The Numerical Evaluation of Double Integrals.

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§ 1. *Introduction.*

The present paper is concerned with formulae by which double integrals of functions of two independent variables may be evaluated approximately. The number of such formulae published hitherto is not great,* and it has seemed desirable both to make a systematic search for new formulae, and to test the comparative merits of these, and of those previously known, by computing the numerical values of certain selected integrals.

§ 2. *General Formulae.*

Take a known formula for the numerical evaluation of single integrals, say

$$\frac{1}{u}\int_a^{a+mu} f(x)\,dx = \sum_r A_r f(a+ru) + \sum_r A'_r \phi_r(\Delta_x) f(a+ru),$$

and similarly

$$\frac{1}{v}\int_b^{b+nv} f(y)\,dy = \sum_s B_s f(b+sv) + \sum_s B'_s \psi_s(\Delta_y) f(b+sv),$$

where the A_r, A'_r, B_s, B'_s are given coefficients, ϕ_r and ψ_s functions of the symbols of partial finite difference operating on the functions to be integrated.

Then a general formula for the numerical evaluation of double integrals is

$$\frac{1}{uv}\int_a^{a+mu}\int_b^{b+nv} f(x,\,y)\,dx\,dy = \frac{1}{u}\int_a^{a+mu}\{\Sigma B_s f(x,\,b+sv) + \Sigma B'_s \psi_s(\Delta_y) f(x,\,b+sv)\}\,dx$$

$$= \Sigma\Sigma A_r B_s f(a+ru,\,b+sv) + \Sigma\Sigma A'_r B_s \phi_r(\Delta_x) f(a+ru,\,b+sv) + \Sigma\Sigma A_r B'_s \psi_s(\Delta_y) f(a+ru,\,b+sv) + \Sigma\Sigma A'_r B'_s \phi_r(\Delta_x)\,\psi_s(\Delta_y) f(a+ru,\,b+sv),$$

a symbolic product of the single formulae from which it is derived.

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* The matter is discussed briefly in § 182 of Whittaker and Robinson's "*Calculus of Observations,*" where references are given.

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Again, if differential coefficients are involved, instead of differences, say

$$\frac{1}{u}\int_a^{a+mu} f(x)\,dx = \Sigma A_r f(a+ru) + \Sigma A'_r\,\phi_r\left(u\frac{\partial}{\partial x}\right) f(a+ru),$$

$$\frac{1}{v}\int_b^{b+nv} f(y)\,dy = \Sigma B_s f(b+sv) + \Sigma B'_s\,\psi_s\left(v\frac{\partial}{\partial y}\right) f(b+sv),$$

it follows in similar fashion that

$$\begin{aligned}\frac{1}{uv}\int_a^{a+mu}\int_b^{b+nv} f(x, y)\,dx\,dy &= \Sigma\Sigma A_r B_s f(a+ru,\ b+sv)\\ &+ \Sigma\Sigma A'_r B_s\,\phi_r\left(u\frac{\partial}{\partial x}\right) f(a+ru,\ b+sv)\\ &+ \Sigma\Sigma A_r B'_s\,\psi_s\left(v\frac{\partial}{\partial y}\right) f(a+ru,\ b+sv)\\ &+ \Sigma\Sigma A_r B'_s\,\phi_r\left(u\frac{\partial}{\partial x}\right)\psi_s\left(v\frac{\partial}{\partial y}\right) f(a+ru,\ b+sv).\end{aligned}$$

In many cases we have

$$\frac{1}{u}\int_a^{a+mu} f(x)\,dx = \Sigma A_r f(a+ru) + R_x,$$

where R_x is a remainder of the form $A'_h\left(\frac{\partial}{\partial x}\right)^p f(a+hu)$, $a+hu$ being some value of x within the range, and likewise

$$\frac{1}{v}\int_b^{b+nv} f(y)\,dy = \Sigma B_s f(b+sv) + B'_k\left(\frac{\partial}{\partial y}\right)^q f(b+kv).$$

Then

$$\begin{aligned}\frac{1}{uv}\int_a^{a+mu}\int_b^{b+nv} f(x, y)\,dx\,dy &= \Sigma\Sigma A_r B_s f(a+ru,\ b+sv)\\ &+ \Sigma A'_h B_s\left(\frac{\partial}{\partial x}\right)^p f(a+hu,\ b+sv)\\ &+ \Sigma A_r B'_k\left(\frac{\partial}{\partial y}\right)^q f(a+ru,\ b+kv)\\ &+ A'_h B'_k\left(\frac{\partial}{\partial x}\right)^p\left(\frac{\partial}{\partial y}\right)^q f(a+hu,\ b+kv).\end{aligned}$$

The last three terms of this expression form the remainder $R_{x,y}$, but in practice the last term is negligible.

§ 3. *Particular Formulae: Extended Newton-Cotes Formulae.*

The simplest and generally the least accurate of the common approximations to a single integral is the "trapezoidal" rule, viz.,

$$\frac{1}{u}\int_a^{a+mu} f(x)\,dx = \tfrac{1}{2}f(a) + \sum_{r=1}^{m-1} f(a+ru) + \tfrac{1}{2}f(a+mu) \text{ nearly.}$$

For a double integral, by § 2, the corresponding formula is

$$\frac{1}{uv}\int_a^{a+mu}\int_b^{b+nv} f(x,\,y)\,dx\,dy = \tfrac{1}{4}f(a,\,b) + \tfrac{1}{2}\sum_{s=1}^{n-1} f(a,\,b+sv)$$

$$+\tfrac{1}{4}f(a,\,b+nv) + \tfrac{1}{2}\sum_{r=1}^{m-1} f(a+ru,\,b) + \sum_{r=1}^{m-1}\sum_{s=1}^{n-1} f(a+ru,\,b+sv)$$

$$+\tfrac{1}{2}\sum_{r=1}^{m-1} f(a+ru,\,b+nv) + \tfrac{1}{4}f(a+mu,\,b) + \tfrac{1}{2}\sum_{s=1}^{n-1} f(a+mu,\,b+sv)$$

$$+\tfrac{1}{4}f(a+mu,\,b+nv) \text{ nearly.}$$

In fact, the coefficients for one variable being given in a row as

$$\tfrac{1}{2},\ 1,\ 1,\ \ldots\ 1,\ \tfrac{1}{2},$$

those for two variables may be tabulated in rows and columns as

$$\begin{array}{|llllll} \tfrac{1}{4}, & \tfrac{1}{2}, & \tfrac{1}{2}, & \cdots & \tfrac{1}{2}, & \tfrac{1}{4}, \\ \tfrac{1}{2}, & 1, & 1, & \ldots & 1, & \tfrac{1}{2}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tfrac{1}{4}, & \tfrac{1}{2}, & \tfrac{1}{2}, & \cdots & \tfrac{1}{2}, & \tfrac{1}{4}\,. \end{array}$$

We shall find it convenient to refer to the sums of values of a function taken with these coefficients as "trapezoidal" sums, denoting by $S_u^m f(a)$ the sum $\tfrac{1}{2}f(a) + \sum_{r=1}^{m-1} f(a+ru) + \tfrac{1}{2}f(a+mu)$, and similarly for $S_u^m S_v^n f(a,\,b)$.

The extension to a triple integral is easily seen. The coefficients of $S_u^m S_v^n S_w^p f(a,\,b,\,c)$ can be represented tier upon tier, forming a cube, the coefficients at the corners being $\tfrac{1}{8}$, along the edges $\tfrac{1}{4}$, in the faces $\tfrac{1}{2}$, and within the cube 1.

The double "Simpson" formula is

$$\frac{1}{uv}\int_a^{a+2u}\int_b^{b+2v} f(x,\,y)\,dx\,dy = \tfrac{1}{9}\{f(a,\,b) + 4f(a+u,\,b)$$

$$+f(a+2u,\,b) + 4f(a,\,b+v) + 16f(a+u,\,b+v)$$

$$+4f(a+2u,\,b+v) + f(a,\,b+2v) + 4f(a+u,\,b+2v)$$

$$+f(a+2u,\,b+2v)\} \text{ nearly.}$$

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Accuracy may generally be increased by having more subdivisions, when coefficients are obtained as below:—

$$\frac{uv}{9}\times\left|\begin{array}{llllllll}
1, & 4, & 2, & 4, & 2, & \ldots & 4, & 1\\
4, & 16, & 8, & 16, & 8, & \ldots & 16, & 4\\
2, & 8, & 4, & 8, & 4, & \ldots & 8, & 2\\
4, & 16, & 8, & 16, & 8, & \ldots & 16, & 4\\
2, & 8, & 4, & 8, & 4, & \ldots & 8, & 2\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
4, & 16, & 8, & 16, & 8, & \ldots & 16, & 4\\
1, & 4, & 2, & 4, & 2, & \ldots & 4, & 1
\end{array}\right.$$

Example.—The true value of $\int_0^1\int_0^1 (1+x^2+y^2)^{-\frac{3}{2}}\,dx\,dy$ is $\frac{\pi}{6}$, or ·523599 to six places.

A first crude application of the elementary Simpson formula for nine points gave the result ·5195.

For 25 points, ·52363 was obtained. 49 points gave the much closer result ·523609, and 121 points ·523602.

Corresponding to the "Three-Eighths" or Simpson's Second Rule there is the scheme

$$\frac{9uv}{64}\times\left|\begin{array}{llllllllll}
1, & 3, & 3, & 2, & 3, & 3, & \ldots & 3, & 1\\
3, & 9, & 9, & 6, & 9, & 9, & \ldots & 9, & 3\\
3, & 9, & 9, & 6, & 9, & 9, & \ldots & 9, & 3\\
2, & 6, & 6, & 4, & 6, & 6, & \ldots & 6, & 2\\
3, & 9, & 9, & 6, & 9, & 9, & \ldots & 9, & 3\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
3, & 9, & 9, & 6, & 9, & 9, & \ldots & 9, & 3\\
1, & 3, & 3, & 2, & 3, & 3, & \ldots & 3, & 1.
\end{array}\right.$$

49 values of the same function as above gave the result by this formula ·523591.

It is possible and sometimes convenient to combine the Simpson with the Three-Eighths Rule, obtaining the scheme

$$\frac{uv}{8}\times\left|\begin{array}{llllllll}
1, & 4, & 2, & 4, & 2, & \ldots & 4, & 1\\
3, & 12, & 6, & 12, & 6, & \ldots & 12, & 3\\
3, & 12, & 6, & 12, & 6, & \ldots & 12, & 3\\
2, & 8, & 4, & 8, & 4, & \ldots & 8, & 2\\
3, & 12, & 6, & 12, & 6, & \ldots & 12, & 3\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
3, & 12, & 6, & 12, & 6, & \ldots & 12, & 3\\
1, & 4, & 2, & 4, & 2, & \ldots & 4, & 1.
\end{array}\right.$$

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$\int_0^{\cdot 8}\int_0^{1\cdot 2} e^{x^2 y}\,dx\,dy$ was calculated by this formula for 117 values, the result 1·10467 being obtained. The true value is 1·10469.

A fairly accurate formula when the ranges may conveniently be subdivided into six intervals or some multiple of six, is the double Weddle formula, viz.:

$$\frac{9uv}{100}\times\left|\begin{array}{ccccccc}1, & 5, & 1, & 6, & 1, & 5, & 1\\ 5, & 25, & 5, & 30, & 5, & 25, & 5\\ 1, & 5, & 1, & 6, & 1, & 5, & 1\\ 6, & 30, & 6, & 36, & 6, & 30, & 6\\ 1, & 5, & 1, & 6, & 1, & 5, & 1\\ 5, & 25, & 5, & 30, & 5, & 25, & 5\\ 1, & 5, & 1, & 6, & 1, & 5, & 1\end{array}\right.$$

This formula gave for $\int_0^1\int_0^1 (1+x^2+y^2)^{-\frac{3}{2}}\,dx\,dy$ the result ·523602, to a degree of accuracy for 49 points equal to that of the Simpson formula for 121 points.

A great number of other formulae may be obtained by combining single formulae in pairs, but it is not necessary to exemplify these.

It is of course possible to obtain analogous formulae for multiple integrals. Thus for a triple integral such as

$$\int_a^{a+2mu}\int_b^{b+2nv}\int_c^{c+2pw} f(x,\,y,\,z)\,dx\,dy\,dz$$

Simpson coefficients may be given in three tiers of nine, one above another:

$$\frac{uvw}{27}\times\left|\begin{array}{ccc}1 & 4 & 1\\ 4 & 16 & 4\\ 1 & 4 & 1\end{array},\quad\begin{array}{ccc}4, & 16, & 4\\ 16, & 64, & 16\\ 4, & 16, & 4\end{array},\quad\begin{array}{ccc}1, & 4, & 1\\ 4, & 16, & 4\\ 1, & 4, & 1\end{array}.\right.$$

It may be remarked that the accurate evaluation of a double or of a multiple integral requires in most cases the addition of a large number of terms. It is here that calculating machines, and adding machines in particular, are of the greatest service and almost indispensable.

§ 4. *Extended Gauss Formulae.*

Combination of Gauss formulae of the types

$$\int_{-1}^{1} f(x)\,dx = f(\sqrt{\tfrac{1}{3}}) + f(-\sqrt{\tfrac{1}{3}})$$
$$= \tfrac{1}{9}\{5f(\sqrt{\tfrac{3}{5}}) + 8f(0) + 5f(-\sqrt{\tfrac{3}{5}})\}$$

gives formulae such as

$$\int_{-1}^{1}\int_{-1}^{1} f(x, y)\,dx\,dy = f(\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}) + f(\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}})$$
$$+ f(-\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}) + f(-\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}})$$
$$= \tfrac{1}{9}[8\{f(0, \sqrt{\tfrac{1}{3}}) + f(0, -\sqrt{\tfrac{1}{3}})\} + 5\{f(\sqrt{\tfrac{3}{5}}, \sqrt{\tfrac{1}{3}})$$
$$+ f(\sqrt{\tfrac{3}{5}}, -\sqrt{\tfrac{1}{3}}) + f(-\sqrt{\tfrac{3}{5}}, \sqrt{\tfrac{1}{3}}) + f(-\sqrt{\tfrac{3}{5}}, -\sqrt{\tfrac{1}{3}})\}]$$
$$= \tfrac{1}{81}[64f(0, 0) + 40\{f(0, \sqrt{\tfrac{3}{5}}) + f(0, -\sqrt{\tfrac{3}{5}}) + f(\sqrt{\tfrac{3}{5}}, 0)$$
$$+ f(-\sqrt{\tfrac{3}{5}}, 0)\} + 25\{f(\sqrt{\tfrac{3}{5}}, \sqrt{\tfrac{3}{5}})$$
$$+ f(\sqrt{\tfrac{3}{5}}, -\sqrt{\tfrac{3}{5}}) + f(-\sqrt{\tfrac{3}{5}}, \sqrt{\tfrac{3}{5}}) + f(-\sqrt{\tfrac{3}{5}}, -\sqrt{\tfrac{3}{5}})\}].$$

The last of these is exact if neither of $f(x, 1)$ and $f(1, y)$ is of higher than the fifth degree.

For $\int_0^1 \int_0^1 (1 + x^2 + y^2)^{-\frac{3}{2}}\,dx\,dy$ this formula gives the result ·5233. For $\int_0^1 \int_0^1 e^{x^2 y}\,dx\,dy$ it gives 1·20698, the true value to five places being 1·20702. A Simpson of 121 points gave 1·20703.

We may mention here a formula of Burnside derived by a different method,

$$\int_{-1}^{1}\int_{-1}^{1} f(x, y)\,dx\,dy = \tfrac{40}{49}\{f(\sqrt{\tfrac{7}{15}}, 0) + f(-\sqrt{\tfrac{7}{15}}, 0)$$
$$+ f(0, \sqrt{\tfrac{7}{15}}) + f(0, -\sqrt{\tfrac{7}{15}})\}$$
$$+ \tfrac{9}{49}\{f(\sqrt{\tfrac{7}{9}}, \sqrt{\tfrac{7}{9}}) + f(\sqrt{\tfrac{7}{9}}, -\sqrt{\tfrac{7}{9}}) + f(-\sqrt{\tfrac{7}{9}}, \sqrt{\tfrac{7}{9}})$$
$$+ f(-\sqrt{\tfrac{7}{9}}, -\sqrt{\tfrac{7}{9}})\}.$$

For $\int_0^1 \int_0^1 (1 + x^2 + y^2)^{-\frac{3}{2}}\,dx\,dy$ this formula gives ·5232. For $\int_0^1 \int_0^1 (3 - x^2 - y^2)^{-\frac{1}{2}}\,dx\,dy$ it gives ·6641, and for $\int_0^1 \int_0^1 (2 - x^2 - y^2)^{-\frac{1}{2}}\,dx\,dy$ ·9262, the true values being ·6638 and ·9202. Results obtained by the third Gauss formula given above are slightly better, being ·6638 and ·9144.

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The amount of computation required by the two formulae is about the same.

All formulae such as the Simpson, Weddle, etc., involve the use of corner points of the range of integration, and therefore fail in such a case as $\int_0^1 \int_0^1 (2-x^2-y^2)^{-\frac{1}{2}}\,dx\,dy$, where the integrand becomes infinite at the point $x=1, y=1$. To evaluate such integrals we may combine a formula involving end points with a formula not involving end points. Thus a combination of elementary Gauss and Simpson formulae gives

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = \tfrac{1}{3}\Big[f\big(\sqrt{\tfrac{1}{3}},\,1\big)+f\big(\sqrt{\tfrac{1}{3}},\,-1\big)+f\big(-\sqrt{\tfrac{1}{3}},\,1\big)$$
$$+f\big(-\sqrt{\tfrac{1}{3}},\,-1\big)+4\big\{f\big(\sqrt{\tfrac{1}{3}},\,0\big)+f\big(-\sqrt{\tfrac{1}{3}},\,0\big)\big\}\Big].$$

This gives for $\int_0^1 \int_0^1 (2-x^2-y^2)^{-\frac{1}{2}}\,dx\,dy$ the value ·9205, but the accuracy of the result is possibly accidental.

§ 5. *Chebyshef Formulae.*

Combination of Chebyshef formulae of the types

$$\int_{-1}^{1} f(x)\,dx = \tfrac{2}{3}\{f(-p)+f(0)+f(p)\}$$
$$= \tfrac{2}{5}\{f(-q)+f(-r)+f(0)+f(r)+f(q)\},$$

where $p = {\cdot}707166\ldots$, $q = {\cdot}832437\ldots$, $r = {\cdot}374542\ldots$, gives formulae such as

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = \tfrac{4}{9}\{f(p,p)+f(p,-p)+f(-p,p)$$
$$+f(-p,-p)+f(0,0)+f(p,0)+f(-p,0)$$
$$+f(0,p)+f(0,-p)\},$$

and other formulae of a greater number of terms.

Such formulae have the advantage that the terms may be summed without first multiplying by various coefficients, but even so the values of the variables required in them are not very convenient for ordinary applications. The simplest of these formulae, written out above, gives fair results, *e.g.* for $\int_0^1\int_0^1(1+x^2+y^2)^{-\frac{3}{2}}\,dx\,dy$ $\int_0^1\int_0^1(2-x^2-y^2)^{-\frac{1}{2}}\,dx\,dy$ and $\int_0^1\int_0^1(3-x^2-y^2)^{-\frac{1}{2}}\,dx\,dy$ the values ·5245, ·9107 and ·6645 respectively.

§ 6. *Formulae in Finite Differences.*

These formulae are of use where the values of the integrand are tabulated for equidistant values of the variables. As in the case of a function of one variable, they consist of terms giving a trapezoidal sum, followed by correction terms, but while in the former case these terms refer to the ends of the linear range of integration, here they refer to the boundary of the rectangular region.

Corresponding to the single formula

$$\frac{1}{u}\int_a^{a+mu} f(x)\,dx = S_u^m f(a) - \frac{\Delta}{12}\{f(a+mu)-f(a)\} + \frac{\Delta^2}{24}\{f(a+mu)-f(a)\}$$
$$-\frac{19\Delta^3}{720}\{f(a+mu)-f(a)\} + \frac{3\Delta^4}{160}\{f(a+mu)-f(a)\} - \text{etc.},$$

we have by § 2 the double formula

$$\frac{1}{uv}\int_a^{a+mu}\int_b^{b+nv} f(x,y)\,dx\,dy = S_u^m S_v^n f(a,b)$$
$$-\frac{\Delta x}{12}\left[S_u^0 S_v^n\{f(a+mu,b)-f(a,b)\}\right]$$
$$-\frac{\Delta y}{12}\left[S_u^m S_v^0\{f(a,b+nv)-f(a,b)\}\right]$$
$$+\frac{\Delta_x^2}{24}\left[S_u^0 S_v^n\{f(a+mu,b)-f(a,b)\}\right]$$
$$+\frac{\Delta_y^2}{24}\left[S_u^m S_v^0\{f(a,b+nv)-f(a,b)\}\right]$$
$$-\frac{19\Delta_x^3}{720}[\text{etc. } \dots] - \dots + \frac{\Delta_x\Delta_y}{144}\left[f(a+mu,b+nv)-f(a,b+nv)\right.$$
$$\left.-f(a+mu,b)+f(a,b)\right]$$
$$-\frac{\Delta_x^2\Delta_y+\Delta_x\Delta_y^2}{288}\left[f(a+mu,b+nv)-\text{etc.}\right]+\dots,$$

where $S_u^0 f(a) = f(a)$.

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The mode of applying the formula can best be illustrated from the actual table below of $z = e^{x^2y}$.

		x				
		·4	·5	·6	·7	·8
	1·3	1·2312	1·3840	1·5968	1·8908	2·2979
	1·4	1·2511	1·4191	1·6553	1·9858	2·4498
	1·5	1·2712	1·4550	1·7160	2·0855	2·6117
y	1·6	1·2918	1·4918	1·7789	2·1902	2·7843
	1·7	1·3126	1·5296	1·8441	2·3002	2·9683
	1·8	1·3338	1·5683	1·9117	2·4157	3·1645

The terms $S_u^0 S_v^n f(a, b)$ and similar ones mean the trapezoidal sums of columns, $S_u^m S_v^0 f(a, b)$ the trapezoidal sums of rows.

Hence the best method is first to take trapezoidal sums of columns and rows, and construct two difference tables of these. (For the above formula columns and rows outside the range are necessary in order to obtain the required differences.)

The trapezoidal sum of either these sums of columns or the sums of rows gives the trapezoidal approximation to the integral, $S_u^m S_v^n f(a, b)$. The more important correction terms are obtained from the two difference tables, just as for single numerical integration. As for the terms in $\Delta_x \Delta_y$, $\Delta_x^2 \Delta_y$, $\Delta_x \Delta_y^2$, etc, they are small corrections taken at the four corners, and are generally negligible.

When values of the integrand within the range of integration only are available, *e.g.* if from the table above we have to find

$\int_{\cdot 4}^{\cdot 8} \int_{1\cdot 3}^{1\cdot 8} e^{x^2y}\, dx\, dy$, we may use the similar formula

$$\frac{1}{uv} \int_a^{a+mu} \int_b^{b+nv} f(x, y)\, dx\, dy = S_u^m S_v^n f(a, b)$$

$$- \frac{\Delta_x}{12} \left[S_u^0 S_v^n \left\{ f(a + \overline{m-1}\, u, b) - f(a, b) \right\} \right]$$

$$- \frac{\Delta_y}{12} \left[S_u^m S_v^0 \left\{ f(a, b + \overline{n-1}\, v) - f(a, b) \right\} \right]$$

$$- \frac{\Delta_x^2}{24} \left[S_u^0 S_v^n \left\{ f(a + \overline{m-2}\, u, b) + f(a, b) \right\} \right]$$

$$-\frac{\Delta_y^2}{24} S_u^m S_v^0 \{ f(a, b+\overline{n-2}\, v) + f(a, b) \}]$$

$$-\frac{19\Delta_x^3}{720} [S_u^0 S_v^n \{ f(a+\overline{m-3}u, b) - f(a, b) \}]$$

$$-\frac{19\Delta_y^3}{720} [\text{etc. } \ldots] + \ldots + \frac{\Delta_x \Delta_y}{144} [f(a+\overline{m-1}u, b+\overline{n-1}v)$$

$$-f(a, b+\overline{n-1}v) - f(a+\overline{m-1}u, b) + f(a, b)] + \ldots$$

Example.—From the given table of values of e^{x^2y} the trapezoidal sums of columns are 6·4092, 7·3716, 8·7485, 10·7149, 13·5453, of rows 6·6361, 6·9106, 7·1979, 7·4989, 7·8143, 8·1448.

The trapezoidal sum of either of these sets, with the differences of successive orders, gives approximations to $\int_{\cdot 4}^{\cdot 8} \int_{1\cdot 3}^{1\cdot 8} e^{x^2y}\, dx\, dy$ of ·36812, ·36652, ·36598, ·36595, ·36591, the last being correct to five places. Corner corrections were neglected.

As a further test, $z = (1+x^2+y^2)^{-\frac{1}{2}}$ was tabulated at intervals of ·1 between the values $x=0, 1, y=0, 1$. To the third order of differences this gives for $\int_0^1 \int_0^1 (1+x^2+y^2)^{-\frac{1}{2}}\, dx\, dy$ the good approximation ·523604.

If values of x outside its range of integration are available, but not of y outside its range, a compromise between the two preceding formulae may be given:

$$\frac{1}{uv}\int_a^{a+mu} \int_b^{b+nv} f(x, y)\, dx\, dy = S_u^m S_v^n f(a, b)$$

$$-\frac{\Delta_x}{12} [S_u^0 S_v^n \{ f(a+mu, b) - f(a, b) \}]$$

$$-\frac{\Delta_y}{12} [S_u^m S_v^0 \{ f(a, b+\overline{n-1}\, v) - f(a, b) \}]$$

$$+\frac{\Delta_x^2}{24} [S_u S_v^n \{ f(a+mu, b) - f(a, b) \}]$$

$$-\frac{\Delta_y^2}{24} [S_u^m S_v^0 \{ f(a, b+\overline{n-2}v) + f(a, b) \}] - \frac{19\Delta_x^3}{720} [\text{etc.}]$$

$$-\frac{19\Delta_y^3}{720} [\text{etc.}] + \ldots + \frac{\Delta_x \Delta_y}{144} [f(a+mu, b+\overline{n-1}v) - f(a, b+\overline{n-1}v)$$

$$-f(a+mu, b) + f(a, b)] + \ldots \text{etc.}$$

From the Bessel central difference formula

$$\frac{1}{u}\int_a^{a+mu} f(x)\,dx = S_u^m f(a) - \frac{\mu\delta}{12}(f_m - f_0) + \frac{11\mu\delta^3}{720}(f_m - f_0) - \frac{191\mu\delta^5}{60480}(f_m - f_0) + \dots,$$

where $\mu\delta f_r = \frac{1}{2}(\delta f_{r+\frac{1}{2}} + \delta f_{r-\frac{1}{2}})$ in the usual notation, we derive by § 2

$$\frac{1}{uv}\int_a^{a+mu}\int_b^{b+nv} f(x, y)\,dx\,dy = S_u^m S_v^n f(a, b)$$

$$- \frac{\mu_x\delta_x}{12}[S_u^0 S_v^n (f_{m,0} - f_{0,0})] - \frac{\mu_y\delta_y}{12}[S_u^m S_v^0 (f_{0,n} - f_{0,0})]$$

$$+ \frac{11\mu_x\delta_x^3}{720}[S_u^0 S_v^n (f_{m,0} - f_{0,0})] + \frac{11\mu_y\delta_y^3}{720}[S_u^m S_v^0 (f_{0,n} - f_{0,0})] - \dots$$

$$+ \frac{\mu_x\delta_x\mu_y\delta_y}{144}(f_{m,n} - f_{0,n} - f_{m,0} + f_{0,0}) + \dots \text{ etc},$$

where $\mu_x\delta_x f_{r,s} = \frac{1}{2}(\delta_x f_{r+\frac{1}{2},s} + \delta_x f_{r-\frac{1}{2},s})$ and so for $\mu_y\delta_y$.

To apply this formula, take trapezoidal sums of columns and rows and form difference tables with these as before. For differences as far as the third, corner terms being neglected, it gave for $\int_{\cdot 4}^{\cdot 8}\int_{1\cdot 3}^{1\cdot 8} e^{x^2 y}\,dx\,dy$ the value correct to five places, ·36591.

Central difference formulae of numerical integration are rapidly convergent, differences of even order not appearing. In the corresponding interpolation formulae no differences of odd order appear, so that, if a function is to be tabulated for the double purpose of interpolation and numerical integration, as useful a table as any is one giving the values of the function and of all the central differences as far as required.

§ 7. *Formulae in Differential Coefficients.*

The double Euler-Maclaurin formula is

$$\frac{1}{uv}\int_a^{a+mu}\int_b^{b+nv} f(x, y)\,dx\,dy = S_u^m S_v^n - \frac{u}{12}\cdot\frac{\partial}{\partial x}[S_u^0 S_v^n \{f(a+mu, b) - f(a, b)\}] - \frac{v}{12}\cdot\frac{\partial}{\partial y}[S_u^m S_v^0 \{f(a, b+nv) - f(a, b)\}]$$

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$$+\frac{u^3}{720}\cdot\frac{\partial^3}{\partial x^3}\left[S_u^0 S_v^n\left\{f(a+mu,\,b)-f(a,\,b)\right\}\right]$$

$$+\frac{v^3}{720}\cdot\frac{\partial^3}{\partial y^3}\left[S_u^m S_v^0\left\{f(a,\,b+nv)-f(a,\,b)\right\}\right]$$

$$-\text{ etc. } \ldots+\frac{uv}{144}\cdot\frac{\partial^2}{\partial x\,.\,\partial y}\left[f(a+mu,\,b+nv)-f(a,\,b+nv)-f(a+mu,\,b)\right.$$

$$\left.+f(a,\,b)\right]+\ldots$$

The correction terms here, as in previous formulae, are taken in columns and rows on the boundary, summed trapezoidally, together with small and generally negligible corner terms.

The formula gave for $\int_0^1\int_0^1(1+x^2+y^2)^{-\frac{3}{2}}\,dx\,dy$ the value ·523604, partial coefficients as far as the third being taken, and 121 values of the integrand being tabled.

$$\int_0^1\int_0^1 e^{x^2y}\,dx\,dy \text{ and } \int_0^{\cdot 8}\int_0^{1\cdot 2} e^{x^2y}\,dx\,dy$$

were also calculated for 121 and 117 points respectively, results being obtained correct to five places, 1·20702 and 1·10469.

§ 8. *General Conclusions.*

Comparisons of the rapidity and accuracy of calculations performed by means of all the foregoing formulae of double numerical integration seem to show that the best method is to apply one or other of the finite difference formulae to trapezoidal sums of the rows and columns concerned. For moderately close approximations, however, a double Simpson or a Gauss formula may serve.

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