APPROXIMATION OF FUNCTIONS BY MEANS OF LIPSCHITZ FUNCTIONS

J. H. MICHAEL

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1. Introduction

Let Q denote the closed unit cube in R^n. The elementary area $A(f)$ of a Lipschitz function f on Q is given by the formula

$$A(f) = \int_Q \left\{1 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2\right\}^{\frac{1}{2}} \mathrm{d}x.$$

In [1], C. Goffman has shown that A is lower semi-continuous with respect to the $\mathscr{L}_1$ norm and admits a lower semi-continuous extension to a functional A defined on the class of all functions summable on Q. Thus for a summable f

$$A(f) = \inf [\liminf_{r\to\infty} A(f^{(r)})],$$

where the infimum is taken over all sequences $\{f^{(r)}\}$ of Lipschitz functions that converge $\mathscr{L}_1$ to f.

Denote by $\mathscr{D}$ the set of all infinitely differentiable functions on R^n with compact support. Let $\mathscr{D}^k$ denote the set of transformations $\psi = (\psi_1, \cdots, \psi_k)$ from R^n to R^k such that each $\psi_i \in \mathscr{D}$.

The functional A can also be characterised by

$$(1) \qquad A(f) = \sup \left[\sum_{i=1}^{n} \int_Q f \frac{\partial \psi_i}{\partial x_i} \mathrm{d}x + \int_Q \psi_{n+1} \mathrm{d}x\right],$$

where the supremum is taken over all $\psi \in \mathscr{D}^{n+1}$ such that spt $\psi \subsetneqq \mathrm{Int}(Q)$ and

$$\sup_x \left[\sum_{i=1}^{n+1} \{\psi_i(x)\}^2\right]^{\frac{1}{2}} \leqq 1.$$

In [2], I proved the following theorem.

Let f be summable on Q and such that $A(f) < \infty$. Then, for each $\varepsilon > 0$, there exists a Lipschitz function g on Q such that the set $\{x; x \in Q$ and $f(x) \neq g(x)\}$ has measure less than ε and $A(g) < A(f) + \varepsilon$.

In the present paper, a similar theorem is proved for a more general functional Ψ, but unfortunately I can only prove the theorem for continuous functions. I take a functional Ψ on the class of Lipschitz functions, extend it by lower semi-continuity to the class of summable functions and then show that for each continuous f on Q, with $\Psi(f) < \infty$ and each $\varepsilon > 0$, there exists a Lipschitz function g on Q which agrees with f except on a set of measure less than ε and is such that $\Psi(g) < \Psi(f) + \varepsilon$.

The functional Ψ is defined on the Lipschitz functions in the following way.

Let ϕ be a non-negative, real-valued continuous function on R^n, ρ be a norm for $\mathscr{D}^{n+1}$, α be an integer that is either 0 or 1 and η be a non-negative, strictly increasing, unbounded, continuous function on the non-negative reals. Let ϕ, ρ, α and η be such that:

(i) $\phi(\xi) \geqq \phi(\xi')$ when $|\xi_1| \geqq |\xi'_1|, \cdots, |\xi_n| \geqq |\xi'_n|$;

(ii) there exist constants A and B such that

$$||\xi|| \leqq A + B\phi(\xi) \quad \text{for all} \quad \xi \in R^n;$$

(iii) there exists a continuous function θ on the $n \times n$ real matrices such that

$$\phi(\xi \cdot X) \leqq \phi(\xi) \cdot \theta(X)$$

for every $\xi \in R^n$ and every $n \times n$ matrix X;

(iv) for every open set U of R^n and every locally Lipschitz function f on U,

$$\int_U \phi(\operatorname{grad} f)\mathrm{d}x = \eta \left[\sup\left\{\sum_{i=1}^n \int_U \frac{\partial f}{\partial x_i}\psi_i \mathrm{d}x + \alpha \int_U \psi_{n+1}\mathrm{d}x\right\}\right], \tag{1}$$

$$= \eta \left[\sup\left\{\sum_{i=1}^n \int_U f\frac{\partial \psi_i}{\partial x_i}\mathrm{d}x + \alpha \int_U \psi_{n+1}\mathrm{d}x\right\}\right], \tag{2}$$

where in each case the supremum is taken over all $\psi \in \mathscr{D}^{n+1}$ with spt $\psi \subseteqq U$ and $\rho(\psi) \leqq 1$;

(v) ρ is translation invariant; i.e., if $\psi \in \mathscr{D}^{n+1}$ and $\nu(\zeta) = \psi(\zeta + a)$, then $\rho(\nu) = \rho(\psi)$;

(vi) $\rho(\psi) = \rho(\varepsilon_1\psi_1, \cdots, \varepsilon_{n+1}\psi_{n+1})$ for all $\psi \in \mathscr{D}^{n+1}$ and all $\varepsilon_1 = \pm 1, \cdots, \varepsilon_{n+1} = \pm 1$.

Define

$$\Psi(f) = \int_Q \phi(\operatorname{grad} f)\mathrm{d}x$$

for every Lipschitz function f on Q. It is shown in [4], that when Ψ is extended to the summable functions by lower semicontinuity, one has for each continuous f,

$$\Psi(f) = \eta \left[\sup\left\{\sum_{i=1}^n \int_Q f\frac{\partial \psi_i}{\partial x_i}\mathrm{d}x + \alpha \int_Q \psi_{n+1}\mathrm{d}x\right\}\right],$$

where the supremum is taken over all $\psi \in \mathscr{D}^{n+1}$ with spt $\psi \subseteqq \operatorname{Int}(Q)$ and $\rho(\psi) \leqq 1$.

A simple example of such a ϕ, ρ, etc. is

$$\phi(\xi) = \left[\sum_{i=1}^{n} \xi_i^2\right]^{\frac{1}{2}} = \|\xi\|,$$

$$\rho(\psi) = \sup_{x} \left[\sum_{j=1}^{n+1} \{\psi_j(x)\}^2\right]^{\frac{1}{2}},$$

$$\alpha = 0, \quad \eta(t) = t \quad \text{and} \quad \theta(X) = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}^2\right]^{\frac{1}{2}}.$$

If one uses the same ρ and η, but puts

$$\phi(\xi) = \left[1 + \sum_{i=1}^{n} \xi_i^2\right]^{\frac{1}{2}}$$

$\alpha = 1$ and $\theta(X) = [1 + \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}^2]^{\frac{1}{2}}$, one obtains the area functional; i.e., $\Psi(f) = A(f)$.

Another example is given by

$$\phi(\xi) = \sum_{i=1}^{n} |\xi_i|^p,$$

where p is a real number > 1,

$$\rho(\psi) = \left[\int_{R^n} \left(\sum_{j=1}^{n+1} |\psi_j(x)|^{p/(p-1)}\right) \mathrm{d}x\right]^{(p-1)/p},$$

$$\alpha = 0, \quad \eta(t) = t^p \quad \text{and} \quad \theta(X) = \sum_{j=1}^{n} \left\{\sum_{i=1}^{n} |X_{ij}|^{p/(p-1)}\right\}^{p-1}.$$

Thus

$$\Psi(f) = \int_Q \left(\sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|^p\right) \mathrm{d}x$$

for a Lipschitz f.

2. Preliminaries

Let U be an open set of R^n. $\mathscr{L}(U)$ denotes the set of all locally summable real-valued functions on U. $\mathscr{K}(U)$ denotes the subset of $\mathscr{L}(U)$ consisting of all locally Lipschitz functions. For each $f \in \mathscr{L}(U)$ and each open subset V of U, define

$$A(f, V) = \sup \left[\sum_{i=1}^{n} \int_V f \frac{\partial \psi_i}{\partial x_i} \mathrm{d}x + \alpha \int_V \psi_{n+1} \mathrm{d}x\right]$$

and

$$\Gamma(f, V) = \sup \left[\sum_{i=1}^{n} \int_V f \frac{\partial \psi_i}{\partial x_i} \mathrm{d}x\right],$$

where in each case the supremum is taken over all $\psi \in \mathscr{D}^{n+1}$ with spt $\psi \subsetneqq V$

and $\rho(\psi) \leqq 1$. The definition of Λ and Γ is extended to arbitrary Borel subsets B of U by putting

$$\Lambda(f, B) = \inf \Lambda(f, V)$$

and

$$\Gamma(f, B) = \inf \Gamma(f, V),$$

where each infimum is taken over all open subsets V of U containing B. For each Borel subset B of U, define

$$\Phi(f, B) = \eta\{\Lambda(f, B)\}.$$

If we put

$$\mu(B) = \Phi(f, B),$$

then we will show in 2.13, that μ is a non-negative completely additive Borel measure.

When $f \in \mathscr{L}(U)$, V is an open subset of U with $\mathrm{d}(V, \sim U) > 0$ and r is a positive integer with $(\sqrt{n}) \cdot r^{-1} < \mathrm{d}(V, \sim U)$, we will use (as in [2]), the symbol $\mathscr{I}_r(f)$ to denote the integral mean

$$\{\mathscr{I}_r(f)\}(x) = r^n \int_0^{1/r} \cdots \int_0^{1/r} f(x+\xi)\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n,$$

which is defined for $x \in V$.

Integral means have the following properties:

2.1 If $f \in \mathscr{L}(U)$, then $\mathscr{I}_r(f)$ is continuous and hence locally summable on V.

2.2 If f is continuous, then $\mathscr{I}_r(f)$ has continuous first order partial derivatives.

2.3 If $f \in \mathscr{L}(U)$ and is bounded, then $\mathscr{I}_r(f)$ is Lipschitz.

2.4 If $f \in \mathscr{K}(U)$, then

$$\frac{\partial}{\partial x_i}\{\mathscr{I}_r(f)\} = \mathscr{I}_r\left(\frac{\partial f}{\partial x_i}\right)$$

everywhere in V.

2.5 If $f \in \mathscr{L}(U)$, then $\mathscr{I}_r(f) \to f$ almost everywhere in V and for every compact set C,

$$\int_C |f - \mathscr{I}_r(f)|\mathrm{d}x \to 0 \quad \text{and} \quad \int_C |f - \mathscr{I}_r\{\mathscr{I}_r(f)\}|\mathrm{d}x \to 0$$

as $r \to \infty$.

Λ, Γ and Φ have the following properties:

2.6 If f, $g \in \mathscr{L}(U)$ and B is a Borel subset of U, then $\Lambda(f+g, B) \leqq \Lambda(f, B) + \Gamma(g, B)$.

2.7 If $f \in \mathscr{L}(U)$, B is a Borel subset of U and β is a real number, then

$$\Gamma(\beta f, B) = |\beta|\Gamma(f, B).$$

2.8 If $f \in \mathscr{K}(U)$ and B is a Borel subset of U, then

$$\Phi(f, B) = \int_B \phi(\operatorname{grad} f)\mathrm{d}x$$

The following theorems establish some further properties of Φ, Γ and Λ.

2.9 THEOREM. *If $f, f^{(r)} \in \mathscr{L}(U)$ and V is an open subset of U such that f and each $f^{(r)}$ is summable on V and if*

$$\int_V |f - f^{(r)}|\mathrm{d}x \to 0$$

as $r \to \infty$, then

$$\liminf_{r\to\infty} \Lambda(f^{(r)}, V) \geqq \Lambda(f, V)$$

and

$$\liminf_{r\to\infty} \Phi(f^{(r)}, V) \geqq \Phi(f, V).$$

PROOF. Take $\varepsilon > 0$ or $N > 0$ according as $\Lambda(f, V)$ is finite or infinite. There exists $\psi \in D^{n+1}$ such that $\rho(\psi) \leqq 1$, spt $\psi \subseteqq V$ and

$$\sum_{i=1}^n \int_V f \frac{\partial \psi_i}{\partial x_i} \mathrm{d}x + \alpha \int_V \psi_{n+1} \mathrm{d}x > \Lambda(f, V) - \varepsilon \text{ or } N.$$

Then

$$\liminf_{r\to\infty} \Lambda\ (f^{(r)}, V) \geqq \lim_{r\to\infty} \left[\sum_{i=1}^n \int_V f^{(r)} \frac{\partial \psi_i}{\partial x_i} \mathrm{d}x + \alpha \int_V \psi_{n+1} \mathrm{d}x\right]$$

$$= \sum_{i=1}^n \int_V f \frac{\partial \psi_i}{\partial x_i} \mathrm{d}x + \alpha \int_V \psi_{n+1} \mathrm{d}x > \Lambda(f, V) - \varepsilon \text{ or } N.$$

2.10 THEOREM. *If $f \in \mathscr{L}(U)$, C is a compact subset of U and $\Phi(f, C)$ is finite, then*

$$\limsup_{r\to\infty} \Lambda\{\mathscr{I}_r(f), C\} \leqq \Lambda(f, C),$$

$$\limsup_{r\to\infty} \Lambda[\mathscr{I}_r\{\mathscr{I}_r(f)\}, C] \leqq \Lambda(f, C),$$

$$\limsup_{r\to\infty} \Phi\{\mathscr{I}_r(f), C\} \leqq \Phi(f, C)$$

and

$$\limsup_{r\to\infty} \Phi[\mathscr{I}_r\{\mathscr{I}_r(f)\}, C] \leqq \Phi(f, C).$$

PROOF. Take $\varepsilon > 0$ and let V be a bounded open subset of U containing C and such that

(1) $$\Lambda(f, V) < \Lambda(f, C) + \tfrac{1}{2}\varepsilon.$$

By the usual procedure for integral means one can easily show that

$$\Lambda\{\mathscr{I}_r(f), C\} \leqq \Lambda(f, V)$$

for sufficiently large r and

$$\Lambda[\mathscr{I}_r\{\mathscr{I}_r(f)\}, C] \leqq \Lambda(f, V)$$

for sufficiently large r. From these inequalities and (1), the theorem immediately follows.

2.11 THEOREM. *If $f \in \mathscr{K}(U)$ and C is a compact subset of U, then*

$$\Lambda\{f - \mathscr{I}_r(f), C\} \to \eta^{-1}\{\phi(0) \cdot m(C)\},$$
$$\Lambda[f - \mathscr{I}_r\{\mathscr{I}_r(f)\}, C] \to \eta^{-1}\{\phi(0) \cdot m(C)\},$$
$$\Phi\{f - \mathscr{I}_r(f), C\} \to \phi(0) \cdot m(C)$$

and

$$\Phi[f - \mathscr{I}_r\{\mathscr{I}_r(f)\}, C] \to \phi(0) \cdot m(C)$$

as $r \to \infty$.

PROOF. It follows from 2.4 and 2.5, that $\partial/\partial x_i\{\mathscr{I}_r(f)\} \to \partial f/\partial x_i$ almost everywhere on C. Also, there exists a constant K such that

$$\left|\frac{\partial f}{\partial x_i} - \frac{\partial}{\partial x_i}\{\mathscr{I}_r(f)\}\right| \leqq K$$

for all sufficiently large r and almost all $x \in C$. Let $L > 0$ be such that $\phi(\xi) \leqq L$ for all ξ for which $|\xi_1| \leqq K, \cdots, |\xi_n| \leqq K$. Then

$$\phi\{\text{grad } f - \text{grad } \mathscr{I}_r(f)\} \leqq L$$

for all sufficiently large r and almost all $x \in C$, and

$$\lim_{r\to\infty} \phi\{\text{grad } f - \text{grad } \mathscr{I}_r(f)\} = \phi(0)$$

for almost all $x \in C$. Therefore, by bounded convergence,

$$\lim_{r\to\infty} \int_C \phi\,\{\text{grad } f - \text{grad } \mathscr{I}_r(f)\}\text{d}x = \phi(0) \cdot m(C).$$

Suppose that $\Phi\{f - \mathscr{I}_r^2(f), C\}$ does not approach $\phi(0) \cdot m(C)$. Then there is an increasing sequence $\{r_s\}$ of positive integers such that

$$\text{(1)} \qquad \lim_{s\to\infty} \Phi\{f - \mathscr{I}_{r_s}^2(f), C\} - \phi(0) \cdot m(C) = \delta \neq 0.$$

But by 2.4 and 2.5

$$\int_C ||\text{grad } f - \text{grad } \mathscr{I}_{r_s}^2(f)||\text{d}x \to 0$$

as $s \to \infty$, so that there exists a subsequence $\{p_s\}$ of $\{r_s\}$ such that

$$\lim_{s\to\infty} [\operatorname{grad} f - \operatorname{grad} \mathscr{I}^2_{p_s}(f)] = 0$$

almost everywhere in C. But there is a constant K' such that $\|\operatorname{grad} f - \operatorname{grad} \mathscr{I}^2_{p_s}(f)\| \leqq K'$ almost everywhere in C.

Hence

$$\lim_{s\to\infty} \int_C \phi\,\{\operatorname{grad} f - \operatorname{grad} \mathscr{I}^2_{p_s}(f)\}\mathrm{d}x = \phi(0)\cdot m(C)$$

contradicting (1).

2.12 Theorem. *If* $g \in \mathscr{K}(U)$ *and* C *is a compact subset of* U, *then*

$$\Gamma[g - \mathscr{I}_r(g), C] \to 0$$

as $r \to \infty$ *and*

$$\Gamma[g - \mathscr{I}_r\{\mathscr{I}_r(g)\}, C] \to 0$$

as $r \to \infty$.

Proof. Let $g^{(r)}$ denote either $\mathscr{I}_r(g)$ or $\mathscr{I}^2_r(g)$ and suppose that $\Gamma[g - g^{(r)}, C]$ does not approach zero. Then there exists an increasing sequence $\{r_s\}$ of positive integers such that

(1) $$\lim_{s\to\infty} \Gamma[g - g^{(r_s)}, C] = \alpha > 0.$$

By 2.11,

$$\lim_{s\to\infty} \int_C \phi\,\{\operatorname{grad} t(g - g^{(r_s)})\}\mathrm{d}x = \phi(0)\cdot m(C)$$

for every positive integer t, hence there exists a subsequence $\{p_t\}$ of $\{r_s\}$ such that

(2) $$\lim_{t\to\infty} \int_C \phi\,\{\operatorname{grad} t(g - g^{(p_t)})\}\mathrm{d}x = \phi(0)\cdot m(C)$$

But

$$\Gamma\{t(g - g^{(p_t)}), C\} = t\Gamma(g - g^{(p_t)}, C) \to \infty$$

as $t \to \infty$, contradicting (2).

2.13 Theorem. *If* $f \in \mathscr{L}(U)$ *and we put*

$$\mu(E) = \Phi(f, E)$$

for every Borel subset E *of* U, *then* μ *is a completely additive Borel measure.*

Proof. We begin by proving

(a) if $V_1, V_2, \cdots$ are open subsets of U, finite or countable in number and $V = V_1 \cup V_2 \cup \cdots$, then

$$\mu(V) \leqq \sum_i \mu(V_i).$$

To prove (a), we take an increasing sequence $\{C_r\}$ of compact sets such that

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$C_r \subseteqq V$ for all r and $\lim_{r\to\infty} \operatorname{Int}(C_r) = V$. Let $\mathscr{V}_r$ be a finite subcollection of the V_i's that covers C_r. We can assign to each $W \in \mathscr{V}_r$ a compact subset $P_r(W)$ of C_r such that $P_r(W) \subseteqq W$ and

$$C_r = \bigcup_{W \in \mathscr{V}_r} P_r(W).$$

Put $\mathscr{C}_r = \{P_r(W); W \in \mathscr{V}_r\}$. Since each of the functions $\mathscr{I}_s^2(f)$ is locally Lipschitz,

$$\begin{aligned}\Phi\{\mathscr{I}_s^2(f), C_r\} &= \int_{C_r} \phi\,\{\operatorname{grad} \mathscr{I}_s^2(f)\}dx \\ &\leqq \sum_{C\in\mathscr{C}_r} \int_C \phi\,\{\operatorname{grad} \mathscr{I}_s^2(f)\}dx = \sum_{C\in\mathscr{C}_r} \Phi\{\mathscr{I}_s^2(f), C\},\end{aligned}$$

so that by 2.9 and 2.10,

$$\Phi\{f, \operatorname{Int}(C_r)\} \leqq \sum_{C\in\mathscr{C}_r} \Phi(f, C)$$

for all r, hence

$$\Phi(f, V) \leqq \sum_i \Phi(f, V_i).$$

Next we prove

(b) if V, W are disjoint open subsets of U, then

$$\mu(V \cup W) = \mu(V) + \mu(W).$$

To prove this, let $\{A_r\}$ be a sequence of compact sets such that $A_r \subseteqq V \cup W$ for all r and $\lim_{r\to\infty} \operatorname{Int}(A_r) = V \cup W$. Then

$$\begin{aligned}\Phi(f, V \cup W) &= \lim_{r\to\infty} \Phi(f, A_r) \\ &\geqq \lim_{r\to\infty} \limsup_{s\to\infty} \Phi\{\mathscr{I}_s^2(f), A_r\} \\ &\geqq \lim_{r\to\infty} \limsup_{s\to\infty} [\Phi\{\mathscr{I}_s^2(f), V \cap \operatorname{Int}(A_r)\} \\ &\qquad\qquad + \Phi\{\mathscr{I}_s^2(f), W \cap \operatorname{Int}(A_r)\}] \\ &\geqq \lim_{r\to\infty} [\Phi\{f, V \cap \operatorname{Int}(A_r)\} + \Phi\{f, W \cap \operatorname{Int}(A_r)\}] \\ &= \Phi(f, V) + \Phi(f, W).\end{aligned}$$

If we now define for every subset A of U,

$$\mu^*(A) = \inf \mu(V),$$

where the infimum is taken over all open subsets V of U containing A, we obtain a Caratheodory outer measure with $\mu^*(V) = \mu(V)$ for open sets V. Thus μ is completely additive on the Borel sets.

2.14 Theorem. *If* $f \in \mathscr{L}(U)$, $N > 0$ *and we put*

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$$\begin{aligned} f_N(x) &= f(x) \quad \textit{if} \quad -N \leqq f(x) \leqq N \\ &= N \quad \textit{if} \quad f(x) > N \\ &= -N \quad \textit{if} \quad f(x) < -N \end{aligned}$$

then $f_N \in \mathscr{L}(U)$ *and* $\Phi(f_N, B) \leqq \Phi(f, B)$ *for every Borel subset* B *of* U.

PROOF. (1) When f is locally Lipschitz on U, the theorem follows immediately from 1 (j).

(ii) When f is arbitrary and B is an open interval such that $\bar{B} \subsetneqq U$ and $\Phi\{f, Fr(B)\} = 0$, we have

$$\Phi(f_N, B) \leqq \liminf_{r\to\infty} \Phi[\{\mathscr{I}_r^2(f)\}_N, B]$$

and by (i), $\leqq \lim_{r\to\infty} \inf \Phi[\mathscr{I}_r^2(f), B] \leqq \Phi(f, B)$.

(iii) When f and B are arbitrary. We can assume $\Phi(f, B) < \infty$. Take $\varepsilon > 0$. There exists an open set V with $B \subseteqq V \subseteqq U$ and $\Phi(f, V) < \Phi(f, B) + \varepsilon$. Let Z_i be the subset of R^1 consisting of all t for which the set

$$A_{it} = \{x; x \in V \text{ and } x_i = t\}$$

has $\Phi(f, A_{it}) = 0$. Put $Z = \bigcap_{i=1}^n Z_i$. Then $R^1 \sim Z$ is countable. There exists a countable collection $\mathscr{J}$ of open intervals with their union containing V, with the coordinates of their vertices all in Z and with

$$\sum_{J\in\mathscr{J}} \Phi(f, J) < \Phi(f, V) + 1.$$

Then

$$\Phi(f_N, V) \leqq \sum_{J\in\mathscr{J}} \Phi(f_N, J)$$

and by (ii)

$$\leqq \sum_{J\in\mathscr{J}} \Phi(f, J) < \Phi(f, V) + 1$$

Thus $\Phi(f_N, V)$ is finite. There now exists a countable collection $\mathscr{J}^*$ of mutually disjoint open intervals with

$$V = \bigcup_{J\in\mathscr{J}^*} J$$

and with $\Phi\{f, Fr(J)\} = \Phi\{f_N, Fr(J)\} = 0$ for all $J \in \mathscr{J}^*$. Now

$$\Phi(f_N, B) \leqq \sum_{J\in\mathscr{J}^*} \Phi(f_N, J)$$

and by (ii)

$$\begin{aligned} &\leqq \sum_{J\in\mathscr{J}^*} \Phi(f, J) \leqq \Phi(f, V) \\ &< \Phi(f, B) + \varepsilon. \end{aligned}$$

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3. Some approximation theorems

3.1. Theorem. *Let C be a compact subset of R^n and f be a locally summable function on R^n such that $\Phi(f, C)$ is finite. Let $\varepsilon > 0$. There exists a Lipschitz function g on R^n with compact support and such that the set*

$$\{x; x \in C \quad \text{and} \quad f(x) \neq g(x)\}$$

has measure less than ε.

Proof. For each positive integer r, put

$$f^{(r)} = \mathscr{I}_r^2(f).$$

Let V be a bounded upen set such that $C \subseteqq V$ and $\Phi(f, \bar{V}) < \infty$. Each $f^{(r)}$ is Lipschitz on V, $f^{(r)} \to f$ in the $\mathscr{L}_1$ topology on V and

$$\limsup_{r\to\infty} \Phi(f^{(r)}, V) < \infty, \tag{1}$$

so that by 1 (ii),

$$\limsup_{r\to\infty} \int_V [1 + \|\text{grad } f^{(r)}\|^2]^{\frac{1}{2}} dx < \infty. \tag{2}$$

Let $J_1, J_2, \cdots, J_p$ be a finite number of mutually non-overlapping closed cubes such that, if we put $W = \bigcup_{j=1}^p J_j$, we have $C \subseteqq \text{Int}(W)$ and $W \subseteqq V$. By (2) and [2] 4.3, each of the functions

$$\begin{aligned} f_j(x) &= f(x) \quad \text{if} \quad x \in J_j \\ &= 0 \quad \text{if} \quad x \notin J_j, \end{aligned}$$

belongs to the class $\mathscr{B}$ of [2]. Hence, the function

$$f^* = \sum_{j=1}^p f_j$$

belongs to $\mathscr{B}$ and by [2] 3.1, there exists a Lipschitz function g on R^n with compact support and agreeing with f^* except on a set of measure less than ϵ. Since f^* agrees with f almost everywhere on C, g is the required function.

3.2 Lemma. *Let f be continuous on R^n with compact support and g be Lipschitz on R^n with compact support. For each $\eta > 0$, put*

$$\begin{aligned} f_\eta(x) &= g(x) & &\text{if} \quad |f(x) - g(x)| \leqq \eta \\ &= f(x) - \eta \operatorname{sgn} \{f(x) - g(x)\} & &\text{if} \quad |f(x) - g(x)| \geqq \eta, \\ B_\eta &= \{x; x \in R^n \quad \text{and} \quad 0 < |f(x) - g(x)| < \eta\}. \end{aligned}$$

Then

$$\Phi(f_\eta, E) \leqq \Phi(f, E) + \Phi(g, B_\eta \cap E).$$

for every Borel set E.

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PROOF. (i) Suppose first of all that E is open. Put

$$f^{(r)} = \mathscr{I}_r(f).$$

Then $f^{(r)} \to f$ uniformly on R^n, hence there exists an increasing sequence $\{r_s\}$ of positive integers such that

$$|f^{(r_s)}(x) - f(x)| < \frac{1}{s}$$

for all $x \in R^n$. Let s_1 be such that $1/s_1 < \frac{1}{2}\eta$ and for each $s \geqq s_1$, put

$$\begin{aligned} p^{(s)}(t) &= t \quad \text{for} \quad |t| \leqq \frac{1}{s} \\ &= \frac{1}{s} \operatorname{sgn} t \quad \text{for} \quad \frac{1}{s} \leqq |t| \leqq \eta - \frac{1}{s} \\ &= t - \left(\eta - \frac{2}{s}\right) \operatorname{sgn} t \quad \text{for} \quad |t| \geqq \eta - \frac{1}{s}, \\ h^{(s)}(x) &= g(x) + p^{(s)}\{f^{(r_s)}(x) - g(x)\} \end{aligned}$$

and

$$G_s = \left\{x; x \in R^n \text{ and } \frac{1}{s} < |f^{(r_s)}(x) - g(x)| < \eta - \frac{1}{s}\right\}.$$

Then $h^{(s)} \to f_\eta$ uniformly on R^n so that for each open set U,

$$\Phi(f_\eta, U) \leqq \liminf_{s\to\infty} \Phi(h^{(s)}, U). \tag{1}$$

But since $h^{(s)}$ is Lipschitz,

$$\begin{aligned} \Phi(h^{(s)}, U) &= \int_U \phi(\operatorname{grad} h^{(s)})dx \\ &= \int_{U\sim G_s} \phi(\operatorname{grad} h^{(s)})dx + \int_{U\cap G_s} \phi(\operatorname{grad} h^{(s)})dx \\ &= \int_{U\sim G_s} \phi(\operatorname{grad} f^{(r_s)})dx + \int_{U\cap G_s} \phi(\operatorname{grad} g)dx \\ &\leqq \Phi(f^{(r_s)}, U) + \Phi(g, U \cap B_\eta). \end{aligned}$$

Thus, it follows from (1), that for every bounded open set U,

$$\Phi(f_\eta, U) \leqq \Phi(f, \overline{U}) + \Phi(g, U \cap B_\eta). \tag{2}$$

Let $\varepsilon > 0$. There exists an increasing sequence $\{U_r\}$ of bounded open sets such that each $\overline{U}_r$ is contained in E and

$$\lim_{r\to\infty} U_r = E.$$

Then

$$\Phi(f_\eta, E) = \lim_{r\to\infty} \Phi(f_\eta, U_r)$$

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and by (2)

$$\leqq \lim_{r\to\infty} [\Phi(f, \bar{U}_r) + \Phi(g, B_\eta \cap U)]$$
$$= \Phi(f, E) + \Phi(g, B_\eta \cap E).$$

(ii) f and E are arbitrary. Take $\varepsilon > 0$ and let V be an open set containing E and such that

$$\Phi(f, V) \leqq \Phi(f, E) + \tfrac{1}{2}\varepsilon$$

and

$$\Phi(g, B_\eta \cap V) \leqq \Phi(g, B_\eta \cap E) + \tfrac{1}{2}\varepsilon.$$

By (i)

$$\Phi(f_\eta, V) \leqq \Phi(f, V) + \Phi(g, B_\eta \cap V),$$

hence

$$\Phi(f, E) \leqq \Phi(f, E) + \Phi(g, B_\eta \cap E) + \varepsilon.$$

3.3 THEOREM. *Let C be a compact subset of an open set U and let f be continuous on U and such that $\Phi(f, C)$ is finite. Let $\varepsilon > 0$. There exists a Lipschitz function f_0 on U such that:*

(i) *the set $\{x; x \in C$ and $f(x) \neq f_0(x)\}$ has measure less than ε, and*

(ii) $\Phi(f_0, C) < \Phi(f, C) + \varepsilon$.

PROOF. Let C_1 be a compact set, contained in U and with C in its interior. There exists a continuous function f_1 on R^n with compact support and agreeing with f on C_1. Evidently

$$\Phi(f_1, C) = \Phi(f, C). \tag{1}$$

By 3.1, there exists a Lipschitz function g on R^n with compact support and such that, if

$$A = \{x; x \in C \quad \text{and} \quad f_1(x) \neq g(x)\},$$

then

$$m(A) < \tfrac{1}{2}\varepsilon. \tag{2}$$

Let $\eta > 0$ be such that, if

$$B = \{x; x \in R^n \quad \text{and} \quad 0 < |f_1(x) - g(x)| < \eta\}$$

and

$$D = \{x; x \in R^n \quad \text{and} \quad |f_1(x) - g(x)| = \eta\},$$

then

$$m(D) = 0 \tag{3}$$

and

$$\Phi(g, B) < \tfrac{1}{2}\varepsilon. \tag{4}$$

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Define

$$(5)\qquad \begin{aligned} f_2(x) &= g(x) \quad \text{if} \quad |f_1(x) - g(x)| \leqq \eta \\ &= f_1(x) - \eta \operatorname{sgn}\{f_1(x) - g(x)\} \quad \text{if} \quad |f_1(x) - g(x)| \geqq \eta. \end{aligned}$$

Then by 3.2,

$$\Phi(f_2, C) \leqq \Phi(f_1, C) + \Phi(g, B \cap C)$$

and therefore by (1) and (4)

$$(6)\qquad \Phi(f_2, C) \leqq \Phi(f, C) + \tfrac{1}{2}\varepsilon.$$

Now it follows from 2.6, that

$$\Phi\{g + \mathscr{I}_r^2(f_2 - g), C\} \leqq \eta[\Lambda\{\mathscr{I}_r^2(f_2), C\} + \Gamma\{g - \mathscr{I}_r^2(g), C\}],$$

hence, by 2.10, 2.12 and (6),

$$(7)\qquad \limsup_{r\to\infty} \Phi\{g + \mathscr{I}_r^2(f_2 - g), C\} \leqq \Phi(f, C) + \tfrac{1}{2}\varepsilon.$$

Put

$$E_r = \{x; x \in R^n \quad \text{and} \quad [\mathscr{I}_r^2(f_2 - g)](x) \neq 0\}$$

and

$$E = \{x; x \in R^n \quad \text{and} \quad f_2(x) \neq g(x)\}.$$

Then $\operatorname{Fr}(E) \subseteqq D$ and, since by (3) D has measure zero,

$$(8)\qquad \limsup_{r\to\infty} m(E_r \sim E) = 0.$$

Thus, by (7) and (8), we can choose an r_1 such that if $f_0 = g + \mathscr{I}_{r_1}^2(f_2 - g)$, then

$$\Phi(f_0, C) < \Phi(f, C) + \varepsilon$$

and

$$(9)\qquad m(E_{r_1} \sim E) < \tfrac{1}{2}\varepsilon.$$

Then

$$\{x; x \in C \quad \text{and} \quad f(x) \neq f_0(x)\} \subseteqq A \cup (E_{r_1} \sim E)$$

and by (2) and (9) has measure less than ε. Since f_0 is Lipschitz, this completes the proof.

4. Approximation of functions on Q

In this section, the final approximation theorem as described in the introduction, is proved (4.2).

The functional Ψ was defined in the introduction for Lipschitz functions on the unit cube Q by

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$$\Psi(f) = \int_Q \phi(\text{grad } f)\mathrm{d}x = \Phi(f, Q) = \Phi\{f, \text{Int}(Q)\}.$$

It follows immediately from 2.9 that Ψ is lower semi-continuous on the Lipschitz functions with respect to $\mathscr{L}_1$ convergence. Therefore, Ψ extends to a lower semicontinuous functional on the set of functions that are summable on Q. Thus, for a function f summable on Q

$$\Psi(f) = \inf [\liminf_{r\to\infty} \Psi(f^{(r)})],$$

where the infimum is taken over all sequences $\{f^{(r)}\}$ of Lipschitz functions that converge $\mathscr{L}_1$ to f. It follows immediately from 2.9, that

$$\Phi\{f, \text{Int}(Q)\} \leqq \Psi(f).$$

4.1 THEOREM. *Let f be continuous on Q and such that $\Phi\{f, \text{Int}(Q)\}$ is finite. Let $\varepsilon > 0$. There exists a Lipschitz function g on Q such that the set*

$$\{x; x \in Q \quad \text{and} \quad f(x) \neq g(x)\}$$

has measure less than ε and

$$\Phi(g, Q) < \Phi\{f, \text{Int}(Q)\} + \varepsilon$$

PROOF. Let $|f(x)| \leqq K$ for all $x \in Q$. Let $a = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ and for each $t \in [0, \frac{1}{2}]$, put

$$Q_t = \{2t(x-a) + a; x \in Q\}.$$

Let D be the set of all $t \in (0, \frac{1}{2})$ for which $\Phi\{f, Fr(Q_t)\} = 0$. The complement of D in $(0, \frac{1}{2})$ is countable. Let $t_0 \in D$ be such that $0 < t_0 < \frac{1}{2}$,

$$m(Q \sim Q_{t_0}) < \tfrac{1}{2}\varepsilon \tag{1}$$

and

$$\Phi\{f, \text{Int}(Q) \sim Q_{t_0}\} < \{1 + \theta(I)\}^{-1} \left(\frac{\sqrt{n}}{2t_0} + 3\right)^{-n} \tfrac{1}{4}\varepsilon. \tag{2}$$

Let $t_1 \in D$ be such that $t_0 < t_1 < \frac{1}{2}$,

$$t_1 - t_0 > \tfrac{1}{2}(\tfrac{1}{2} - t_0) \tag{3a}$$

and

$$\theta(X) \leqq 1 + \theta(I) \tag{3b}$$

for all matrices X such that

$$|X_{ij} - \delta_{ij}| \leqq \frac{(\frac{1}{2} - t_1)}{t_0(\frac{1}{2} - t_0)}$$

for all i, j. By 3.3, there exists for each positive integer r a Lipschitz function $g^{(r)}$ on $\text{Int}(Q)$ such that

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$$m\{x; x \in Q_{t_1} \quad \text{and} \quad f(x) \neq g^{(r)}(x)\} < r^{-1}$$

and

(4) $$\Phi(g^{(r)}, Q_{t_1}) < \Phi(f, Q_{t_1}) + r^{-1}.$$

We can assume that $|g^{(r)}(x)| \leqq K$, for all $x \in \text{Int}(Q)$. Then $g^{(r)} \to f$ in the $\mathscr{L}_1$ topology so that by 2.9,

$$\liminf_{r\to\infty} \Phi(g^{(r)}, Q_{t_0}) \geqq \Phi(f, Q_{t_0})$$

and

$$\liminf_{r\to\infty} \Phi(g^{(r)}, Q_{t_1}) \geqq \Phi(f, Q_{t_1}).$$

But by (4),

$$\limsup_{r\to\infty} \Phi(g^{(r)}, Q_{t_1}) \leqq \Phi(f, Q_{t_1})$$

so that

$$\limsup_{r\to\infty} \Phi(g^{(r)}, Q_{t_1} \sim Q_{t_0}) \leqq \Phi(f, Q_{t_1} \sim Q_{t_0}).$$

Hence, one can choose a large r, put $h = g^{(r)}$ and obtain

(5) $$m\{x; x \in Q_{t_1} \quad \text{and} \quad f(x) \neq h(x)\} < \tfrac{1}{2}\varepsilon,$$

(6) $$\Phi(h, Q_{t_1}) < \Phi(f, Q_{t_1}) + \tfrac{1}{2}\varepsilon$$

and

(7) $$\Phi(h, Q_{t_1} \sim Q_{t_0}) < \{1 + \theta(I)\}^{-1} \left(\frac{\sqrt{n}}{2_{t_0}} + 3\right)^{-n} \tfrac{1}{2}\varepsilon.$$

For each $x \in Q$ define $\nu(x)$ by

$$x \in \text{Fr}\{Q_{\nu(x)}\}.$$

Then

$$|\nu(x) - \nu(x')| \leqq ||x - x'||$$

for all $x, x' \in Q$. For $x \in Q \sim Q_{t_0}$ define

$$p(x) = \left[\frac{t_0(\frac{1}{2} - t_1)}{\nu(x)(\frac{1}{2} - t_0)} + \frac{t_1 - t_0}{\frac{1}{2} - t_0}\right] (x - a) + a.$$

Then p maps $Q \sim Q_{t_0}$ onto $Q_{t_1} \sim Q_{t_0}$. Also

$$p^{-1}(y) = \left[\frac{t_0(t_1 - \frac{1}{2})}{\nu(y)(t_1 - t_0)} + \frac{\frac{1}{2} - t_0}{t_1 - t_0}\right] (y - a) + a.$$

Then

(8) $$||p(x) - p(x')|| \leqq \left(\frac{\sqrt{n}}{4t_0} + 1\right) ||x - x'||$$

for all $x, x' \in Q \sim Q_{t_0}$ and

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$$(9) \qquad ||p^{-1}(y) - p^{-1}(y')|| \leqq \left(\frac{\sqrt{n}}{2t_0} + 3\right) ||y - y'||$$

for all $y, y' \in Q_{t_1} \sim Q_{t_0}$. Define

$$\begin{aligned} g(x) &= h(x) && \text{if } x \in Q_{t_0} \\ &= h\{p(x)\} && \text{if } x \in Q \sim Q_{t_0}. \end{aligned}$$

Then g is Lipschitz on Q and by (1) and (5) the set $\{x; x \in Q \text{ and } f(x) \neq g(x)\}$ has measure less than ε. For almost all $x \in Q \sim Q_{t_0}$, we have

$$\phi(\text{grad } g) = \phi\{(\text{grad } h) \cdot J(x)\},$$

where $J(x)$ denotes the Jacobian matrix of p. Therefore, by 1 (iii),

$$(10) \qquad \phi(\text{grad } g) \leqq \phi[\{\text{grad } h(y)\}_{y=p(x)}] \cdot \theta\{J(x)\}.$$

But

$$\frac{\partial p_i}{\partial x_j} = \left[\frac{t_0(\frac{1}{2} - t_1)}{\nu(x)(\frac{1}{2} - t_0)} + \frac{t_1 - t_0}{\frac{1}{2} - t_0}\right] \delta_{ij} - \frac{t_0(\frac{1}{2} - t_1)}{\{\nu(x)\}^2(\frac{1}{2} - t_0)} \frac{\partial \nu}{\partial x_j} (x_i - a_i),$$

hence

$$\left|\frac{\partial p_i}{\partial x_j} - \delta_{ij}\right| \leqq \frac{(\frac{1}{2} - t_1)}{t_0(\frac{1}{2} - t_0)},$$

so that by (3b) and (10)

$$\int_{Q \sim Q_{t_0}} \phi(\text{grad } g) dx \leqq \{1 + \theta(I)\} \int_{Q \sim Q_{t_0}} \phi[\{\text{grad } h(y)\}_{y=p(x)}] dx$$

and by (9)

$$\begin{aligned} &\leqq \{1 + \theta(I)\} \left(\frac{\sqrt{n}}{2t_0} + 3\right)^n \int_{Q \sim Q_{t_0}} \phi[\{\text{grad } h(y)\}_{y=p(x)}] \frac{\partial(p)}{\partial x} dx \\ &= \{1 + \theta(I)\} \left(\frac{\sqrt{n}}{2t_0} + 3\right)^n \int_{Q_{t_1} \sim Q_{t_0}} \phi(\text{grad } h) dy \end{aligned}$$

which by (7), $< \frac{1}{2}\varepsilon$. Then

$$\Phi(g, Q) < \Phi(h, Q_{t_0}) + \tfrac{1}{2}\varepsilon \leqq \Phi(h, Q_{t_1}) + \tfrac{1}{2}\varepsilon$$

and by (6), $< \Phi\{f, \text{Int}(Q)\} + \varepsilon$.

4.2 Corollary. *If f is continuous on Q and such that $\Psi(f)$ is finite and if $\varepsilon > 0$, then there exists a Lipschitz function g on Q such that the set*

$$\{x; x \in Q \quad \text{and} \quad f(x) \neq g(x)\}$$

has measure less than ε and

$$\Psi(g) < \Psi(f) + \varepsilon.$$

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4.3 THEOREM. *If f is continuous on Q, then*

$$\Phi\{f, \operatorname{Int}(Q)\} = \Psi(f).$$

PROOF. It is sufficient to prove that $\Psi(f) \leqq \Phi\{f, \operatorname{Int}(Q)\}$. We can assume that $\Phi\{f, \operatorname{Int}(Q)\}$ is finite. Let $|f(x)| \leqq K$ for all $x \in Q$. By 4.1, there exists for each r a Lipschitz function $g^{(r)}$ on Q such that

$$m\{x; x \in Q \quad \text{and} \quad f(x) \neq g^{(r)}(x)\} < r^{-1}$$

and

$$\Phi(g^{(r)}, Q) < \Phi\{f, \operatorname{Int}(Q)\} + r^{-1}.$$

Because of 2.14, we can assume that $|g^{(r)}(x)| \leqq K$ for all $x \in Q$. Then $g^{(r)} \to f$ in the $\mathscr{L}_1$ topology so that

$$\Psi(f) \leqq \liminf_{r \to \infty} \Phi(g^{(r)}, Q)$$

hence

$$\Psi(f) \leqq \Phi\{f, \operatorname{Int}(Q)\}.$$

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University of Adelaide
South Australia.

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