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THE AVERAGE DISTANCE PROPERTY OF CLASSICAL BANACH SPACES II

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A Banach space X has the average distance property if there exists a unique real number r such that for each positive integer n and all $x_1, \ldots, x_n$ in the unit sphere of X there is some x in the unit sphere of X such that

$$\frac{1}{n}\sum_{k=1}^{n}\|x_k - x\| = r.$$

We show that l_p does not have the average distance property if $p > 2$. This completes the study of the average distance property for l_p spaces.

1. INTRODUCTION

The aim of this note is to finish the study of the average distance property of l_p and $L_p[0,1]$ for $1 \leqslant p \leqslant \infty$ using and refining the method introduced in [**1**]. We start by giving a short review of that method. The reader is referred to [**1**] for further information and to the pointers to the literature therein.

A rendezvous number of a metric space (M, d) is a real number r with the property that for each positive integer n and $x_1, \ldots, x_n \in M$ there exists $x \in M$ such that

$$\frac{1}{n}\sum_{k=1}^{n} d(x_k, x) = r.$$

We say that a (real or complex) Banach space X has the average distance property if its unit sphere has a unique rendezvous number. It is known that l_2 and $L_2[0,1]$ have the average distance property ([**4**]) and that l_p and $L_p[0,1]$ do not have the average distance property if $1 \leqslant p < 2$ and if $p \geqslant 3$, see [**3**] and [**1**], respectively. Here we prove the following result.

THEOREM 1. *For $p > 2$, l_p and $L_p[0,1]$ do not have the average distance property.*

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In [**1**], using an improved Clarkson inequality, the study of the average distance property for l_p and L_p in the case $p > 2$ was reduced to the study of a scalar function as follows. For $n \in \mathbb{N}$, $p > 2$ and $x, y_1, \ldots, y_n \in l_p$ or L_p such that $\|x\|^p = 1/n$ and $\sum_{i=1}^{n} \|y_i\|^p = 1$ define

$$\sigma_i := \frac{\|x - y_i\|^p}{\|x\|^p + \|y_i\|^p} \quad \text{and} \quad \alpha_i := \frac{\|x\|^p + \|y_i\|^p}{2}. \tag{1}$$

It follows that

$$\frac{1}{2n} \leqslant \alpha_i \leqslant \frac{n+1}{2n} \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

Let $u_i \in [-1, +1]$ be defined by the relation

$$\sigma_i = \frac{(1 - u_i)^p}{1 + |u_i|^p}. \tag{2}$$

and let

$$\varphi(x, y_1, \ldots, y_n) := 2^{-n} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \left(\sum_{i=1}^{n} \alpha_i \frac{(1 + \varepsilon_i u_i)^p}{1 + |u_i|^p} \right)^{1/p}.$$

As pointed out in [**1**], in order to prove Theorem 1 for a fixed $p > 2$, it suffices to find n such that $\varphi > 1$ for $(u_1, \ldots, u_n) \neq (0, \ldots, 0)$.

Considering the case $u_i = 1$ for $i = 1, \ldots, n$, $\alpha_i = 1/(2n)$ for $i = 1, \ldots, n-1$, and $\alpha_n = (n+1)/(2n)$ yields that

$$\begin{aligned}
&2^{-n} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \left(\sum_{i=1}^{n} \alpha_i \frac{(1 + \varepsilon_i u_i)^p}{1 + |u_i|^p} \right)^{1/p} \\
&\quad = 2^{1-2/p} 2^{-n} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \left(1 + \frac{1}{2n} \sum_{i=1}^{n-1} \varepsilon_i + \frac{n+1}{2n} \varepsilon_n \right)^{1/p} \\
&\quad = 2^{-2/p} \sum_{\varepsilon_n = \pm 1} \left(2^{-n+1} \sum_{\varepsilon_1, \ldots, \varepsilon_{n-1} = \pm 1} \left(1 + \frac{1}{2n} \sum_{i=1}^{n-1} \varepsilon_i + \frac{n+1}{2n} \varepsilon_n \right)^{1/p} \right) \\
&\quad \leqslant 2^{-2/p} \sum_{\varepsilon_n = \pm 1} \left(1 + 2^{-n+1} \sum_{\varepsilon_1, \ldots, \varepsilon_{n-1} = \pm 1} \frac{1}{2n} \sum_{i=1}^{n-1} \varepsilon_i + \frac{n+1}{2n} \varepsilon_n \right)^{1/p} \\
&\quad = 2^{-2/p} \left(\left(\frac{3}{2} + \frac{1}{n} \right)^{1/p} + \left(\frac{1}{2} - \frac{1}{n} \right)^{1/p} \right) \\
&\quad \leqslant 2^{-2/p} \left(\left(\frac{3}{2} \right)^{1/p} + \left(\frac{1}{2} \right)^{1/p} \right) = \frac{3^{1/p} + 1}{8^{1/p}}
\end{aligned}$$

which is smaller than 1 for $p < 2.10528\ldots$.

This shows that, in contrast to [**1**], we have to take into account the concrete definition of the u_i's and α_i's to be able to cover also the cases where p is close to 2. This will be done in Proposition 2.

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The remaining part of the paper is the proof of Theorem 1, which follows from the upcoming Propositions 6 and 8.

2. The relation of α_i and u_i

We begin by providing an auxiliary estimate.

Lemma 1.
$$\frac{1+u}{(1+u^p)^{1/p}} \geqslant 1 + (2^{1-1/p} - 1)u$$
for $u \in [0,1]$.

Proof: Let $g(u) := (1+u)/(1+u^p)^{1/p}$. Note that
$$g'(u) = \frac{1-u^{p-1}}{(1+u^p)^{1+1/p}} \geqslant 0$$
while
$$g''(u) = -\frac{(p+1)(1-u^{p-1})u^{p-1} + (p-1)(1+u^p)u^{p-2}}{(1+u^p)^{2+1/p}} \leqslant 0.$$
This means that g is a concave function on $[0,1]$ and therefore $g(u) \geqslant g(0) + \big(g(1)-g(0)\big)u$. This proves the assertion. ∎

Proposition 2. *If α_i and u_i are defined by* (1) *and* (2), *then*
$$|u_i| \leqslant c_1 n^{-1/p} \alpha_i^{-1/p},$$
where $c_1 = \max\big(2^{1-1/p}, 1/(2-2^{1/p})\big)$.

Proof: We split the proof into three cases.

First case.
$$\alpha_i \leqslant \frac{1}{n}.$$
Since $c_1 \geqslant 1$, in this case
$$|u_i| \leqslant 1 \leqslant (n\alpha_i)^{-1/p} \leqslant c_1 n^{-1/p} \alpha_i^{-1/p}.$$

Second case.
$$\alpha_i > \frac{1}{n} \quad \text{and} \quad u_i \geqslant 0.$$
Then
$$\frac{\|y_i\|}{(2\alpha_i)^{1/p}} = \left(1 - \frac{1}{2\alpha_i n}\right)^{1/p} > \left(\frac{1}{2\alpha_i n}\right)^{1/p} = \frac{\|x\|}{(2\alpha_i)^{1/p}}$$
and it follows from the definition (2) of u_i that
$$\begin{aligned}(1-u_i)^p \geqslant \frac{(1-u_i)^p}{1+u_i^p} = \sigma_i = \frac{\|x-y_i\|^p}{\|x\|^p + \|y_i\|^p} &= \left\|\frac{x}{(2\alpha_i)^{1/p}} - \frac{y_i}{(2\alpha_i)^{1/p}}\right\|^p \\ &\geqslant \left(\left(1 - \frac{1}{2\alpha_i n}\right)^{1/p} - \left(\frac{1}{2\alpha_i n}\right)^{1/p}\right)^p.\end{aligned}$$

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Now, using the relations

$$1-\frac{1}{2\alpha_i n}\leqslant\Big(1-\frac{1}{2\alpha_i n}\Big)^{1/p}\quad\text{and}\quad\frac{1}{2\alpha_i n}\leqslant\Big(\frac{1}{2\alpha_i n}\Big)^{1/p}$$

which follow from $\alpha_i \geqslant 1/(2n)$, we obtain

$$u_i\leqslant 1-\Big(1-\frac{1}{2\alpha_i n}\Big)^{1/p}+\Big(\frac{1}{2\alpha_i n}\Big)^{1/p}\leqslant\frac{1}{2\alpha_i n}+\Big(\frac{1}{2\alpha_i n}\Big)^{1/p}\leqslant 2\Big(\frac{1}{2\alpha_i n}\Big)^{1/p}.$$

From this we get

$$|u_i|=u_i\leqslant 2^{1-1/p}n^{-1/p}\alpha_i^{-1/p}.$$

THIRD CASE.

$$\alpha_i>\frac{1}{n}\quad\text{and}\quad u_i<0.$$

It follows from Lemma 1 for $u=-u_i$ that

$$1-(2^{1-1/p}-1)u_i\leqslant\sigma_i^{1/p}\leqslant\Big(1-\frac{1}{2\alpha_i n}\Big)^{1/p}+\Big(\frac{1}{2\alpha_i n}\Big)^{1/p}\leqslant 1+\Big(\frac{1}{2\alpha_i n}\Big)^{1/p}.$$

Finally in this case

$$|u_i|=-u_i\leqslant\frac{1}{2-2^{1/p}}n^{-1/p}\alpha_i^{-1/p}.$$

∎

With this proposition in hand, we can forget about the concrete nature of the α_i's and u_i's. All we have to show is that for given n and $\alpha_1,\dots,\alpha_n$ such that

$$\frac{1}{2n}\leqslant\alpha_i\leqslant\frac{n+1}{2n}\quad\text{and}\quad\sum_{i=1}^n\alpha_i=1$$

the function

$$\varphi(u_1,\dots,u_n):=2^{-n}\sum_{\varepsilon_1,\dots,\varepsilon_n=\pm1}\Big(\sum_{i=1}^n\alpha_i\frac{(1+\varepsilon_i u_i)^p}{1+|u_i|^p}\Big)^{1/p}$$

is bigger than one as long as

$$|u_i|\leqslant c_1 n^{-1/p}\alpha_i^{-1/p}\tag{3}$$

and $(u_1,\dots,u_n)\neq(0,\dots,0)$.

Since all relations on the u_i's are symmetric and since the function φ is symmetric in u_i, we can henceforth assume that $u_i\geqslant 0$.

3. PROOF OF $\varphi>1$, THE CASE OF MANY LARGE u_i'S

COROLLARY 3.

$$\Big(\sum_{i=1}^n(\alpha_i u_i)^2\Big)^{1/2}\leqslant c_1 n^{-1/p}.$$

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PROOF: It follows from (3) that

$$\Big(\sum_{i=1}^{n}(\alpha_i u_i)^2\Big)^{1/2} \leqslant c_1 n^{-1/p}\Big(\sum_{i=1}^{n}\alpha_i^{2-2/p}\Big)^{1/2}.$$

Since $2-2/p>1$ and $\alpha_i<1$ we have

$$\sum_{i=1}^{n}\alpha_i^{2-2/p} \leqslant \sum_{i=1}^{n}\alpha_i = 1,$$

which proves the assertion. ∎

LEMMA 4. *We have*

(4) $$v(u) := \frac{(1+u)^p+(1-u)^p}{2(1+u^p)} \geqslant 1 + c_2 u^p$$

and

(5) $$w(u) := \frac{(1+u)^p-(1-u)^p}{2(1+u^p)} \leqslant c_3 u$$

for $u\in[0,1]$, *where* $c_2 := 2^{p-2}-1$ *and* $c_3 := p2^{p-1}$.

PROOF: To see (4), we let

$$g(u) := \frac{(1+u)^p+(1-u)^p-2}{u^p}$$

and use the fact that $(1+u)^{p-1}+(1-u)^{p-1}$ is non-increasing for $p>2$, to compute

$$g'(u) = \frac{p}{u^{p+1}}\big(2-(1+u)^{p-1}-(1-u)^{p-1}\big) \leqslant 0.$$

Therefore $g(u)\geqslant g(1)=2^p-2$, which yields

$$(1+u)^p+(1-u)^p \geqslant 2+(2^p-2)u^p = 2(1+u^p)+(2^p-4)u^p.$$

Division by $2(1+u^p)$ and $1+u^p\leqslant 2$ proves (4).

Since $2u/(1+u)\leqslant 1$, Bernoulli's inequality states

$$\frac{(1-u)^p}{(1+u)^p} = \Big(1-\frac{2u}{1+u}\Big)^p \geqslant 1-\frac{2pu}{1+u}.$$

It follows that

$$\frac{(1+u)^p-(1-u)^p}{1+u^p} = \frac{(1+u)^p}{1+u^p}\Big(1-\frac{(1-u)^p}{(1+u)^p}\Big) \leqslant 2pu\frac{(1+u)^{p-1}}{1+u^p} \leqslant p2^p u$$

which proves (5). ∎

The following Lemma is known as a subgaussian tail estimate for Rademacher averages and is by now classical. A proof can be found for example, in [**2**, p. 90].

Lemma 5. *For a given vector $x = (\xi_1, \ldots, \xi_n)$, let $\|x\|_2 := \left(\sum_{i=1}^n |\xi_i|^2\right)^{1/2}$ and $\mathbb{B} := \left\{(\varepsilon_1, \ldots, \varepsilon_n) : \sum_{i=1}^n \varepsilon_i \xi_i > t\|x\|_2\right\}$, then*

$$2^{-n}|\mathbb{B}| \leqslant e^{-t^2/2}.$$

We are now ready to tackle the case, where 'many' of the u_i's are bigger than $1/2$.

Proposition 6. *There exists n_1 such that for all $n > n_1$ we have*

$$\varphi(u_1, \ldots, u_n) > 1$$

if $|\mathbb{A}| > n/2$, where $\mathbb{A} := \{i : u_i > 1/2\}$.

Proof: With v and w defined as in Lemma 4, observe that

$$v(u) + \varepsilon w(u) = \frac{(1 + \varepsilon u)^p}{1 + u^p}$$

for $\varepsilon = \pm 1$. Put

$$\mathbb{B} := \left\{(\varepsilon_1, \ldots, \varepsilon_n) : -\sum_{i=1}^n \alpha_i \varepsilon_i w(u_i) \leqslant (2 \log n)^{1/2} c_3 c_1 n^{-1/p}\right\}.$$

Since by (5) and Corollary 3

$$\left(\sum_{i=1}^n \left(\alpha_i w(u_i)\right)^2\right)^{1/2} \leqslant c_3 \left(\sum_{i=1}^n \left(\alpha_i u_i\right)^2\right)^{1/2} \leqslant c_3 c_1 n^{-1/p}$$

it follows from Lemma 5 that

$$2^{-n}|\mathbb{B}| \geqslant 1 - \frac{1}{n}.$$

With these preliminaries we can estimate φ as follows

$$\begin{aligned}\varphi(u_1, \ldots, u_n) &\geqslant 2^{-n} \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{B}} \left(\sum_{i=1}^n \alpha_i v(u_i) + \sum_{i=1}^n \alpha_i \varepsilon_i w(u_i)\right)^{1/p} \\ &\geqslant \left(1 - \frac{1}{n}\right) \left(\sum_{i=1}^n \alpha_i v(u_i) - (2 \log n)^{1/2} c_3 c_1 n^{-1/p}\right)^{1/p}.\end{aligned}$$

From (4) and the assumption on $\mathbb{A}$ it follows that

$$\sum_{i=1}^n \alpha_i v(u_i) \geqslant \sum_{i=1}^n \alpha_i + \sum_{i \in \mathbb{A}} \alpha_i c_2 u_i^p \geqslant 1 + \frac{n}{2} \frac{1}{2n} c_2 2^{-p} = 1 + c_4,$$

where $c_4 := c_2 2^{-p-2}$.

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Since $c_4 > 0$, we can now choose n_1 so that for all $n > n_1$

$$(2\log n)^{1/2} c_3 c_1 n^{-1/p} < \frac{c_4}{2} \quad \text{and} \quad \left(1-\frac{1}{n}\right)\left(1+\frac{c_4}{2}\right)^{1/p} > \left(1+\frac{c_4}{4}\right)^{1/p}.$$

By these assumptions on n

$$\begin{aligned}\varphi(u_1,\dots,u_n) &\geqslant \left(1-\frac{1}{n}\right)\left(1+c_4-(2\log n)^{1/2}c_3c_1n^{-1/p}\right)^{1/p}\\ &\geqslant \left(1-\frac{1}{n}\right)\left(1+\frac{c_4}{2}\right)^{1/p}\\ &\geqslant \left(1+\frac{c_4}{4}\right)^{1/p}.\end{aligned}$$

This proves the assertion. ∎

4. Proof of $\varphi > 1$, the case of few large u_i's

From now on, we shall only deal with the case $|\mathbb{A}| \leqslant n/2$. So for the rest of this section, we assume that

$$(6) \qquad |\mathbb{A}| \leqslant \frac{n}{2}, \quad \text{where } \mathbb{A} = \{i : u_i > 1/2\}.$$

Lemma 7. *Denote*

$$(7) \qquad f(u) := \frac{(1-u^2)^p}{1+u^p}\,\frac{(1+u^{p-1})^{p/(p-1)} - (1-u^{p-1})^{p/(p-1)}}{(1+u)^p(1-u^{p-1})^{p/(p-1)} - (1-u)^p(1+u^{p-1})^{p/(p-1)}}.$$

Then $\lim_{u\to 0} f(u) = \lim_{u\to 1} f(u) = 0$ *and* f *is bounded on* $[0,1]$.

Proof: Note that the derivative of the function $(1\pm u)^p(1\mp u^{p-1})^{p/(p-1)}$ is

$$\pm p(1\pm u)^{p-1}(1\mp u^{p-1})^{p/(p-1)} \mp p(1\pm u)^p(1\mp u^{p-1})^{1/(p-1)}u^{p-2}.$$

Since $p > 2$ we therefore have

$$\lim_{u\to 0}\frac{d}{du}(1+u)^p(1-u^{p-1})^{p/(p-1)} - \lim_{u\to 0}\frac{d}{du}(1-u)^p(1+u^{p-1})^{p/(p-1)} = 2p.$$

By l'Hopital's rule

$$\begin{aligned}\lim_{u\to 0} f(u) &= \lim_{u\to 0}\frac{(1+u^{p-1})^{p/(p-1)} - (1-u^{p-1})^{p/(p-1)}}{(1+u)^p(1-u^{p-1})^{p/(p-1)} - (1-u)^p(1+u^{p-1})^{p/(p-1)}}\\ &= \lim_{u\to 0}\frac{p(1+u^{p-1})^{1/(p-1)}u^{p-2} + p(1-u^{p-1})^{1/(p-1)}u^{p-2}}{\frac{d}{du}(1+u)^p(1-u^{p-1})^{p/(p-1)} - \frac{d}{du}(1-u)^p(1+u^{p-1})^{p/(p-1)}}\\ &= 0.\end{aligned}$$

On the other hand, again by l'Hopital's rule it follows that

$$\lim_{u\to 1}\frac{(1-u^{p-1})^{1/(p-1)}}{1-u} = \lim_{u\to 1}\frac{u^{p-2}}{(1-u^{p-1})^{(p-2)/(p-1)}} = +\infty.$$

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Therefore

$$\begin{aligned}\lim_{u\to 1} f(u) &= \lim_{u\to 1} \frac{2^{1/(p-1)}(1-u^2)^p}{(1+u)^p(1-u^{p-1})^{p/(p-1)} - (1-u)^p(1+u^{p-1})^{p/(p-1)}} \\ &= \lim_{u\to 1} \frac{2^{1/(p-1)}}{\dfrac{(1-u^{p-1})^{p/(p-1)}}{(1-u)^p} - \dfrac{(1+u^{p-1})^{p/p-1)}}{(1+u)^p}} \\ &= 0.\end{aligned}$$

The boundedness of f on $[0,1]$ now follows from its continuity in $(0,1)$ and the boundedness of the limits of $f(u)$ for $u \to 0$ and $u \to 1$. ∎

We can now also treat the remaining case, where only 'few' of the u_i's are bigger than $1/2$. In this case, the next proposition shows that $\varphi(u_1,\dots,u_n) > \varphi(0,\dots,0) = 1$, provided that n is big enough. This completes the proof of Theorem 1.

PROPOSITION 8. *There exists $n_2 \geqslant n_1$ such that for all $n > n_2$ we have*

$$\frac{\partial\varphi}{\partial u_j}(u_1,\dots,u_n) > 0$$

for all $j = 1,\dots,n$ and all $u_1,\dots,u_n$ satisfying (6).

PROOF: Note that

$$\frac{\partial\varphi}{\partial u_j}(u_1,\dots,u_n) = \frac{\alpha_j}{(1+u_j^p)^2} 2^{-n} \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_j \frac{(1+\varepsilon_j u_j)^{p-1}(1-\varepsilon_j u_j^{p-1})}{\left(\sum_{i=1}^n \alpha_i \dfrac{(1+\varepsilon_i u_i)^p}{1+u_i^p}\right)^{1-1/p}}.$$

We shall show that for every $\varepsilon_1,\dots,\varepsilon_{j-1},\varepsilon_{j+1},\dots\varepsilon_n$ the summand

$$\sum_{\varepsilon_j=\pm 1} \varepsilon_j \frac{(1+\varepsilon_j u_j)^{p-1}(1-\varepsilon_j u_j^{p-1})}{\left(\sum_{i=1}^n \alpha_i \dfrac{(1+\varepsilon_i u_i)^p}{1+u_i^p}\right)^{1-1/p}}$$

is positive.

To this end we denote

$$a_j(u_1,\dots,u_n) := \sum_{\substack{i=1\\ i\neq j}}^n \alpha_i \frac{(1+\varepsilon_i u_i)^p}{1+u_i^p}$$

and show that

$$\frac{(1+u_j)^{p-1}(1-u_j^{p-1})}{\left(a_j(u_1,\dots,u_n) + \alpha_j \dfrac{(1+u_j)^p}{1+u_j^p}\right)^{1-1/p}} > \frac{(1-u_j)^{p-1}(1+u_j^{p-1})}{\left(a_j(u_1,\dots,u_n) + \alpha_j \dfrac{(1-u_j)^p}{1+u_j^p}\right)^{1-1/p}}.$$

Some manipulations show that this is equivalent to

$$a_j(u_1, \dots, u_n) > \alpha_j f(u_j),$$

where f is the function defined in (7) in Lemma 7.

Using (6), we see that

$$a_j(u_1, \dots, u_n) \geqslant \sum_{\substack{i \notin A \\ i \neq j}} \alpha_i \frac{(1-u_i)^p}{1+u_i^p} \geqslant \left(\frac{n}{2} - 1\right) \frac{1}{2n} \frac{2^{-p}}{1+2^{-p}} \geqslant \frac{1}{8} \frac{1}{1+2^p} = c_5,$$

if $n \geqslant 4$ and $c_5 := 1/(8 + 2^{p+3})$. It is hence enough to show that

$$c_5 > \alpha_j f(u_j). \tag{8}$$

Since $\lim_{u \to 0} f(u) = 0$ by Lemma 7, we can find $\delta > 0$ small enough such that

$$f(u) < c_5$$

for $u^p < \delta$. Since f is also bounded by Lemma 7, we can choose

$$n \geqslant n_2 := \max\left(\frac{c_1^p \|f\|_\infty}{c_5 \delta}, n_1, 4\right).$$

If $\alpha_j < c_5 / \|f\|_\infty$ then obviously (8) holds.

If on the other hand $\alpha_j \geqslant c_5 / \|f\|_\infty$ then

$$\alpha_j n > \frac{c_5}{\|f\|_\infty} \frac{c_1^p \|f\|_\infty}{c_5 \delta} = \frac{c_1^p}{\delta}$$

and by (3)

$$u_j^p \leqslant \frac{c_1^p}{n \alpha_j} < \delta.$$

Consequently

$$\alpha_j f(u_j) < \alpha_j c_5 \leqslant c_5,$$

since $\alpha_j \leqslant 1$.

This proves the assertion. ∎

REMARK. Using the methods developed in Sections 3 and 4, it can be shown that without Relation (3) one can prove the result of the main theorem for all $p > p_0$, where

$$p_0 := \inf\{p > 2 : g \geqslant 2^{(1+1/p)}\} = 2.2751\dots$$

and

$$g(u) := \left(1 + \frac{(1+u)^p}{1+u^p}\right)^{1/p} + \left(1 + \frac{(1-u)^p}{1+u^p}\right)^{1/p}.$$

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