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SOME THEOREMS ON STRONG NÖRLUND SUMMABILITY

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1. **Preliminaries.** Throughout this paper H, H_1, etc. will denote positive constants which will not necessarily be the same at different occurrences.

If $\sum_{n=0}^{\infty} a_n$ is a series, we shall use the notation $s_n=\sum_{r=0}^{n} a_r$. For α real, define

$$\epsilon_0^\alpha = 1, \qquad \epsilon_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} \quad (n = 1, 2, \ldots).$$

Let $\{p_n\}$ be a sequence with $p_0>0$ and $p_n \geq 0$ for $n>0$. Define

$$p_n^\alpha = \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} p_r.$$

The following identities are immediate:

$$\sum_{r=0}^{n} \epsilon_{n-r}^{\beta-1} p_r^\alpha = p_n^{\alpha+\beta};$$

$$P_n^\alpha = p_n^{\alpha+1} = \sum_{r=0}^{n} p_r^\alpha;$$

where

$$P_n = \sum_{r=0}^{n} p_r.$$

Definition 1. Nörlund summability (N, p_n^α).

For $\alpha>-1$ and a series $\sum_{n=0}^{\infty} a_n$, let

$$t_n^{(\alpha)} = (1/P_n^\alpha) \sum_{r=0}^{n} p_{n-r}^\alpha s_r. \tag{1.1}$$

If $t_n^{(\alpha)} \to s$ as $n \to \infty$, we write

$$\sum_{n=0}^{\infty} a_n = s(N, p_n^\alpha) \quad \text{or} \quad s_n \to s(N, p_n^\alpha).$$

Definition 2. Strong Nörlund summability $[N, p_n^{\alpha+1}]_\lambda$.

For $\alpha>-1$ and $\lambda>0$, we say that the series $\sum_{n=0}^{\infty} a_n$ is *strongly summable* $(N, p_n^{\alpha+1})$ *with index* λ *to* s if

$$\sum_{r=0}^{n} p_r^{\alpha+1} |t_r^{(\alpha)} - s|^\lambda = o(P_n^{\alpha+1}) \quad \text{as } n \to \infty; \tag{1.2}$$

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and we write

$$\sum_{n=0}^{\infty} a_n = s[N, p_n^{\alpha+1}]_\lambda \quad \text{or} \quad s_n \to s[N, p_n^{\alpha+1}]_\lambda.$$

This definition of strong Nörlund summability was first given by Borwein and Cass [2], where it is investigated in some detail.

If we take $p_0=1$ and $p_n=0$ for $n>0$, then Definition 1 yields the standard definition of Cesàro summability of order α; while Definition 2 yields a definition of strong Cesàro summability of order $\alpha+1$ and index λ. That this definition is equivalent to the standard definition of strong Cesàro summability of order $\alpha+1$ is proved in Borwein and Cass [2, p. 99]. We denote strong Cesàro summability of order $\alpha+1$ and index λ by $[C, \alpha+1]_\lambda$.

2. **Known results and statement of new theorems.** The following three theorems concerning strong Cesàro summability have been established by Flett [5, see Theorems 2 and 3]. See also Borwein [1, Theorems 10, 11, and Corollary (V)].

(I) *If* $\alpha>-1$, $\lambda\geq 1$ *and* $\delta>1/\lambda$, *then* $\sum_{n=0}^{\infty} a_n=s[C, \alpha+1]_\lambda$ *implies that*

$$\sum_{n=0}^{\infty} a_n = s(C, \alpha+\delta).$$

(II) *If* $\alpha>-1$, $\mu>\lambda\geq 1$ *and* $\delta>1/\lambda-1/\mu$, *then* $\sum_{n=0}^{\infty} a_n=s[C, \alpha+1]_\lambda$ *implies that*

$$\sum_{n=0}^{\infty} a_n = s[C, \alpha+\delta+1]_\mu.$$

(III) *If* $\alpha>-1$, $\mu>\lambda>1$ *and* $\delta=1/\lambda-1/\mu$, *then* $\sum_{n=0}^{\infty} a_n=s[C, \alpha+1]_\lambda$ *implies that*

$$\sum_{n=0}^{\infty} a_n = s[C, \alpha+\delta+1]_\mu.$$

The purpose of this paper is to extend these theorems to include certain other families of Nörlund summability methods. We establish three theorems which include as special cases the above theorems concerning Cesàro summability.

The technique employed in the proofs of our theorems requires some restriction on the sequence $\{P_n^\zeta\}$. We shall impose the following condition:

For each $\zeta>-1$ there are positive constants H_1 and H_2 (which may depend on ζ but not on n) such that

$$H_1 n^\zeta \leq P_n^\zeta/P_n \leq H_2 n^\zeta, \tag{2.1}$$

for n sufficiently large. In case of (2.1) we see that for $\zeta>-1$ and $\delta>0$

$$H_1 n^\delta \leq P_n^{\zeta+\delta}/P_n^\zeta \leq H_2 n^\delta, \tag{2.2}$$

for n sufficiently large. We make frequent but tacit use of inequality (2.2) in the proofs of our theorems.

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That (2.1) does not hold in general can be seen by taking $P_n = e^{\sqrt{n}}$; for in this case $P_n^1 \sim 2\sqrt{n}\, e^{\sqrt{n}}$, so $P_n^1/P_n \sim 2\sqrt{n}$. It is satisfied, however, by a reasonably large class of sequences; in fact if the sequence $\{P_n\}$ satisfies $P_{2n} = O(P_n)$ and $p_n/P_n = O(1/n)$, it also satisfies (2.1) for $\zeta > -1$.

In Theorem 3 we impose the further condition "$(C, 1)$ is equivalent to $(\overline{N}, P_n^\alpha)$ for $-1 < \alpha < 0$"(1). In the presence of (2.1), it is a consequence of Hardy [7, Theorem 14] that $(C, 1)$ and $(\overline{N}, P_n^\alpha)$ are equivalent for $\alpha \geq 0$. Under these circumstances

$$\sum_{n=0}^{\infty} a_n = s[N, p_n^{\alpha+1}]_\lambda$$

if and only if

$$\sum_{n=0}^{m} |t_r^{(\alpha)} - s|^\lambda = o(m). \tag{2.3}$$

This casts the condition of strong Nörlund summability into a form receptive to an application of the deep but special inequality of Hardy Littlewood and Pólya which we state as our Lemma 2 below. In view of the standard definition of strong Cesàro summability (see for example Borwein [1]), it may appear that (2.3) would give an appropriate definition of strong Nörlund summability; however an examination of Borwein and Cass ([2], [3]), particularly of Theorems 1 and 6 in [2], shows that using the weighted mean in place of the arithmetic mean yields a much more satisfactory theory.

The statements of our theorems follow.

Suppose throughout that $\{p_n\}$ is a sequence with $p_0 > 0$ and $p_n \geq 0$ for $n > 0$.

THEOREM 1. *If $\alpha > -1$, $\lambda \geq 1$, $\delta > 1/\lambda$ and (2.1) holds for $\zeta > -1$; then $\sum_{n=0}^{\infty} a_n = s[N, p_n^{\alpha+1}]_\lambda$ implies that $\sum_{n=0}^{\infty} a_n = s(N, p_n^{\alpha+\delta})$.*

THEOREM 2. *If $\alpha > -1$, $\mu > \lambda \geq 1$, $\delta > 1/\lambda - 1/\mu$ and (2.1) holds for $\zeta > -1$; then $\sum_{n=0}^{\infty} a_n = s[N, p_n^{\alpha+1}]_\lambda$ implies that $\sum_{n=0}^{\infty} a_n = s[N, p_n^{\alpha+\delta+1}]_\mu$.*

THEOREM 3. *If $\mu > \lambda > 1$, $\delta = 1/\lambda - 1/\mu$, (2.1) holds for $\zeta > -1$ and either $\alpha \geq 0$, or $-1 < \alpha < 0$ and $(C, 1)$ is equivalent to $(\overline{N}, P_n^\alpha)$; then $\sum_{n=0}^{\infty} a_n = s[N, p_n^{\alpha+1}]_\lambda$ implies that $\sum_{n=0}^{\infty} a_n = s[N, p_n^{\alpha+\delta+1}]_\mu$.*

3. **Proofs of theorems.**

Proof of Theorem 1. If $\delta \geq 1$ or $\lambda = 1$, the result follows without (2.1) from Theorems 11 and 13 of [2]. We now suppose that $0 < \delta < 1$ and note that it is sufficient to prove our theorem for the case $s = 0$. We are given

$$\sum_{\nu=0}^{n} P_\nu^\alpha |t_\nu^{(\alpha)}|^\lambda = o(P_n^{\alpha+1}) \quad \text{as } n \to \infty$$

(1) Two methods of summability are equivalent if every series summable by the one method is summable to the same sum by the other method. For the definition of the method $(\overline{N}, p_n)$ see Hardy [7, p. 57]. Necessary and sufficient conditions for inclusion or equivalence between $(\overline{N}, p_n)$ means are given by Garabedian and Randels [6].

and are required to show that $t_n^{(\alpha+\delta)}=o(1)$ as $n\to\infty$.

Now

$$|t_n^{(\alpha+\delta)}| \le \{1/P_n^{\alpha+\delta}\} \sum_{\nu=0}^{n} \epsilon_{n-\nu}^{\delta-1} P_\nu^\alpha \, |t_\nu^{(\alpha)}|.$$

Applying Hölder's inequality, we find that

$$|t_n^{(\alpha+\delta)}| \le \{1/P_n^{\alpha+\delta}\}\left\{\sum_{\nu=0}^{n} P_\nu^\alpha \, |t_\nu^{(\alpha)}|^\lambda\right\}^{1/\lambda}\left\{\sum_{\nu=0}^{n} (\epsilon_{n-\nu}^{\delta-1})^{\lambda'} P_\nu^\alpha\right\}^{1/\lambda'}$$

where $1/\lambda+1/\lambda'=1$. Now $\delta>1/\lambda$ yields $(\delta-1)\lambda'>-1$, giving

$$(\epsilon_{n-\nu}^{\delta-1})^{\lambda'} \le H\epsilon_{n-\nu}^{\lambda'(\delta-1)},$$

so

$$|t_n^{(\alpha+\delta)}| = o\left(\{1/P_n^{\alpha+\delta}\}\{P_n^{\alpha+1}\}^{1/\lambda}\left\{\sum_{\nu=0}^{n} \epsilon_{n-\nu}^{\lambda'(\delta-1)} P_\nu^\alpha\right\}^{1/\lambda'}\right);$$

and finally applying (2.1) we find that

$$t_n^{(\alpha+\delta)} = o(1) \quad \text{as } n\to\infty, \text{ as required.}$$

Proof of theorem 2. If $\delta\ge 1$, the result follows without (2.1) from Theorems 12, 13 and 15 in [2]. We suppose now that $0<\delta<1$ and note that it is sufficient to prove our theorem for the case $s=0$. We are given

$$\sum_{\nu=0}^{n} P_\nu^\alpha \, |t_\nu^{(\alpha)}|^\lambda = o(P_n^{\alpha+1}) \quad \text{as } n\to\infty,$$

and we must prove that

$$\sum_{\nu=0}^{n} P_\nu^{\alpha+\delta} \, |t_\nu^{(\alpha+\delta)}|^\mu = o(P_n^{\alpha+\delta+1}) \quad \text{as } n\to\infty.$$

Now

$$\{P_n^{\alpha+\delta} \, |t_n^{(\alpha+\delta)}|\} \le \left\{\sum_{\nu=0}^{n} \epsilon_{n-\nu}^{\delta-1} P_\nu^\alpha \, |t_\nu^{(\alpha)}|\right\}$$

$$= \left\{\sum_{\nu=0}^{n} (\epsilon_{n-\nu}^{\delta-1})^{p/\mu}(\epsilon_{n-\nu}^{\delta-1})^{p(1-1/\lambda)}(P_\nu^\alpha)^{1/\lambda}(P_\nu^\alpha)^{1-1/\lambda}|t_\nu^{(\alpha)}|\right\}$$

where $1/p=1+1/\mu-1/\lambda$. Applying Hölder's inequality in the indicated manner with index λ we obtain

$$\{P_n^{\alpha+\delta}|t_n^{(\alpha+\delta)}|\}^\lambda \le \left\{\sum_{\nu=0}^{n} (\epsilon_{n-\nu}^{\delta-1})^{p(\lambda/\mu)} P_\nu^\alpha \, |t_\nu^{(\alpha)}|^\lambda\right\}\left\{\sum_{\nu=0}^{n} (\epsilon_{n-\nu}^{\delta-1})^p P_\nu^\alpha\right\}^{\lambda-1}$$

$$\le H\left\{\sum_{\nu=0}^{n} (\epsilon_{n-\nu}^{\delta-1})^{p(\lambda/\mu)} P_\nu^\alpha \, |t_\nu^{(\alpha)}|^\lambda\right\}\{P_n^{\alpha+p(\delta-1)+1}\}^{\lambda-1},$$

since $\delta>1/\lambda-1/\mu$, which is the same as $p(\delta-1)>-1$, giving

$$(\epsilon_{n-\nu}^{\delta-1})^p \le H\epsilon_{n-\nu}^{p(\delta-1)}.$$

Therefore

$$\{P_n^{\alpha+\delta}\,|t_n^{(\alpha+\delta)}|\}^\mu \le H\left\{\sum_{\nu=0}^{n} (\epsilon_{n-\nu}^{\delta-1})^{p(\lambda/\mu)} P_\nu^\alpha\,|t_\nu^{(\alpha)}|^\lambda\right\}^{(\mu/\lambda)} \{P_n^{\alpha+p(\delta-1)+1}\}^{\mu-\mu/\lambda}.$$

Now applying Hölder's inequality with index μ/λ we find

$$\{P_n^{\alpha+\delta}\,|t_n^{(\alpha+\delta)}|\}^\mu \le H\left\{\sum_{\nu=0}^{n} (\epsilon_{n-\nu}^{\delta-1})^p P_\nu^\alpha |t_\nu^{(\alpha)}|^\lambda\right\}$$
$$\times\left\{\sum_{\nu=0}^{n} P_\nu^\alpha |t_\nu^{(\alpha)}|^\lambda\right\}^{\mu/\lambda-1} \{P_n^{\alpha+p(\delta-1)+1}\}^{\mu-\mu/\lambda}.$$

Now using (2.1) and the fact that $s_n \to 0[N, p_n^{\alpha+1}]_\lambda$ we see

$$P_n^{\alpha+\delta}\,|t_n^{(\alpha+\delta)}|^\mu = o\left(\left\{\sum_{\nu=0}^{n} (\epsilon_{n-\nu}^{\delta-1})^p P_\nu^\alpha\,|t_\nu^{(\alpha)}|^\lambda\right\} n^{(1-\delta)(p-1)}\right) = o(\phi_n) \text{ say.}$$

From this order relation and the Toeplitz theorem (see, for example, Hardy [7, Theorem 2]) applied to the matrix $[c_{m,n}]$ with

$$c_{m,n} = \phi_n \Big/ \sum_{r=0}^{m} \phi_r \text{[2]} \quad \text{for } 0 \le n \le m$$

and

$$c_{m,n} = 0 \qquad \text{for } n > m;$$

then using the fact that $\{n^{(1-\delta)(p-1)}\}$ is monotonic nondecreasing we obtain

$$\sum_{n=0}^{m} P_n^{\alpha+\delta}\,|t_n^{(\alpha+\delta)}|^\mu = o\left(m^{(1-\delta)(p-1)} \sum_{n=0}^{m}\sum_{\nu=0}^{n} P_\nu^\alpha\,|t_\nu^{(\alpha)}|^\lambda \epsilon_{n-\nu}^{(\delta-1)p}\right)$$
$$= o\left(m^{(1-\delta)(p-1)} \sum_{\nu=0}^{m} P_\nu^\alpha\,|t_\nu^{(\alpha)}|^\lambda \sum_{n=\nu}^{m} \epsilon_{n-\nu}^{p(\delta-1)}\right)$$
$$= o(m^{(1-\delta)(p-1)+p(\delta-1)+1} P_m^{\alpha+1}),$$

since $p(\delta-1)+1>0$. Thus

$$(1/P_m^{\alpha+\delta+1}) \sum_{n=0}^{m} P_n^{\alpha+\delta}\,|t_n^{(\alpha+\delta)}|^\mu = o(m^{(1-\delta)(p-1)+p(\delta-1)+1-\delta}) = o(1)$$

as required.

Before proving Theorem 3, we state two lemmas.

Lemma 1. *Let* $0 \le \mu < 1$, *let* $s_n \ge 0$ *for all* n, $t_n = (n+1)^{-1}\sum_{\nu=0}^{n} s_\nu$ *and* $t = \sup t_n$, *then*

$$\sum_{\nu=0}^{n} (\nu+1)^{-\mu} s_\nu \le H(n+1)^{1-\mu} t.$$

If further, $t_n = o(1)$, *then*

$$\sum_{\nu=0}^{n} (\nu+1)^{-\mu} s_\nu = o(n^{1-\mu}) \quad \text{as } n \to \infty.$$

This follows easily by partial summation.

(2) Note that if ϕ_n is identically zero, then so is $t_n^{(\alpha+\delta)}$ so there is nothing to prove.

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The next lemma is due to Hardy, Littlewood, and Pólya. See [9], [8, Theorem 4], [10, Theorems 381, 382, and 383].

LEMMA 2. *If* $1<\lambda<\mu<\infty$, $\delta=1/\lambda-1/\mu$, $c_n\geq 0$, $C_n=\sum_{\nu<n}(n-\nu)^{\delta-1}c_\nu$, *then*

$$\{\sum C_n^\mu\}^{1/\mu} \leq H\{\sum c_n^\lambda\}^{1/\lambda}.$$

Proof of Theorem 3. We may suppose as before that $0<\delta<1$ and that $s=0$. Since, when $\alpha\geq 0$ it follows from Hardy [7, Theorem 14] that $(C, 1)$ and $(\bar{N}, P_n^\alpha)$ are equivalent, it is sufficient to show that if

$$\sum_{n=0}^{m} |t_n^{(\alpha)}|^\lambda = o(m),$$

then

$$\sum_{n=0}^{m} |t_n^{(\alpha+\delta)}|^\mu = o(m).$$

Now

$$\begin{aligned} |t_n^{(\alpha+\delta)}| &\leq (1/P_n^{\alpha+\delta}) \sum_{\nu=0}^{n} \epsilon_{n-\nu}^{\delta-1} P_\nu^\alpha\, |t_\nu^{(\alpha)}| \\ &\leq (1/P_n^{\alpha+\delta}) \sum_{0\leq\nu\leq n/2} \epsilon_{n-\nu}^{\delta-1} P_\nu^\alpha\, |t_\nu^{(\alpha)}| \\ &\quad +(1/P_n^{\alpha+\delta}) \sum_{n/2\leq\nu<n} \epsilon_{n-\nu}^{\delta-1} P_\nu^\alpha\, |t_\nu^{(\alpha)}| + P_n^\alpha\, |t_n^{(\alpha)}|/P_n^{\alpha+\delta} \\ &= Q_n+R_n+S_n \text{ say.} \end{aligned}$$

Thus, applying Minkowski's inequality, to complete the proof we have to show that

$$\left\{\sum_{n=0}^{m} W_n^\mu\right\}^{1/\mu} = o(m^{1/\mu})$$

in each of the cases where W_n is replaced by Q_n, R_n, or S_n.

$$\begin{aligned} Q_n &\leq (H/\epsilon_n^{\alpha+\delta} P_n) \sum_{0\leq\nu\leq n/2} \epsilon_{n-\nu}^{\delta-1} P_\nu\, |t_\nu^{(\alpha)}|\, \epsilon_\nu^\alpha \\ &\leq \frac{H\epsilon_{n/2}^{\delta-1}}{\epsilon_n^{\alpha+\delta}} \sum_{0\leq\nu\leq n/2} \epsilon_\nu^\alpha\, |t_\nu^{(\alpha)}| \leq \frac{H}{\epsilon_n^{\alpha+1}} \sum_{\nu=0}^{n} \epsilon_\nu^\alpha\, |t_\nu^{(\alpha)}| \\ &\leq H_1\left\{(1/\epsilon_n^{\alpha+1}) \sum_{\nu=0}^{n} \epsilon_\nu^\alpha\, |t_\nu^{(\alpha)}|^\lambda\right\}^{1/\lambda} \end{aligned}$$

by Hölder's inequality. So by Hardy [7, Theorem 14], $Q_n=o(1)$. Thus

$$\left\{\sum_{n=0}^{m} Q_n^\mu\right\}^{1/\mu} = o(m^{1/\mu}).$$

Now

$$\begin{aligned} R_n &\leq (H/\epsilon_n^{\alpha+\delta} P_n) \sum_{n/2\leq\nu<n} \epsilon_{n-\nu}^{\delta-1} \epsilon_\nu^\alpha P_\nu\, |t_\nu^{(\alpha)}| \\ &\leq H \sum_{0\leq\nu<n} (n-\nu)^{\delta-1}(\nu+1)^{-\delta}\, |t_\nu^{(\alpha)}|. \end{aligned}$$

Now let

$$c_n = (n+1)^{-\delta}\,|t_n^{(\alpha)}| \quad \text{for } n \le m$$

and

$$c_n = 0 \qquad \text{for } n > m;$$

also let

$$C_n = \sum_{0 \le \nu < n} (n-\nu)^{\delta-1} c_\nu .$$

Then, by Lemma 2,

$$\left\{\sum_{n=0}^{m} R_n^{\mu}\right\}^{1/\mu} \le H\left\{\sum_{n=0}^{m} C_n^{\mu}\right\}^{1/\mu} \le H\left\{\sum_{n=0}^{m} c_n^{\lambda}\right\}^{1/\lambda}$$
$$= H\left\{\sum_{n=0}^{m} (n+1)^{-\lambda\delta}\,|t_n^{(\alpha)}|^{\lambda}\right\}^{1/\lambda}.$$

Since $\lambda\delta < 1$ we may apply Lemma 1 to this last term to find that it is

$$o(m^{(1-\lambda\delta)(1/\lambda)}) = o(m^{1/\mu}) \quad \text{as } m \to \infty.$$

Finally $S_n \le H(n+1)^{-\delta}\,|t_n^{(\alpha)}|$, so by Jensen's inequality (see for example Wilansky [12, p. 7]),

$$\left\{\sum_{n=0}^{m} S_n^{\mu}\right\}^{1/\mu} \le H\left\{\sum_{n=0}^{m} (n+1)^{-\mu\delta}\,|t_n^{(\alpha)}|^{\mu}\right\}^{1/\mu} \le H\left\{\sum_{n=0}^{m} (n+1)^{-\lambda\delta}\,|t_n^{(\alpha)}|^{\lambda}\right\}^{1/\lambda}$$

and again applying Lemma 1 we obtain that the final term is $o(m^{1/\mu})$. The desired conclusion now follows.

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