in the statement of the result, let

$$E'_k = \{Z' : A(u, x_n, \psi)(Z') < +\infty \text{ and } |Z'| \leqslant k\}$$

and let $\alpha < \beta$, $r < t$ and $\varepsilon > 0$. We can apply Lemma 2.2 with

$$\Phi(X) = x_n \left|\nabla\left(\frac{u(X)}{x_n}\right)\right|^2 \psi\left(\frac{u(X)}{x_n}\right)$$

to see that there is a compact subset F' of E'_k satisfying $\lambda_{n-1}(F') > \lambda_{n-1}(E'_k) - \varepsilon$ and such that

$$\int_U x_n^2 \left|\nabla\left(\frac{u(X)}{x_n}\right)\right|^2 \psi\left(\frac{u(X)}{x_n}\right) \mathrm{d}\lambda_n(X) < +\infty, \tag{4.2}$$

where $U = \bigcup_{Z' \in F'} \Gamma(Z'; \alpha, r)$. The number of components in U is finite since F' is compact, so, without loss of generality, we can assume that U is connected. It is not difficult to check that U is a Lipschitz domain. Taking standard estimates for the Green function on D (and noting that the Green function for D at $(X, Y) \in U \times U$ majorizes the Green function for U at (X, Y)), it is clear from (4.2) that condition (1.5) of Corollary 1.5 holds with $h(X) = x_n$ and $\Omega = U$. Since

$$\mu_{x_n}(F' \times \{0\}) = \int_{F' \times \{0\}} x_n \, \mathrm{d}\mu_1(X) = 0,$$

it follows from Corollary 1.5 that $u_{\mathrm{nt}}(Z') = 0$ for λ_{n-1}-almost every $Z' \in F'$. Since ε and k are arbitrary, the result follows.

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