ON THE GENERALIZED CAUCHY EQUATION

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It is the purpose of this note to prove the following theorem: *Let* $f: G \to R$ *be a non-constant continuous function with* G *a locally compact connected topological group and with* R *the real numbers. Let* $C = f(G)$ *and suppose that* $F: C \times C \to C$ *is a function such that*

$$(*) \qquad f(xy) = F(f(x), f(y)), \qquad \textit{all } x, y \in G.$$

Then f *is monotone and open and* F *is continuous.*

This is a generalization of a well-known proposition about functional equations when G is taken as the real numbers; see, for example, Aczel **(1, 2)**. It is a modest step toward the formulation of the principal results of the theory of functional equations in the modern guise of topological algebra. To emphasize the connection with topological algebra, let us write

$$u \circ v = F(u, v), \qquad \text{all } u, v \in C.$$

It can then be seen from $(*)$ that $(C, \circ)$ is a group, that f is a homomorphism from G onto C relative to $\circ$, and since f is open and continuous that $(C, \circ)$ is a topological group.

In the course of proving the theorem, we use a series of lemmas which may not be entirely new but which we have been unable to find elsewhere in the form in which we use them.

Before going into the details of the proof it should be remarked that a function $f: X \to Y$ is open (closed) if $f(A)$ is open (closed) for each open (closed) set $A \subset X$ and $f: X \to Y$ is monotone if $f^{-1}(y)$ is connected for each $y \in Y$. A function is locally one-to-one if every point in the domain has a neighbourhood on which the function is one-to-one. Throughout this paper, all spaces will be considered Hausdorff, a neighbourhood of a point will be a set which contains the point in its interior and the boundary of a set A will be denoted by $\partial(A)$. For general results on topology we refer to Kelley **(3)** and Whyburn **(6)** and for results on topological groups to Montgomery and Zippin **(4)**.

The following proposition is crucial.

PROPOSITION. *If a function which maps a connected locally compact space into the real numbers is locally one-to-one and continuous, then the function is a homeomorphism.*

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It is convenient to prove the following four lemmas.

LEMMA 1. *If a connected, locally compact space can be mapped into the real numbers by a locally one-to-one continuous function, then the space is locally connected.*

Proof. Let X be such a space and $f: X \to R$ be continuous and locally one-to-one. Assume that X is not locally connected at x and let U be a compact neighbourhood of x such that f is one-to-one on U. Then f is a homeomorphism on U so that $f(U)$ is not locally connected at $f(x)$. Now $f(x) \notin f(\partial(U))$ which is compact so there is an open interval N such that $f(x) \in N$ and $N \cap f(\partial(u)) = \emptyset$. Since $f(U)$ is not locally connected at $f(x)$, $N \cap f(U)$ must have an infinite number of components. Therefore, there is an interval $[a, b] \subset E$ such that $a \notin f(U)$ and $b \notin f(U)$ but $[a, b] \cap f(U) \neq \emptyset$. If

$$E = [a, b] \cap f(U) = (a, b) \cap f(U),$$

then $f^{-1}(E) \cap U$ is closed in U, a closed set, and $f^{-1}(E) \cap U$ is open in U^o, an open set, the latter following from the fact that $E \cap f(\partial(U)) = \emptyset$. Thus $f^{-1}(E) \cap U$ is both open and closed, which contradicts the fact that X is connected. Therefore, X is locally connected.

LEMMA 2. *If a locally one-to-one continuous function maps a connected, locally compact space into the real numbers, then the function is one-to-one.*

Proof. By Lemma 1, we may assume that the space is locally connected. Let X be the space and f be the function. It is convenient to prove the following statement.

If U_1 and U_2 are two open connected subsets of X such that $U_1 \cap U_2 \neq \emptyset$, $\overline{U_1}$ and $\overline{U_2}$ are compact, and f is one-to-one on both $\overline{U_1}$ and $\overline{U_2}$, then f is one-to-one on $\overline{U_1} \cup \overline{U_2}$.

Assume that $f(x) = f(y)$ for some $x \in \overline{U_1}$, $y \in \overline{U_2}$, and $x \neq y$, and let $a = f(x) = f(y)$ and $b = f(z)$ for some $z \in U_1 \cap U_2$. Since both $\overline{U_1}$ and $\overline{U_2}$ are connected, $f(\overline{U_1}) \supset [a, b]$ and $f(\overline{U_2}) \supset [a, b]$. Moreover, since f is homeomorphic on both $\overline{U_1}$ and $\overline{U_2}$, there exist homeomorphisms

$$g_1: [a, b] \to \overline{U_1} \text{ and } g_2: [a, b] \to \overline{U_2}$$

such that $f(g_i(x)) = x$ for $i = 1$ and 2. Let $S = \{r | g_1(r) = g_2(r)\}$. Then S is closed and since $g_1(a) = x \neq y = g_2(a)$ while $g_1(b) = g_2(b)$, S is a non-empty proper subset of $[a, b]$. Now $[a, b]$ is connected so $\partial(S) \cap (a, b) \neq \emptyset$. Let $s \in \partial(S) \cap (a, b)$, $t = g_1(s) = g_2(s)$, and W be a neighbourhood of t on which f is one-to-one. Then, since f is homeomorphic on both $\overline{U_1}$ and $\overline{U_2}$, $f(W \cap \overline{U_1}) \cap f(\overline{U_2} \cap W) \cap [a, b]$ is a neighbourhood of s in $[a, b]$ and thus contains a point $r \in [a, b] \setminus S$ so that $g_1(r) \neq g_2(r)$. But both $g_1(r)$ and $g_2(r)$ are elements of W and $f(g_1(r)) = r = f(g_2(r))$, which contradicts the fact that f is one-to-one on W. Thus f is one-to-one on $\overline{U_1} \cup \overline{U_2}$.

To prove the lemma, let $\mathfrak{C}$ be an open cover of X consisting of connected sets, U, such that $\bar{U}$ is compact and f is one-to-one on $\bar{U}$. Then if two sets

U_1 and U_2 from $\mathfrak{C}$ intersect, f is one-to-one on $\overline{(U_1 \cup U_2)}$ by the lemma. By induction we have that if $\bigcup_{i=1} U_i$ is connected, then f is one-to-one on $\bigcup_{i=1}^{n} \bar{U}_i$. However, if x and y are any two distinct points in X, then there exists a finite collection of sets from $\mathfrak{C}$ whose union is connected and contains both x and y. Then $f(x) \neq f(y)$ so that f is one-to-one on X.

LEMMA 3. *If a monotone continuous function maps a connected space into the real numbers, then the inverse of every connected set is connected.*

Proof. Let X and f satisfy the conditions of the lemma. Since any connected set in R is the union of closed intervals, it is sufficient to show that the inverse of every closed interval contained in the image of f is connected. Let A be the closed interval $[a_1, a_2]$ and assume that $f^{-1}(A) = S_1 \cup S_2$, where $\bar{S}_1 \cap S_2 = \emptyset = S_1 \cap \bar{S}_2$. Since f is continuous, $f^{-1}(A)$ is closed so that S_1 and S_2 are disjoint closed sets. From the fact that f is monotone it follows that $f(S_1) \cap f(S_2) = \emptyset$. If both a_1 and a_2 are elements of $f(S_1)$, say, let

$$T = f^{-1}(R \setminus (a_1, a_2)).$$

Then T is closed and $f(T) \cap f(S_2) = \emptyset$ so that T and S_2 are disjoint closed sets. But this would imply that $S_1 \cup T$ and S_2 are disjoint closed sets and since $S_1 \cup T \cup S_2 = X$, this contradicts the connectivity of the space. If $a_1 \in f(S_1)$, say, and $a_2 \in f(S_2)$, then let $T_1 = f^{-1}\{x|x \leqslant a_1\}$ and $T_2 = f^{-1}\{x|x \geqslant a_2\}$. Then since $f(T_1) \cap A = a_1 \in f(S_1)$, T_1 and S_2 are disjoint closed sets. Likewise S_1 and T_2 are disjoint closed sets so that $T_1 \cup S_1$ and $T_2 \cup S_2$ are disjoint closed sets which cover the space, which also contradicts the connectivity of the space. Thus the inverse of every connected set is connected.

LEMMA 4. *If a one-to-one closed function maps a locally connected space into a locally compact space and the image of every connected set is connected, then the function is continuous.*

Proof. Let $f: X \to Y$ satisfy the conditions of the lemma. Consider any $x \in X$ and let $\mathfrak{V}$ be a basis of the neighbourhood system of x consisting of connected sets and $\mathfrak{U}$ a basis of the neighbourhood system of $f(x)$ consisting of compact sets. We must show that for any $U \in \mathfrak{U}$, there is a $V \in \mathfrak{V}$ such that $f(V) \subset U$. Assume that this is not the case for some $U \in \mathfrak{U}$. Then if $V \in \mathfrak{V}$, $f(V) \cap Y \setminus U \neq \emptyset$ so that $f(V) \cap \partial(U) \neq \emptyset$ since the image of every connected set is connected. Observe also that for any finite collection V_i, $i = 1, 2, \ldots, n$, of sets from $\mathfrak{V}$, there is a set $V_0 \in \mathfrak{V}$ such that

$$V_0 \subset \bigcap_{i=1}^{n} V_i$$

and since $f(V_0) \cap \partial(U) \neq \emptyset$,

$$\bigcap_{i=1}^{n} f(V_i) \cap \partial(U) \neq \emptyset.$$

Thus the set $\mathfrak{C} = \{f(\bar{V}) \cap \partial(U) | V \in \mathfrak{B}\}$ has the finite intersection property, and since $\partial(U)$ is compact, $\cap\mathfrak{C}$ contains some element z. But this would mean that $f^{-1}(z)$ is in the closure of every neighbourhood of x, which is impossible since the space is Hausdorff and f is one-to-one. Thus f is continuous at x.

For some related material, see **(5)**.

We can now prove the proposition.

If a locally one-to-one continuous function maps a connected locally compact space into the real numbers, then by Lemma 2, the function is one-to-one. By Lemma 3, the inverse of every connected set is connected and by Lemma 4 the inverse of f is continuous.

THEOREM. *Let f: $G \to R$ be a non-constant continuous function where G is a locally compact connected topological group and R is the real numbers. Let $C = f(g)$ and suppose that there is a binary operation "$\circ$" on C such that*

$$f(x \cdot y) = f(x) \circ f(y), \qquad \text{all } x, y \in G.$$

Then f is monotone and open and $(C, \circ)$ is a topological group.

Proof. If e is the identity element of G and $H = f^{-1}(f(e))$, then since f is a group homomorphism and is continuous, H is a closed normal subgroup of G. Let Q be the component of H which contains e so that Q is also a closed normal subgroup of G. We shall show that f is monotone by showing that $H = Q$. Let π_1 be the canonical map from $G \to G/Q = G'$ and g: $G' \to C$ be the homomorphism induced by f:

$$\begin{tikzcd}
G \arrow[d, "\pi_1"'] \arrow[dr, "f"] & \\
G' \arrow[r] \arrow[d, "\pi_2"'] & C \\
G'' \arrow[ur, "g", "h"'] &
\end{tikzcd}$$

Thus $f = g \circ \pi_1$, where π_1 is open and continuous so that G' is locally compact and connected and g is continuous. If $H \neq Q$, then $H' = H/Q$ is a non-trivial totally disconnected normal subgroup of the connected group G'. These conditions imply that H' lies in the centre of G' and that H' has a proper open (in H') subgroup V; see **(3**, pp. 39, 54**)**. The fact that H' is in the centre of G' makes V a normal subgroup of G'. Let π_2: $G' \to G'/V = G''$ be the canonical map so that G'' is locally compact and connected and h: $H'' \to C$ be the homomorphism induced by g. Now h is locally one-to-one, since the kernel of h is H'/V, which is discrete because V is open in H'. Thus there is a neighbourhood, U, of the identity element e'' of G'' such that $U \cap H'/V = e''$.

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Let U' be a symmetric open neighbourhood of e'' such that $U'^2 \subset U$. If $x, y \in U'$ and $h(x) = h(y)$, then $xy^{-1} \in U'^2 \subset U \cap H'/V = e''$ so $x = y$. Thus h is locally one-to-one on U'. Now we may use the preceding proposition to assert that h is one-to-one, which implies that $H' = V$ or that H' is trivial. Thus $H = Q$ so that f is monotone. Moreover, since g is one-to-one, the preceding proposition also asserts that g is a homeomorphism and is thus open. Since $(C, \circ)$ is an iseomorphic image of the topological group G', $(C, \circ)$ is a topological group. Since $f = g \circ \pi_1$ and g and π_1 are open, f is open.

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