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# MODULAR FORMS OF DEGREE $n$ AND REPRESENTATION BY QUADRATIC FORMS V

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Let $A$, $B$ be integral symmetric positive definite matrices of degree $m$ and two, respectively, and suppose that $A[X] = B$ is soluble over $Z_p$ for every prime $p$. If $m \geq 7$ and $\min B = \min B[x]$ $(Z^2 \ni x \neq 0)$ is sufficiently large, then $A[X] = B$ is soluble over $Z$. We gave a conditional result for $m = 6$ in [7] under an assumption on the estimate from above of a kind of generalized Weyl sums. Here we give an unconditional result for special sequences of $B$.

THEOREM. *Let $A$ (resp. $B$) be an even integral positive definite symmetric matrix of degree six (resp. two). For any sufficiently large integer $m$ which is prime to* $\det A$, *$A[X] (= {}^tXAX) = mB$ has an integral solution $X \in M_{6,2}(Z)$ if $A[X] = mB$ has an integral solution over $Z_p$ for every prime $p$ dividing* $\det A$.

As usual $Z$, $Z_p$ denote the ring of rational integers, and $p$-adic integers, respectively. A symmetric matrix is called even integral if all entries are integral and moreover diagonals are even.

## § 1.

Let $A$ be an even integral positive definite matrix of degree six. We denote by $q$ the level of $A$, that is, $q$ is the minimum positive integer such that $qA^{-1}$ is even. $q$ and $\det A$ have the same prime divisors. We consider the theta function

$$\theta(Z, A) = \sum_{G \in M_{6,2}(Z)} \exp(\pi i \operatorname{tr} A[G]Z),$$

where $Z$ is a symmetric complex matrix of degree two with positive definite imaginary part. It is clear that

$$\theta(Z, A) = \sum r(T, A) \exp(\pi i \operatorname{tr} TZ)$$

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for

$$r(T, A) = \sharp\{G \in M_{6,2}(Z) \mid A[G] = T\},$$

and it is a modular form of degree two, weight three and level $q$ ([3]). Decompose $\theta(Z, A)$ as $\theta(Z, A) = E(Z) + g(Z)$ where $E(Z)$ is the Siegel's weighted sum of theta functions for quadratic forms in the genus of $A$, that is, $E(Z) = (\sum \sharp O(A_i)^{-1})^{-1} \sum \sharp O(A_i)^{-1}\theta(Z, A_i)$ where $\{A_i\}$ is a complete set of representatives of isometry classes in the genus of $A$ and $\sharp O(A_i)$ is the order of the unit group of $A_i$, and the constant term of the Fourier expansion of $g(Z)$ vanishes at every cusp. Put

$$E(Z) = \sum a(T) \exp(\pi i \operatorname{tr} TZ),$$
$$g(Z) = \sum b(T) \exp(\pi i \operatorname{tr} TZ).$$

Then from [9], we know that for $T > 0$,

$$a(T) = \pi^{11/2}\Gamma(3)^{-1}\Gamma(5/2)^{-1}|A|^{-1}|T|^{3/2} \prod_p \alpha_p(A, T),$$

where $\Gamma$ is the gamma function and $\alpha_p$ is the so-called local density. We fix even positive definite integral matrix $T_0$. Here we note that if $q$ is odd, then $A$ represents primitively every regular even binary symmetric matrix $T$ over $Z_2$ (c.f. the proof of Lemma 6 in [6]) and hence $\alpha_2(A, T) > c_0$ for some positive $c_0$ independent of $T$. Therefore Remark on p.p. 142~143 in [6] implies that $a(mT_0) > cm^{3-\varepsilon}$ for any $\varepsilon > 0$ if $a(mT_0) \neq 0$ and $m$ is prime to $q$, where $c$ is a positive constant independent of $m$. It is easy to see that the condition $a(mT_0) \neq 0$ is valid if and only if $A[X] = mT_0$ is soluble over $Z_p$ for every prime divisor of $q$. It remains to show that $b(mT_0) = O(m^{3-\kappa})$ for some $\kappa > 0$ if $(m, q) = 1$, which is obviously sharper than the theorem.

## § 2.

We recall results of Evdokimov which we need here. Let

$$\Gamma_2 = \{M \in M_4(Z) \mid {}^tMJM = J\} \qquad \text{for } J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

with the identity matrix $1_2$ of degree 2 and $\Gamma_2(q)$ be a principal congruence subgroup $\{M \in \Gamma_2 \mid M \equiv 1_4 \bmod q\}$ for a natural number $q$. A holomorphic function $F$ on the Siegel upper half-plane of genus 2, which is the set of all complex symmetric binary matrices with positive definite imaginary part, is called a modular form of weight 3 and level $q$ if

$$(F|M)(Z) := |CZ + D|^{-3}F(M\langle Z\rangle) = F(Z)$$

for every matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2(q)$, where $M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$. $\mathfrak{M}_3^2(q)$ denotes the set of all modular forms of weight 3 and level $q$. Let $\xi$, $\psi$ be Dirichlet characters modulo $q$. We denote by $\mathfrak{M}_3^2(q; \xi, \psi)$ the set of modular forms $F \in \mathfrak{M}_3^2(q)$ which satisfies

$$(F|\gamma(m, n))(Z) = \xi(m)\psi(n)F(Z) \qquad \text{for integers } m, n \text{ prime to } q$$

where $\gamma(m, n)$ is any element of $\Gamma_2$ such that $\gamma(m, n) \equiv \begin{pmatrix} m^{-1} & & & \\ & n^{-1} & & \\ & & m & \\ & & & n \end{pmatrix} \bmod q$.
Then from [5] we know that $g(Z)$ in Section 1, for which the constant term of the Fourier expansion of $(g|N)(Z)$ vanishes for every $N \in \Gamma_2$, is a linear combination of modular forms $F$ such that $F$ is in $\mathfrak{M}_3^2(q; \xi, \psi)$, is a common eigen function of so called Hecke operators and satisfies $\Phi^2(F|N) = 0$ for every $N \in \Gamma_2$ for the Siegel's $\Phi$ operator. We have only to show that for such a modular form $F(Z) = \sum c(T) \exp(2\pi i q^{-1} \operatorname{tr} TZ)$, $c(mT_0) = O(m^{3-\kappa})$ for some $\kappa > 0$ is valid if $(m, q) = 1$ with $T_0$ fixed $(\det T_0 \neq 0)$.

First suppose that $F(Z)$ is a cusp form, that is, $\Phi(F|N) = 0$ for every $N \in \Gamma_2$. Then it is shown in [8] that $c(T) = O(|T|^{3/2-\kappa/2})$ is valid for some $\kappa > 0$ if $\det T \neq 0$.

Hence the case that $F$ is not a cusp form remains and it is treated in the next section.

## § 3.

Let $F = \sum c(T) \exp(2\pi i q^{-1} \operatorname{tr} TZ)$ be an element in $\mathfrak{M}_3^2(q; \xi, \psi)$ and suppose that $F$ is an eigen function of all Hecke operators,

$$\begin{aligned} &F|T(m) = \lambda(m)F \quad \text{for } (m, q) = 1, \quad \text{and} \\ &\Phi^2(F|M) = 0 \quad \text{for every } M \in \Gamma_2 \quad \text{but} \\ &\Phi(F|N) \neq 0 \quad \text{for some } N \in \Gamma_2 . \end{aligned}$$

LEMMA 1. *Let $m$ be a natural number relatively prime to $q$. Suppose that $M \in M_4(Z)$ satisfies $M \equiv \begin{pmatrix} 1_2 & \\ & m1_2 \end{pmatrix} \bmod q$, ${}^tMJM = mJ$ for $J$ defined above. Denoting by $K^{(m)}$ any element of $\Gamma_2$ which satisfies*

$$K^{(m)} \equiv \begin{pmatrix} 1_2 & \\ & m1_2 \end{pmatrix}^{-1} K \begin{pmatrix} 1_2 & \\ & m1_2 \end{pmatrix} \bmod q \qquad \textit{for } K \in \Gamma_2 ,$$

*we have*

$$K\Gamma_2(q)M\Gamma_2(q) = \Gamma_2(q)M\Gamma_2(q)K^{(m)}.$$

*Proof.* Let $\{M_i\}$ be representatives of left cosets of $\Gamma_2(q)M\Gamma_2(q)$ by $\Gamma_2(q)$, i.e. $\Gamma_2(q)M\Gamma_2(q) = \cup \Gamma_2(q)M_i$. Since $\Gamma_2 M\Gamma_2 = \cup \Gamma_2 M_i$ is disjoint, $M_iK^{(m)} = g_iM_{s(i)}$ for some $g_i \in \Gamma_2$ and some permutation $s$. Considering both sides modulo $q$, we have $K \equiv g_i \bmod q$, and hence $g_iK^{-1} \in \Gamma_2(q)$. Thus we have $\Gamma_2(q)M\Gamma_2(q)K^{(m)} = \cup \Gamma_2(q)M_iK^{(m)} = \cup \Gamma_2(q)g_iM_{s(i)} = \cup \Gamma_2(q)(g_iK^{-1})KM_{s(i)} = \cup K\Gamma_2(q)M_{s(i)} = K\Gamma_2(q)M\Gamma_2(q)$.

LEMMA 2. *For Dirichlet characters $\chi_1, \chi_2, \chi_3$ modulo $q$, we put $F(Z;\chi_1, \chi_2, \chi_3) = \sum \chi_1(m_1)\chi_2(m_2)\chi_3(m_3)F|N^{(m_1)}\gamma(m_2, m_3)$ where $m_i$ $(i = 1, 2, 3)$ runs over a complete set of residue classes prime to $q$ and $\gamma(m_2, m_3)$ is an element in $\Gamma_2$ defined above. Then there are characters $\chi_i$ $(i = 1, 2, 3)$ which satisfy that*

1) $\Phi F(Z;\chi_1, \chi_2, \chi_3) \neq 0$
2) $F(Z;\chi_1, \chi_2, \chi_3)|T(m) = \lambda(m)\overline{\chi_1(m)}F(Z;\chi_1, \chi_2, \chi_3)$ *if* $(m, q) = 1$
3) $F(Z;\chi_1, \chi_2, \chi_3) \in \mathfrak{M}_3^2(q;\bar{\chi}_2, \bar{\chi}_3,)$.

*Proof.* By virtue of the previous lemma, we have, for $m$ prime to $q$,

$$\begin{aligned} F(Z;\chi_1, \chi_2, \chi_3)|T(m) &= \sum \chi_1(m_1)\chi_2(m_2)\chi_3(m_3)F|T(m)(N^{(m_1)}\gamma(m_2, m_3))^{(m)} \\ &= \lambda(m)\sum \chi_1(m_1)\chi_2(m_2)\chi_3(m_3)F|N^{(mm_1)}\gamma(m_2, m_3) \\ &= \lambda(m)\overline{\chi_1(m)}F(Z;, \chi_1, \chi_2, \chi_3). \end{aligned}$$

The last assertion is obvious. Suppose that $\Phi F(Z;\chi_1, \chi_2, \chi_3) = 0$ for all $\chi_i$; then we have

$$\varphi(q)^3\Phi(F|N) = \Phi(\sum_{\chi_i} F(Z;\chi_1, \chi_2, \chi_3)) = 0.$$

This contradicts $\Phi(F|N) \neq 0$.

LEMMA 3. $\lambda(m) = O(m^{5/2+\varepsilon})$ *for any* $\varepsilon > 0$.

*Proof.* By the previous lemma, we may assume $\Phi F \neq 0$; then $f = \Phi F$ is a non-zero cusp form of weight 3 and level $q$ on the classical upper half-plane and is an eigen function of all Hecke operators $T(m)$, $(m, q) = 1$([5]). Put $f|T(m) = t(m)f$. By easy calculations we have

$$\begin{aligned} &\lambda(p)f = \Phi(F|T(p)) = (1 + \psi(p)p)t(p)f, \\ &\lambda(p^2)f = \Phi(F|T(p^2)) = \{(1 + \psi(p)p + \psi(p^2)p^2)t(p^2) + \xi\psi(p)p^3(1 - p^{-1})\}f. \end{aligned}$$

$f \neq 0$ implies

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$$\lambda(p) = (1 + \psi(p)p)t(p)\,,$$
$$\lambda(p^2) = (1 + \psi(p)p + \psi(p^2)p^2)t(p^2) + \xi\psi(p)p^3(1 - p^{-1})$$

and

$$\begin{aligned} Q_{p,F}(x) := {} & 1 - \lambda(p)x + (\lambda(p)^2 - \lambda(p^2) - \xi\psi(p)p^2)x^2 - \xi\psi(p)p^3\lambda(p)x^3 \\ & + \xi\psi(p^2)p^6x^4 \\ = {} & (1 - t(p)x + \xi(p)p^2x^2)(1 - t(p)\psi(p)px + \xi(p)p^2(\psi(p)px)^2)\,, \end{aligned}$$

noting

$$t(p)^2 = t(p^2) + p^2\xi(p)\,.$$

It follows from Section 7 in [4] that

$$\begin{aligned} \sum_{(m,q)=1} \lambda(m)m^{-s} &= L(2s - 2, \xi\psi)^{-1} \prod_{(p,q)=1} Q_{p,F}(p^{-s})^{-1} \\ &= \Big(\sum_{m\geq 1} \frac{\xi\psi(m)\mu(m)}{m^{2s-2}}\Big)\Big(\sum_{(m,q)=1} \frac{t(m)}{m^s}\Big)\Big(\sum_{(m,q)=1} \frac{t(m)\psi(m)}{m^{s-1}}\Big), \end{aligned}$$

where $\mu$ is the möbius function.

Thus we have, for $m$ relatively prime to $q$

$$\begin{aligned} \lambda(m) &= \sum_{m_1^2m_2m_3=m} \xi\psi(m_1)\mu(m_1)m_1^2t(m_2)t(m_3)\psi(m_3)m_3 \\ &= O(\sum_{m_1^2m_2m_3=m} m_1^2m_2^{3/2}m_3^{5/2}) \\ &= m^{5/2}O(\sum_{m_1^2m_2m_3=m} m_1^{-3}m_2^{-1}) \\ &= m^{5/2}O(\sum_{m_2|m} m_2^{-1}) \\ &= O(m^{5/2+\varepsilon}) \qquad \text{for any } \varepsilon > 0\,. \end{aligned}$$

PROPOSITION. *For fixed* $T_0 > 0$, *we have* $c(mT_0) = O(m^{5/2+\varepsilon})$ *for any* $\varepsilon > 0$ *if* $m$ *is relatively prime to* $q$.

*Proof.* For our modular form $F$ (which does not necessarily satisfy $\Phi F \neq 0$), $L(2s - 2, \xi\psi) \sum_{(m,q)=1} \lambda(m)m^{-s} = \prod_{(p,q)=1} Q_{p,F}(p^{-s})^{-1}$ holds (§7 in [4]). By Theorem 7.1 and Remark 1 in [4] we know that

$$\sum_{(m,q)=1} c(mT_0)m^{-s}$$

is a finite sum of products of finite Dirichlet series and

$$\prod_{(p,q)=1} Q_{p,F}(p^{-s})^{-1}L_{df^2}(s - 1, X)^{-1}\,,$$

where $X$ is a character of ideal classes with respect to the order $o_f$ of conducter $f$ in the quadratic field $Q(\sqrt{-|T_0|})$, $(df^2 = -4|T_0|)$, and $L_{df^2}(s, X) =$

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$\prod_{\mathfrak{P}}(1 - X(\mathfrak{P})N(\mathfrak{P})^{-s})^{-1}$ where $\mathfrak{P}$ runs through all regular prime ideals of $o_f$ whose norm $N(\mathfrak{P})$ is prime to $q$. We note that the above Remark 1 without proof is based on the fact that the left hand of (7.3) there is multiplied by $X(b)\bar{\chi}(b)$, taking $\mathfrak{U}b$ as representatives instead of $\mathfrak{U}$ for $b$ prime to $q$. Putting

$$\begin{aligned}\sum_{(m,q)=1} \lambda'(m)m^{-s} &:= \prod_{(p,q)=1} Q_{p,F}(p^{-s})^{-1}L_{df^2}(s-1, X)^{-1} \\ &= \sum_{(m,q)=1} \lambda(m)m^{-s}L(2s-2, \xi\psi)L_{df^2}(s-1, X)^{-1},\end{aligned}$$

Lemma 3 implies $\lambda'(m) = O(m^{5/2+\varepsilon})$ for any positive $\varepsilon$.

Thus we have proved the proposition and hence the theorem.

*Remark.* It is easy to see that the same estimate from below of the expected main term as in the text holds if either $mB$ is primitively represented by $A$ over $Z_p$ at every prime $p$ dividing $\det A$, or $p^{t_p}$ does not divide $m$ with $t_p$ arbitrarily fixed at each prime $p$ where the Witt index of the $p$-adic completion of a quadratic space corresponding to $A$ is less than 2 (this implies $p|\det A$). Evolving the Hecke theory at primes dividing the level, we may get the same estimate of the error term and improve our theorem. If the relations due to Andrianov-Evdokimov between Fourier coefficients and eigenvalues of Hecke operators are also valid for primitive Fourier coefficients and eigenvalues, then our theorem is valid for primitive representations. Here the primitive Fourier coefficients $a^*(T)$ is defined from Fourier coefficients $a(T)$ as follows, following Böcherer-Ragahvan: On Fourier coefficients of Siegel modular forms:

$$a(T) = \sum_{G \in GL_2(\boldsymbol{Z}) \backslash M_2(\boldsymbol{Z})} a^*(T[G^{-1}]),$$

where $\det G \neq 0$ and $T[G^{-1}]$ is half-integral. (c.f., Theorem 1.7.2 on p. 162 in [7]).

*Remark.* When we consider a similar representation problem of binary quadratic forms $T$ by a quinary positive definite quadratic form, the essential part of the expected main term is $\det T$ apart from the infinite product of local densities. Hence we need the estimate $a(T) = O(|T|^{1-\varepsilon})$ for some $\varepsilon > 0$ for Fourier coefficients $a(T)$ of cusp forms of weight 5/2 and degree 2.

Let $M(\simeq A_4 \perp \langle 4 \rangle)$ be an even quinary decomposable positive definite quadratic form of det $= 20$. The class number of gen $M$ is two and another form is given by

$$2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + 3x_5^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_3x_5)\,.$$

Using a computer, we know that all even binary quadratic form of $\det \leq 1000$ except $4(x^2 + xy + y^2)$ is represented by $M$. Does this suggest that with respect to the representation problem of binary forms whose minimum is sufficiently large, by this quinary form we can expect an existence of a solution or more strongly the asymptotic formula of solutions? Are there any examples like $M$, that is, is there, for $2 \leq n \leq m$ a positive definite quadratic form $M$ of rank $= m$ such that any positive definite quadratic form of rank $= n$ represented by gen $M$ is represented by $M$?

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