CENTRE-BY-METABELIAN LIE ALGEBRAS

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If V is a variety of metabelian Lie algebras then V has a finite basis for its laws [3]. The proof of this result is similar to Cohen's proof that varieties of metabelian groups have the finite basis property [1]. However there are centre-by-metabelian Lie algebras of characteristic 2 which do not have a finite basis for their laws [4]; this contrasts with McKay's recent result that varieties of centre-by-metabelian groups do have the finite basis property [2]. The following theorem shows that once again "2" is the odd man out.

THEOREM. *If V is a variety of centre-by-metabelian Lie algebras over a field K, and if the characteristic of K is not 2, then V has a finite basis for its laws.*

The notation will follow [4]. Throughout this paper K will denote a field whose characteristic is not 2.

Let X be the free Lie algebra over K freely generated by $x_1, x_2, \cdots$. Then the variety of centre-by-metabelian Lie algebras over K is determined by the law $((x_1x_2)(x_3x_4))x_5$. Let $F = X/(X^2)^2X$ and for $i = 1, 2, \cdots$ let y_i denote the image of x_i under the canonical epimorphism from X onto F. Then F is the free centre-by-metabelian Lie algebra over K freely generated by $y_1, y_2, \cdots$.

The theorem is equivalent to the following proposition.

PROPOSITION. *F satisfies the ascending chain condition on fully invariant ideals.*

Now if V is a variety of metabelian Lie algebras then V has a finite basis for its laws [3], and so $F/(F^2)^2$ satisfies the ascending chain condition on fully invariant ideals. It follows that to prove the proposition it is sufficient to show that F satisfies the ascending chain condition on fully invariant ideals of F contained in $(F^2)^2$. The proof follows the method developed by Cohen in [1].

For each element $g \in (F^2)^2$ I shall define the weight of g, an element $wt\, g \in (F^2)^2$. I shall define a partial well ordering, $\preccurlyeq$, and a well ordering, $\leqq$, on the set S of weights of elements of $(F^2)^2$. (A partially ordered set $(S, \preccurlyeq)$ is said to be partially well ordered if every infinite sequence of elements of S contains an ascending subsequence. This is equivalent to the property that for every subset $T \subseteq S$ there is a finite subset $T_0 \subseteq T$ such that for each $t \in T$ there is an element $s \in T_0, s \preccurlyeq t$.) The partial well ordering $\preccurlyeq$ and the well ordering $\leqq$ will be used to show that fully invariant ideals of F contained in $(F^2)^2$ are finitely generated as fully invariant ideals. This is equivalent to the ascending chain condition on fully invariant ideals of F contained in $(F^2)^2$.

All products will be left-normed; thus abc denotes $(ab)c$.

If a, b are elements of a Lie algebra then let $ab^0 = a, (ab^{i-1})b$ for $i = 1, 2, \cdots$.

Let Φ be the set of one-one order preserving maps of the positive integers into the positive integers.

Let A be the set of infinite sequences of finite support of non-negative integers. Addition of elements of A is defined componentwise, i.e.

$$(\alpha_1, \alpha_2, \cdots) + (\beta_1, \beta_2, \cdots) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \cdots).$$

Define a partial ordering $\leqq$ on A by

$$(\alpha_1, \alpha_2, \cdots) \leqq (\beta_1, \beta_2, \cdots)$$

if $\alpha_i \leqq \beta_i$ for $i = 1, 2, \cdots$. If $\phi \in \Phi$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots) \in A$ let

$$\boldsymbol{\alpha}\phi = (\beta_1, \beta_2, \cdots)$$

where $\beta_i = 0$ if $i \notin \operatorname{Im}\phi$, $\beta_{i\phi} = \alpha_i$ for $i = 1, 2, \cdots$.

If i, j, k, l are positive integers, and if $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m, 0, 0, \cdots) \in A$ let

$$(i, j, k, l; \boldsymbol{\alpha})$$

denote the element

$$(y_i y_j y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_m^{\alpha_m})(y_k y_l)$$

of $(F^2)^2$. By 2.8 of [4] the set

$$S = \{(i, j, k, l; \boldsymbol{\alpha}) \colon i, j, k, l \text{ positive integers}, \boldsymbol{\alpha} \in A\}$$

spans $(F^2)^2$ as a vector space over K.

Define a partial ordering $\preccurlyeq$ on S as follows. Let

$$(i, j, k, l; \boldsymbol{\alpha}) \preccurlyeq (p, q, r, s; \boldsymbol{\beta})$$

if there is an element $\phi \in \Phi$ such that

(1) $i\phi = p, j\phi = q, k\phi = r, l\phi = s$,
(2) $\boldsymbol{\alpha}\phi \leqq \boldsymbol{\beta}$,

and if

(3) $\sum_{n=1}^{\infty} \alpha_n \equiv \sum_{n=1}^{\infty} \beta_n \bmod 2$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots)$.

Let $\preccurlyeq *$ denote the partial ordering of S determined by properties (1) and (2). Then by Proposition 4.4 [5] $(S, \preccurlyeq *)$ is partially well ordered. Hence every infinite sequence of elements of S contains a subsequence which is ascending with respect to $\preccurlyeq *$. This subsequence must contain a subsequence which also satisfies property (3). Hence $(S, \preccurlyeq)$ is partially well ordered.

Defined a full ordering $\leqq$ on S as follows.

Let

$$(i,j,k,l;(\alpha_1,\alpha_2,\cdots)) < (p,q,r,s;(\beta_1,\beta_2,\cdots))$$

if one of the following conditions holds

(a) $i < p$.
(b) $i = p, j < q$.
(c) $i = p, j = q, k < r$.
(d) $i = p, j = q, k = r, l < s$.
(e) $i = p, j = q, k = r, l = s$, and, for some n, $\alpha_n < \beta_n$, $\alpha_m = \beta_m$ for $m > n$.

Then $(S, \leqq)$ is well ordered.

Let $g \in (F^2)^2$, $g \neq 0$. Then g can be written as a linear combination

$$\lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_n s_n$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are non-zero elements of K and $s_1, s_2, \cdots, s_n$ are distinct elements of S. Let the weight of g, $wt\ g$, be the greatest element under $\leqq$ of the set $\{s_1, s_2, \cdots, s_n\}$. (Strictly speaking I have defined the weight of the particular epresentation $\lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_n s_n$ of g.)

LEMMA 1. *If* $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots) \in A$ *and* $\sum_{n=1}^{\infty} \beta_n \equiv 0 \bmod 2$ *and if* $(i,j,k,l;\boldsymbol{\alpha}) \in S$ *then the fully invariant ideal of* F *generated by* $(i,j,k,l;\boldsymbol{\alpha})$ *contains* $(i,j,k,l;\boldsymbol{\alpha}+\boldsymbol{\beta})$.

PROOF. Let θ be the endomorphism of F determined by

$$y_r\theta = y_r + y_r y_n$$

for $r = 1, 2, \cdots$. Then

$$\begin{aligned}(y_1 y_2)\theta &= (y_1 + y_1 y_n)(y_2 + y_2 y_n)\\ &= y_1 y_2 + (y_1 y_n) y_2 + y_1 (y_2 y_n) + (y_1 y_n)(y_2 y_n)\\ &= y_1 y_2 + y_1 y_2 y_n + (y_1 y_n)(y_2 y_n) \text{ by the Jacobi identity}\\ &= y_1 y_2 + y_1 y_2 y_n \text{ modulo } (F^2)^2.\end{aligned}$$

By induction

$$(y_1 y_2 \cdots y_m)\theta = y_1 y_2 \cdots y_m + y_1 y_2 \cdots y_m y_n \text{ modulo } (F^2)^2.$$

Hence

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$$
\begin{aligned}
&((y_1y_2\cdots y_m)(y_{m+1}\,y_{m+2}))\theta \\
&\quad = (y_1y_2\cdots y_m + y_1y_2\cdots y_my_n + g)(y_{m+1}\,y_{m+2} + y_{m+1}\,y_{m+2}y_n + h) \\
&\qquad\qquad\qquad\qquad\text{where } g,\, h\in (F^2)^2 \\
&\quad = (y_1y_2\cdots y_m)(y_{m+1}\,y_{m+2}) \\
&\qquad + (y_1y_2\cdots y_my_n)(y_{m+1}\,y_{m+2}) + (y_1y_2\cdots y_m)(y_{m+1}\,y_{m+2}\,y_n) \\
&\qquad + (y_1y_2\cdots y_my_n)(y_{m+1}\,y_{m+2}\,y_n) \\
&\qquad\qquad\qquad\qquad\text{since } (F^2)^3 = 0 \\
&\quad = (y_1y_2\cdots y_m)(y_{m+1}\,y_{m+2}) \\
&\qquad + (y_1y_2\cdots y_m)(y_{m+1}\,y_{m+2})y_n \\
&\qquad - (y_1y_2\cdots y_my_ny_n)(y_{m+1}\,y_{m+2}) + (y_1y_2\cdots y_my_n)(y_{m+1}\,y_{m+2})y_n \\
&\qquad\qquad\qquad\qquad\text{by the Jacobi identity} \\
&\quad = (y_1y_2\cdots y_m)(y_{m+1}\,y_{m+2}) \\
&\qquad - (y_1y_2\cdots y_my_ny_n)(y_{m+1}\,y_{m+2}) \\
&\qquad\qquad\qquad\qquad\text{since } (F^2)^2F = 0.
\end{aligned}
$$

Hence $(y_1y_2\cdots y_my_ny_n)(y_{m+1}\,y_{m+2})$ is in the fully invariant ideal generated by $(y_1y_2\cdots y_m)\,(y_{m+1}\,y_{m+2})$. Suppose that $n > m+2$ and substitute $y_{n+1}+y_{n+2}$ for y_n. We obtain

$$
\begin{aligned}
&(y_1y_2\cdots y_m\,y_{n+1}\,y_{n+1})(y_{m+1}\,y_{m+2}) \\
&\qquad + (y_1y_2\cdots y_m\,y_{n+2}\,y_{n+2})(y_{m+1}\,y_{m+2}) \\
&\qquad + (y_1y_2\cdots y_m\,y_{n+1}\,y_{n+2})(y_{m+1}\,y_{m+2}) \\
&\qquad + (y_1y_2\cdots y_m\,y_{n+2}\,y_{n+1})(y_{m+1}\,y_{m+2}) \\
&\quad = (y_1y_2\cdots y_m\,y_{n+1}\,y_{n+1})(y_{m+1}\,y_{m+2}) \\
&\qquad + (y_1y_2\cdots y_m\,y_{n+2}\,y_{n+2})(y_{m+1}\,y_{m+2}) \\
&\qquad + 2(y_1y_2\cdots y_m\,y_{n+1}y_{n+2})(y_{m+1}\,y_{m+2}) \\
&\qquad\qquad\text{by the Jacobi identity, since } (F^2)^3 = 0.
\end{aligned}
$$

Now the characteristic of K is not 2, and so the fully invariant ideal of F generated by $(y_1y_2\cdots y_m)(y_{m+1}\,y_{m+2})$ contains $(y_1y_2\cdots y_m\,y_{n+1}\,y_{n+2})(y_{m+1}\,y_{m+2})$. By induction, if $r \equiv 0 \bmod 2$ the fully invariant ideal of F generated by $(y_1y_2\cdots y_m)$ $(y_{m+1}\,y_{m+2})$ contains $(y_1y_2\cdots y_m\,y_{n+1}\cdots y_{n+r})(y_{m+1}\,y_{m+2})$.

Now let

$$(i,j,k,l;\boldsymbol{\alpha}) = (y_i, y_j y_1^{\alpha_1} \cdots y_m^{\alpha_m})(y_k y_l)$$

and let $\sum_{n=1}^{\infty} \beta_n = r$. Then by the above remarks, provided $n > i,j,k,l,m$, the fully invariant ideal generated by $(i,j,k,l;\boldsymbol{\alpha})$ contains

$$(y_i y_j y_1^{\alpha_1} \cdots y_m^{\alpha_m} y_{n+1} y_{n+2} \cdots y_{n+r})(y_k y_l)$$

and so contains

$$(i,j,k,l;\boldsymbol{\alpha}+\boldsymbol{\beta}) = (y_i y_j y_1^{\alpha_1} \cdots y_m^{\alpha_m} y_1^{\beta_1} \cdots y_s^{\beta_s})(y_k y_l)$$

where s is chosen so that $\beta_r = 0$ for $r > s$.

COROLLARY. *If* $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots) \in A$ *and if* $\sum_{n=1}^{\infty} \beta_n \equiv 0$ *mod* 2 *then the fully invariant ideal generated by*

$$g = \sum_{m=1}^{n} \lambda_m (i_m, j_m, k_m, l_m; \boldsymbol{\alpha}_m)$$

contains

$$\sum_{m=1}^{n} \lambda_m (i_m, j_m, k_m, l_m; \boldsymbol{\alpha}_m + \boldsymbol{\beta})$$

PROOF. Apply the proof of Lemma 1 to g.

LEMMA 2. (a) *If* $\phi \in \Phi$ *and if* $s, t \in S$, $s < t$ *then* $s\phi^* < t\phi^*$, *where* ϕ^* *is the endomorphism of* F *given by* $y_r\phi^* = y_{r\phi}$ *for* $r = 1, 2, \cdots$.

(b) *If* $(i,j,k,l;\boldsymbol{\alpha}) < (p,q,r,s;\,\boldsymbol{\beta})$ *then* $(i,j,k,l;\boldsymbol{\alpha}+\gamma) < (p,q,r,s;\,\boldsymbol{\beta}+\gamma)$ *for all* $\gamma \in A$.

The proof of Lemma 2 is straightforward.

LEMMA 3. *If* $g, h \in (F^2)^2$ *and if* $wt\, g \preccurlyeq wt\, h$ *then there is an element* g^* *in the fully invariant ideal of* F *generated by* g *such that* $wt\, g^* = wt\, h$.

PROOF. Let $wt\ g = (i,j,k,l;\boldsymbol{\alpha})$, $wt\ h = (p,q,r,s;\,\boldsymbol{\beta})$ and let $\phi \in \Phi$ satisfy

(1) $i\phi = p$, $j\phi = q$, $k\phi = r$, $l\phi = s$,

(2) $\boldsymbol{\alpha}\phi \leqq \boldsymbol{\beta}$,

(3) $\sum_{n=1}^{\infty} \alpha_n \equiv \sum_{n=1}^{\infty} \beta_n \mod 2$.

Let ϕ^* be the endomorphism of F determined by $y_n\phi^* = y_{n\phi}$ for $n = 1, 2, \cdots$. Then

$$(i,j,k,l;\boldsymbol{\alpha})\phi^* = (p,q,r,s;\boldsymbol{\alpha}\phi)$$

and by Lemma 2 this is $wt(g\phi^*)$.

Since $\sum_{n=1}^{\infty} \alpha_n \equiv \sum_{n=1}^{\infty} \beta_n \mod 2$, by the corollary to Lemma 1, and by Lemma 2, the fully invariant ideal generated by $g\phi^*$ contains an element with weight

$$\begin{aligned}(p,q,r,s;\boldsymbol{\alpha}\phi + (\boldsymbol{\beta}-\boldsymbol{\alpha}\phi)) \\ &= (p,q,r,s;\boldsymbol{\beta}) \\ &= wt\ h.\end{aligned}$$

This completes the proof of Lemma 3.

Let I be an ideal of F contained in $(F^2)^2$. Since the set of weights of elements of $(F^2)^2$ is partially well ordered by $\preccurlyeq$ there is a finite subset $G \subseteq I$ with the property that for each $h \in I$ there is an element $g \in G$ such that $wt\ g \preccurlyeq wt\ h$.

Let $h \in I$ and let $g \in G$, $wt\ g \preccurlyeq wt\ h$. Then by Lemma 3 there is an element g^* of the fully invariant ideal generated by g such that $wt\ g^* = wt\ h$. But then for some $\lambda \in K$ $wt(h+\lambda g^*) < wt\ h$. Since $\leqq$ is a well ordering on S it follows, by induction on $wt\ h$, that h is in the fully invariant ideal generated by G. This completes the proof of the proposition.

With minor modifications this proof gives the following result.

If V is a variety of Lie algebras over a field K, if the characteristic of K is not 2, and if V satisfies the law

$$(x_1x_2)(x_3x_4)x_5x_5\cdots x_n$$

for some n, then V has a finite basis for its laws.

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