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Classification of subgroups isomorphic to $\mathrm{PSL}_2(27)$ in the Monster

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Abstract

As a contribution to an eventual solution of the problem of the determination of the maximal subgroups of the Monster we prove that the Monster does not contain any subgroup isomorphic to $\mathrm{PSL}_2(27)$.

Supplementary materials are available with this article.

1. *Introduction*

The Monster is the largest of the 26 sporadic simple groups. The maximal subgroups of the other 25 are all known, so it would be satisfying to complete this project also for the Monster. The problem of determining the maximal subgroups of the Monster has a long history (see for example [**3**, **5**, **7**, **8**, **10**–**14**]). The cases left open by previous published work are normalizers of simple subgroups with trivial centralizer, and isomorphic to one of the groups

$$\mathrm{PSL}_2(8), \mathrm{PSL}_2(13), \mathrm{PSL}_2(16), \mathrm{PSL}_2(27), \mathrm{PSU}_3(4), \mathrm{PSU}_3(8), \mathrm{Sz}(8).$$

Of these, $\mathrm{PSL}_2(8)$ and $\mathrm{PSL}_2(16)$ have apparently been classified in unpublished work of Holmes. The cases $\mathrm{PSL}_2(27)$ and $\mathrm{Sz}(8)$ are particularly interesting because no subgroup isomorphic to $\mathrm{PSL}_2(27)$ or $\mathrm{Sz}(8)$ is known. Here we consider the case $\mathrm{PSL}_2(27)$, and show that in fact there is no subgroup isomorphic to $\mathrm{PSL}_2(27)$ in the Monster. The methods are partly theoretical and partly computational, as is usual with problems of this nature. In § 2 we apply some local analysis to obtain some restrictions on putative subgroups isomorphic to $\mathrm{PSL}_2(27)$. In § 3 we outline the computational methods and strategy adopted, while the rest of the paper is devoted to the calculations themselves. The computer programs and data necessary to repeat or check our calculations are available as online supplementary material available for download from the publisher's website. For all necessary facts about finite simple groups we refer to [**2**, **15**].

2. *Theoretical results*

The overall strategy is the 'obvious' one for $\mathrm{PSL}_2(q)$, namely to classify the possibilities for the BN-pair, consisting in this case of $B \cong 3^3{:}13$ and $N \cong D_{26}$ intersecting in the maximal torus of order 13. The first step is therefore to use the 3-local and 13-local structure of the Monster (see [**14**]) to analyse the various possibilities for B and N.

First we use the 3-local analysis to classify subgroups of the Monster isomorphic to $3^3{:}13$. Since neither $3A$-elements nor $3C$-elements can form a pure 3^3, the 3-elements in any $3^3{:}13$ must be in class $3B$. Moreover, the 13 subgroups of order 3^2 in 3^3 are all conjugate in $3^3{:}13$, and there are just three classes of $3B$-pure 3^2 in the Monster, which are labelled (i)–(iii) in [**14**, § 4].

Theorem 1. *There are exactly two conjugacy classes of subgroups isomorphic to* $3^3{:}13$ *in the Monster.*

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(i) *In one case, the group contains* $13A$*-elements, has centralizer* $3^3{:}D_8$ *and normalizer* $(3^3{:}13{:}3 \times 3^2{:}D_8).2$.
(ii) *In the other case, the group contains* $13B$*-elements, has centralizer of order* 3*, and normalizer* $3 \times 3^3{:}(2 \times 13{:}3)$.

Proof. Clearly all the 3^2 subgroups of the 3^3 must be of the same type. Consider first the case when they are of type $3B_4$(i) in the notation of [**14**]. Such a 3^2 has centralizer

$$3^2.3^5.3^{10}.\mathrm{M}_{11}.$$

It is shown in [**14**, Theorem 6.5] that there is a unique conjugacy class of such 3^3. The normalizer of this 3^3 has the shape

$$3^3.3^2.3^6.3^6.(\mathrm{PSL}_3(3) \times SD_{16}).$$

Hence there is, up to conjugacy, a unique group $3^3{:}13$ of this type. It has centralizer $3^2{:}D_8$, and its normalizer is a group of shape

$$(3^3{:}(2 \times 13{:}3) \times 3^2{:}SD_{16}).\tfrac{1}{2} = (3^3{:}13{:}3 \times 3^2{:}D_8).2,$$

that is, a subgroup of index 2 in $3^3{:}(2 \times 13{:}3) \times 3^2{:}SD_{16}$. This $3^3{:}13$ contains $13A$-elements, since it lies inside a copy of $3{\cdot}\mathrm{Fi}_{24}$.

Next consider the cases $3B_4$(ii) and $3B_4$(iii). By [**14**, Corollary 6.3], the whole of the 3^3 lies inside a unique 3^{1+12}. Therefore $3^3{:}13$ lies inside $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$. Now in Suz:2 the Sylow 13-normalizer has the shape 13:12. Therefore the 13-element normalizes just four subgroups of shape 3^3 in the 3^{1+12}, and these are permuted by the 13-normalizer. It follows that there is a unique class of $3^3{:}13$ of this type in the Monster. Such a subgroup has centralizer of order 3, and normalizer of shape $3 \times 3^3{:}(2 \times 13{:}3)$. Since the 13-elements lie in $6{\cdot}\mathrm{Suz}$ they are in class $13B$. □

THEOREM 2. *There is no* $\mathrm{PSL}_2(27)$ *in the Monster containing* $13A$*-elements.*

Proof. The invertilizer of a $13A$-element has shape $13{:}2 \times \mathrm{PSL}_3(3)$. Hence there are just 118 copies of D_{26} containing a given element of class $13A$, corresponding to the 117 involutions and the identity element of $\mathrm{PSL}_3(3)$. Elementary calculations show that the subgroup $3^2{:}D_8$ of $\mathrm{PSL}_3(3)$ has orbits of lengths

$$9, 6, 6, 12, 12, 36, 36$$

on the 117 involutions in $\mathrm{PSL}_3(3)$, and in particular has no regular orbit. It follows that any $\mathrm{PSL}_2(27)$ of this type would have a non-trivial centralizer. However, this is impossible, as none of the element-centralizers in the Monster contains $\mathrm{PSL}_2(27)$. □

The corresponding method applied in the $13B$-case yields a less satisfactory result.

THEOREM 3. *There are at most five conjugacy classes of* $\mathrm{PSL}_2(27)$ *containing* $13B$*-elements in the Monster, comprising:*
(i) *at most four classes of self-normalizing subgroups* $\mathrm{PSL}_2(27)$*; and*
(ii) *at most one class of subgroups with normalizer* $\mathrm{P\Gamma L}_2(27) \cong \mathrm{PSL}_2(27){:}6$.

Proof. The invertilizer of a $13B$-element is $13^{1+2}{:}4A_4$, and therefore there are exactly 78 copies of D_{26} containing a given $13B$-element. Moreover, the group $3^3{:}13$ has index 18 in its normalizer, which is $3 \times 3^3{:}(2 \times 13{:}3)$, and therefore there is a group of shape $6 \times 13{:}3$ permuting these 78 copies of D_{26}. It is easy to see that there are two orbits of length 3 and four orbits of length 18.

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In particular, the group generated by $3^3{:}13$ and a D_{26} in one of the orbits of length 3 is normalized by a group of order 6, in such a way that the normalizer contains both types of D_{26}. If either of these groups was isomorphic to $\mathrm{PSL}_2(27)$, then the other would be $\mathrm{PGL}_2(27)$, and their common normalizer would be $\mathrm{P\Gamma L}_2(27)$.

On the other hand, if any of the other groups generated by $3^3{:}13$ and D_{26} is isomorphic to $\mathrm{PSL}_2(27)$, then it is self-normalizing. □

3. *Computational techniques and strategy*

At this point the theoretical methods appear to be exhausted, and it is necessary to resort to computer calculations to finish the job. In fact, I first did such calculations about ten years ago, using the mod 2 construction of the Monster [**9**], but they have subsequently been lost. The calculations were therefore repeated, as described here, using the mod 3 construction [**6**]. The general methods of computation in the Monster are described in [**5**, **6**, **8**], and summarized in [**12**], which also contains some improved methods. As in these references, take a, b as generators for the subgroup $2^{1+24}{\cdot}\mathrm{Co}_1$, and T as a 'triality element', cycling the three central involutions in a subgroup $2^2.2^{11}.2^{22}.\mathrm{M}_{24}$ of $2^{1+24}{\cdot}\mathrm{Co}_1$.

3.1. *Changing post*

One of the crucial tools required is a method of conjugating $2B$-elements in $2^{1+24}{\cdot}\mathrm{Co}_1$ to the central involution z. This method is explained in [**12**], but was not carried out systematically there. To avoid repetitious calculation, it is worth carrying out the procedure once and for all for a standard representative of each conjugacy class of $2B$-elements in $C(z)$ outside the normal 2^{1+24}.

There is a unique conjugacy class of involutions in $C(z)$ which map to Co_1-class $2C$. These are all in Monster-class $2B$. We make our 'standard' involution in this conjugacy class in $2^{1+24}{\cdot}\mathrm{Co}_1$ as follows. As in [**12**] make

$$\begin{aligned}
h &= (ab)^{34}(abab^2)^3(ab)^6,\\
i &= (ab^2)^{35}((ababab^2)^2ab)^4(ab^2)^5,\\
k_1 &= hihi^2,\\
k_2 &= hihihi^2,\\
k &= (k_1k_2)^3k_2k_1k_2,\\
k' &= ((a^2)^{(ab)^3}k^8)^{11}k^{11},
\end{aligned}$$

and choose k' as the standard involution in this conjugacy class. This element is carefully chosen so that $T^{-1}k'T$ is an element of the normal 2^{1+24}. We calculate once and for all an element which conjugates this element to z. (A similar calculation was already done in [**12**], but with a different involution.) Specifically, we found that if

$$k_3 = (ab)^3(ab^2)^{20}(ababab^2abab^2)^8(ababab^2ab)^{12}(ababab^2)^5$$

then

$$(k')^{Tk_3T} = z.$$

The other class which will be needed in this paper is one of the classes which maps to Co_1-class $2A$. Specifically, it is a $2B$-element whose product with z is a $2A$-element. A representative of this class can be made as

$$j_0 = ((hi)^4i)^{15}.$$

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The same process as described in [**12**] for the previous case is applied here, and produces the element

$$j_4 = (ab)^{27}(ab^2)^4(abab^2)^4(ababab^2abab^2)^{13}(ababab^2ab)^9(ababab^2)^4$$

with the property that

$$(j_0)^{Tj_4T^{-1}} = z.$$

3.2. *Shortening words*

After 'changing post' as above, we typically have a number of elements from the old involution centralizer which we want to use in the new involution centralizer. Such an element is expressed as a word of the form

$$x' = x^{T^{\pm 1}yT^{\pm 1}},$$

but in order to use it effectively we must express it as a word in a and b.

This is a classic 'membership-testing' problem. A good method for solving this problem in a simple group is Ryba's algorithm, described in [**4**], which essentially reduces the problem to three instances of the same problem in involution centralizers. However, this method does not work so well in groups with large normal 2-subgroups, which is the situation here. Nevertheless, it is possible to use Ryba's algorithm in the quotient Co_1, and then to lift to $2^{1+24}{}^{\cdot}\mathrm{Co}_1$ using other methods.

We shall use slightly different methods, however, which exploit the fact that the $3A$-elements of Co_1 form a very small conjugacy class with very nice properties. Thus the methods are special to Co_1 and not susceptible to generalization on the scale of Ryba's algorithm.

First, we use the method described in [**12**] to find a 24×24 matrix over $\mathbb{F}_3$, which gives the action of our element x' in the standard copy $\langle a, b\rangle$ of $2{}^{\cdot}\mathrm{Co}_1$, modulo the central involution. Then to find the centralizer in Co_1 of x', we search for $3A$-elements in $\langle a, b\rangle$ which centralize it. Assuming the centralizer contains a reasonably large number of $3A$-elements, and is generated by a reasonably small number of them, this method is effective, and the element itself can then be found as a word in the generators of its centralizer. Various refinements of this general method, together with methods for lifting to $C(z)$, are described below.

3.3. *The main steps of the calculation*

We break the calculations down into a number of steps. First we shall make the part of the 13-normalizer that can be easily found inside $2^{1+24}{}^{\cdot}\mathrm{Co}_1$. This is done in § 4, where we obtain a group

$$13{:}(3 \times 4A_4) \cong (13{:}3 \times 2A_4){:}2.$$

Then in § 5 we pick a non-central involution in this group, and find an element of the Monster conjugating it to the central involution. This allows us in § 6 to find another element of order 13 commuting with the first one, thereby extending the subgroup to a group $13^{1+2}{:}(3 \times 4A_4)$ which has index 2 in the full $13B$-normalizer. This group contains all the involutions which extend 13 to D_{26}.

Then in § 7 we find an element of order 3 which, together with our original element of order 13, generates $3^3{:}13$. Finally, in § 8 the six cases are each tested to see if the group so generated is $\mathrm{PSL}_2(27)$.

4. *Finding* $(13{:}3 \times 2A_4){:}2$

The strategy here is to work first in Co_1, to find enough of the centralizer of a $2B$-element to obtain a group $2^2 \times G_2(4)$. Then we conjugate one of the central involutions to the other, in

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such a way as to obtain generators for $A_4 \times G_2(4)$. Within this subgroup a copy of $\mathrm{PSL}_2(13)$ is located, by a random search, and then a copy of 13:6 inside $\mathrm{PSL}_2(13)$. Finally we apply the standard method known colloquially as 'applying the formula' in order to lift to elements of $2^{1+24}{\cdot}\mathrm{Co}_1$ which normalize a particular chosen element of order 13.

4.1. *Constructing* $A_4 \times G_2(4)$ *in* Co_1

We take a, b to be the original pair of generators of $2^{1+24}{\cdot}\mathrm{Co}_1$, and first work in the quotient Co_1 to make the element

$$c_1 = (ab)^4(ab^2)^3$$

of order 26, so that

$$i_1 = (c_1)^{13}$$

is an element of class $2B$ in Co_1. We make

$$c_2 = abi_1[ab, i_1]^5$$

which centralizes i_1, and let

$$i_2 = (c_2)^{13}.$$

The elements c_1, c_2 then generate $2^2 \times G_2(4)$, in which the central 2^2 is generated by i_1, i_2. Then let

$$\begin{aligned} n_1 &= (ai_1)^5(ab)^{-2}i_2(ab)^2a(ab)^{-2}, \\ n_2 &= (ai_1)^5(i_1i_2a)^5, \end{aligned}$$

to give elements which normalize the 2^2 and generate a group $A_4 \times G_2(4)$.

The normal subgroup A_4 is generated by

$$\begin{aligned} a_1 &= i_1, \\ a_2 &= (n_1n_2)^{13}, \end{aligned}$$

while the normal $G_2(4)$ is generated by

$$\begin{aligned} g_1 &= (c_1)^2, \\ g_2 &= (n_1n_2)^3. \end{aligned}$$

4.2. *Constructing a subgroup* $A_4 \times 13{:}6$

Standard generators of $G_2(4)$, as defined in [**16**], can be made as

$$\begin{aligned} g_3 &= (g_1^4g_2)^4, \\ g_4 &= ((g_1g_2g_1g_2^2)^3)^{g_2^4}, \end{aligned}$$

and generators for a subgroup $\mathrm{PSL}_2(13)$ can then be read off from [**16**] as

$$\begin{aligned} g_5 &= ((g_3g_4)^3g_4)^3((g_3g_4)^4g_4g_3g_4(g_3g_4^2)^2)^3((g_3g_4)^3g_4)^{-3}, \\ g_6 &= (g_3g_4g_3g_4^2)^{-2}(g_3g_4(g_3g_4g_3g_4^2)^2)^5(g_3g_4g_3g_4^2)^2. \end{aligned}$$

Inside here we find that a subgroup 13:6 is generated by g_5 and

$$g_7 = (g_6)^{g_5g_6^2},$$

and we may take the element of order 13 to be

$$g_8 = g_5g_7g_5g_7^2.$$

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4.3. *Lifting to* $2^{1+24}{\cdot}\mathrm{Co}_1$

Now we 'apply the formula' to lift to $2^{1+24}{\cdot}\mathrm{Co}_1$. That is, the elements a_1, a_2, g_5, g_7 are replaced by new elements, in the same cosets of 2^{1+24}, which normalize the subgroup $\langle g_8\rangle$ of order 13. Specifically,

$$\begin{aligned}
a_1' &= g_8 a_1 (g_8 a_1^{-1} g_8 a_1)^6,\\
a_2' &= (g_8 a_2 (g_8 a_2^{-1} g_8 a_2)^6)^2,\\
g_5' &= g_8 g_5 (g_8^{-1} g_5^{-1} g_8 g_5)^6,\\
g_7' &= (g_8 g_7 (g_8^9 g_7^{-1} g_8 g_7)^6)^2,
\end{aligned}$$

so that a_1', a_2' generate $2A_4$ and g_5', g_7' generate $26{\cdot}6 = (2\times 13{:}3){\cdot}2$, commuting with each other. Thus they together generate

$$2{\cdot}(A_4 \times 13{:}6).$$

An element of order 12 normalizing $\langle g_8\rangle$ and commuting with $\langle a_1', a_2'\rangle \cong 2A_4$ and with g_5' may be obtained as

$$g_9 = g_5' g_8^3 g_7' g_8^{10}.$$

5. *Changing post and shortening words*

The process of 'changing post' really consists of two parts. The first part consists of finding a word x in the generators of the Monster, which conjugates a given involution in $C(z)$, to z. This part is more or less algorithmic. Here the involution which we want to map to z is

$$i_3 = a_1' g_5'.$$

The second part consists of 'shortening the word' for an element g^x, where both g and g^x lie in $C(z)$. This part is more *ad hoc*, and involves often quite laborious search for a word in a and b which is equal to the desired element. In this section, the element we want to write as a word in a and b is the appropriate conjugate of the element g_9 of order 12. The strategy is to first find a word for its image in Co_1, and then to lift to $2^{1+24}{\cdot}\mathrm{Co}_1$. Even within Co_1, the search is not easy, and we perform it in stages, first dealing with the involution which is its sixth power, and then its fourth power, before finally reaching the element itself. In the course of these calculations, we shall also identify two useful elements which centralize the given element of order 12.

5.1. *Conjugating* i_3 *to* z

Since we have already conjugated k', the 'standard' involution in this conjugacy class, to z, it only remains to conjugate i_3 to k'. Now if two elements of Co_1-class $2C$ have product of order 13 or 35, then this product is fixed-point-free in its action on $2^{24} = 2^{1+24}/2$, and hence when we lift to $2^{24}{\cdot}\mathrm{Co}_1$ the product remains of odd order. Thus we can conjugate one to the other in $2^{24}{\cdot}\mathrm{Co}_1$ using the standard formula.

Lifting to $2^{1+24}{\cdot}\mathrm{Co}_1$ is then straightforward: if it had turned out that we conjugated i_3 to zk' instead of k', we simply need to replace T by T^{-1} in the appropriate place in the formula.

So we search for conjugates of i_3 and k' whose product has order 13 or 35, and thereby find that if

$$l_3 = (ab^2)^4 (k'(i_3)^{(ab^2)^4})^6$$

then l_3 conjugates i_3 to k', and therefore $l_3 T k_3 T$ conjugates i_3 to z.

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5.2. *Finding the centralizer of* $(g_9')^2$ *in* Co_1

The above conjugation takes the element g_9 to an element

$$g_9' = g_9^{l_3 T k_3 T}$$

which has order 12 in the quotient Co_1. Using the procedure described in [**12**], we can obtain a 24×24 matrix over $\mathbb{F}_3$ which gives the action of g_9' as an element of $2{\cdot}\mathrm{Co}_1$, whose standard generators we again denote a, b. We are now faced with the 'membership-testing' problem, to find a word in a, b which gives the element g_9'. After solving this problem in $2{\cdot}\mathrm{Co}_1$ (modulo the central involution) we have to lift the solution to $2^{1+24}{\cdot}\mathrm{Co}_1$.

In this subsection we obtain a word for an element which is congruent to $(g_9')^2$ modulo 2^{1+24}, by first finding the centralizer of the involution $g_9'^6$ and then the element $g_9'^4$ of order 3. First note that $g_9^6 = z$, so that $(g_9')^6$ is (modulo 2^{1+24}) in the normal 2^{11} subgroup of the standard copy $\langle h, i\rangle$ of $2^{11}{:}\mathrm{M}_{24}$. By a random search we find a subgroup $2^{11}{:}\mathrm{M}_{12}$ centralizing $(g_9')^6$, generated by

$$\begin{aligned}
t_1 &= ((hi^2)^6(hi)^{17})^{27} i^2 (hi^2)^6 (hi)^{17},\\
t_2 &= ((hi^2)^5(hi)^{10})^{20} i^2 (hi^2)^5 (hi)^{10},\\
t_3 &= ((hihihi^2hi)^3(hi)^{21})^{11} i^2 (hihihi^2hi)^3 (hi)^{21}.
\end{aligned}$$

Moreover, the central involution of this group is

$$t_0 = (t_1t_3t_1t_3t_1t_3^2)^{11}.$$

Then we conduct another random search in this subgroup for elements commuting with the element $(g_9')^4$ of order 3. Writing

$$\begin{aligned}
u &= t_1,\\
v &= t_2t_3,\\
v_2 &= (uvuvuv^2uvuv^2)^9(uvuvuv^2uv)^{10},\\
v_3 &= (uvuvuv^2)^{10}(uvuvuv^2uvuv^2)^2,\\
t_4 &= v_2^{11}(uv)^4 v_2,\\
t_5 &= v_3^5 (uv)^4 v_3,\\
t_6 &= (t_4t_5t_4t_5t_4t_5^2(t_4t_5)^7)^2 t_4^2 t_5t_4t_5t_4t_5^2(t_4t_5)^7,
\end{aligned}$$

we have that t_6 is in fact the inverse of $(g_9')^4$, modulo 2^{1+24}.

5.3. *Finding the centralizer of* g_9' *in* Co_1

Working first in the M_{12} quotient of $\langle u, v\rangle$ we find that the following elements commute with t_6 modulo the 2-group:

$$\begin{aligned}
t_8 &= u^{(uvuvuv^2)^3 v^5},\\
t_9 &= u^{(uvuvuv^2uvuv^2)^7 v^6}.
\end{aligned}$$

(The conjugating elements have orders 24 and 32 respectively.) Applying the formula we obtain

$$\begin{aligned}
t_8' &= t_6^2 t_8 t_6^2 t_8^2 t_6^2 t_8,\\
t_9' &= t_6^2 t_9 t_6^2 t_9^2 t_6^2 t_9,
\end{aligned}$$

and then

$$t_{10} = (t_9' t_8' t_9')^3$$

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is congruent to $(g_9')^3$, modulo 2^{1+24}. We also make some elements commuting with g_9' modulo the 2-group, as follows:

$$\begin{aligned} t_{11} &= ((t_1t_3)^8t_6^2)^3t_8'(t_9')^2t_{10}, \\ t_{12} &= (t_8')^2((t_1t_3)^8t_6^2)^3(t_8')^2(t_9')^2. \end{aligned}$$

5.4. *Lifting to* $2^{1+24}{\cdot}\mathrm{Co}_1$

Now we know that $t_{10}t_6$ is congruent modulo 2^{1+24} to the inverse of g_9'. It remains to find the correct element of 2^{1+24} to multiply by. Using the method explained in [**12**], with the generators $p_1, \ldots, p_{12}, d_1, \ldots, d_{12}$ of 2^{1+24} defined there, we obtain the element

$$w = t_{10}t_6p_1d_3d_4d_5d_7d_8d_9d_{10}d_{11}d_1p_3p_4p_6p_7p_8.$$

Thus w^{-1} is actually equal to g_9' in the Monster.

We next lift the elements t_{11}, t_{12} to elements which commute with w, by the following method. First apply the formula, to get elements t_{11}', t_{12}' which commute with w^4:

$$\begin{aligned} t_{11}' &= w^4t_{11}w^4t_{11}^{-1}w^4t_{11}, \\ t_{12}' &= w^4t_{12}w^4t_{12}^{-1}w^4t_{12}. \end{aligned}$$

Then make the part of 2^{1+24} which commutes with w^4: by computing the nullspace of $1 - w^4$ in the 24-dimensional $\mathbb{F}_2$-representation of Co_1, whose 24 coordinates correspond to the given generators $p_1, \ldots, p_{12}, d_1, \ldots, d_{12}$ of 2^{1+24}, we find that this is generated by

$$\begin{aligned} q_1 &= d_9d_{12}, \\ q_2 &= d_1d_4d_5d_6d_{10}d_{11}, \\ q_3 &= d_3d_5d_7d_{10}d_{12}, \\ q_4 &= d_2d_6d_7d_8d_{10}d_{12}, \\ q_5 &= p_4p_6p_9p_{12}d_5d_8d_{10}, \\ q_6 &= p_3p_6p_7p_9d_4d_7d_{12}, \\ q_7 &= p_2p_5p_6p_7p_{10}p_{12}d_7d_{11}, \\ q_8 &= p_1p_7p_8p_9p_{10}p_{12}d_4d_8. \end{aligned}$$

Finally we test all multiples of t_{11}' and t_{12}' by products of the q_i. We find the following elements which commute with w:

$$\begin{aligned} t_{11}'' &= q_4q_5q_6q_7t_{11}', \\ t_{12}'' &= q_5q_6q_8t_{12}'. \end{aligned}$$

Note also that w commutes with $q_2q_3q_4$, and, modulo the central involution, also with q_4.

6. *Finding the full* $13B$*-centralizer*

In order to extend $(13{:}3 \times 2A_4){:}2$ to $13^{1+2}{:}(3 \times 2A_4){:}2$, we now seek an element of order 13 which is normalized by w. First we work in the quotient Co_1, and afterwards lift to $2^{1+24}{\cdot}\mathrm{Co}_1$.

6.1. *Extending* 12 *to* 13:12 *in* Co_1

Now the element t_{11} maps to a $2B$-involution in the quotient Co_1, and the element of order 13 we are looking for centralizes *either* this involution, *or* t_0t_{11}. However, conjugating by t_{12}

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interchanges these two cases, so we can assume the former. We therefore begin by making the centralizer of t_{11} in the quotient Co_1. Let

$$\begin{aligned} h_5 &= ((at_{11})^6 t_0 t_6)^4, \\ h_6 &= (t_0 t_6 (at_{11})^6)^4, \end{aligned}$$

which are elements of order 21 generating $G_2(4)$ in this centralizer. We search for a subgroup $\mathrm{PSL}_2(13)$ containing t_6, and find that $\langle h_7, t_6\rangle$ is such a subgroup, where

$$h_7 = (h_5 h_6^7)^{11} (h_5 h_6 h_5 h_6^2)^5 h_5 h_6^7.$$

Inside this copy of $\mathrm{PSL}_2(13)$, we find an element of order 13,

$$h_8' = (h_7 t_6^2)^4 t_6^2 h_7 t_6^2 (h_7 t_6^4)^2,$$

and the one normalized by t_6 is

$$h_8 = (h_7 h_8'^7)^{11} h_8' h_7 (h_8')^7.$$

Then we work with the centralizer of t_6 in $G_2(4)$ to conjugate this 13-element to one which is normalized also by t_{10}. We first make this centralizer by a random search through $3A$-elements of $G_2(4)$ to find some which commute. We find

$$\begin{aligned} h_{10} &= ((h_5 h_6 h_5 h_6 h_5 h_6^2 h_5 h_6)^7)^{h_5^{18} h_6^{10}}, \\ h_{11} &= ((h_5 h_6 h_5 h_6 h_5 h_6^2 h_5 h_6)^7)^{h_5^{18} h_6^{15}}, \end{aligned}$$

which generate A_5. Conjugating by random elements of this centralizer we quickly find one of the 13-elements we are looking for, namely

$$h_{12} = (h_{10} h_{11}^2 h_{10})^4 h_8 h_{10} h_{11}^2 h_{10}.$$

6.2. *Lifting the* 13*-element*

The main lifting problem is to lift the element of order 13 to one which is normalized by w. Since there are 2^{24} elements of order 13 in the given coset of 2^{1+24}, only two of which are normalized by w, a brute force search is out of the question (or, at least, unwieldy). We therefore do this in two stages, first finding an appropriate conjugating element to get a 13-element inverted by w^6. Since w^6 centralizes just 2^{12} out of the 2^{24} factor, this divides the problem into two searches, each in a population of size 2^{12}.

First we work in the 24-dimensional $\mathbb{F}_2$-representation of Co_1 and find that the fixed space of w^6 is spanned by vectors which lift to the following elements of 2^{1+24}: all even products of the d_i, together with

$$d_1 p_1 p_3 p_6 p_8 p_{10} p_{12}.$$

Hence in the first search we may test conjugates just by the p_i. We find that the correct conjugating element is

$$p_1 p_3 p_5 p_6 p_7 p_{12}.$$

In the second search we test conjugates by $d_i d_{i+1}$ and $d_1 p_1 p_3 p_6 p_8 p_{10} p_{12}$. We find that exactly two conjugating elements work:

$$\begin{aligned} &d_1 d_2 d_3 d_5 d_8 d_9 d_{11} p_1 p_3 p_6 p_8 p_{10} p_{12}, \\ &d_1 d_3 d_5 d_6 d_7 d_9 d_{10} d_{11} d_{12} p_1 p_3 p_6 p_8 p_{10} p_{12}. \end{aligned}$$

Let h_{12}' and h_{12}'' be the respective conjugates of h_{12}, that is

$$\begin{aligned} h_{12}' &= (h_{12})^{d_1 d_2 d_3 d_5 d_8 d_9 d_{11} p_5 p_7 p_8 p_{10}}, \\ h_{12}'' &= (h_{12})^{d_1 d_3 d_5 d_6 d_7 d_9 d_{10} d_{11} d_{12} p_5 p_7 p_8 p_{10}}. \end{aligned}$$

Then h_{12}' and h_{12}'' are elements of order 13 which are normalized by w in the Monster.

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6.3. *Finding the* 12*-normalizer*

In order to obtain all possible 13-elements normalized by the element w of order 12, it is necessary to conjugate not just by elements of its centralizer, but by elements of its normalizer. Indeed it turns out that w must be conjugated to its seventh power.

Such a normalizing element can be found inside the centralizer of w^4 as follows. First we conjugate a_1' to t_{11}', by conjugating by $(t_{11}'a_1')^2$, so that the full $A_4 \times G_2(4)$ is available. In particular, the element

$$(t_{11}'a_1')^3(a_2'^2a_1'a_2')(t_{11}'a_1')^2$$

is an involution in the A_4, but not equal to t_{11}'.

Within the A_5 generated by h_{10}, h_{11} the 15 involutions may be made as conjugates, by powers of $h_{10}h_{11}$, of

$$(h_{10}h_{11})^2h_{11},$$
$$h_{11}h_{10}h_{11}^2h_{10}$$

and their product. We find that the first involution conjugated by $(h_{10}h_{11})^2$ commutes with w, and therefore the normalizing element we want is (modulo the 2-group)

$$t_{14} = (t_{11}'a_1')^3(a_2'^2a_1'a_2')(t_{11}'a_1')^2(h_{10}h_{11})^3(h_{11}h_{10}h_{11}^2h_{10})(h_{10}h_{11})^2.$$

To lift to $2^{1+24}{\cdot}\mathrm{Co}_1$, we first apply the formula to get an element which commutes with w^4:

$$t_{14}' = w^4t_{14}w^4t_{14}^3w^4t_{14}.$$

Finally multiplying by combinations of the q_i we find that the element we want can be taken to be

$$t_{14}'' = q_3q_7q_8t_{14}'.$$

We also made an element t_{13}' which conjugates w to its fifth power, but this turned out not to be necessary.

6.4. *Testing commuting with the first* 13*-element*

We are aiming to find the normalizer of g_8, so the candidate elements of order 13 have to be tested to see which one(s) commute with

$$g_8' = g_8^{l_3Tk_3T}.$$

The candidate elements are conjugates of h_{12}' and h_{12}'' by combinations of t_{11}'', t_{12}'', t_{13}' and t_{14}''. Of these, we found that the one which works is

$$w_1 = (h_{12}'')^{t_{11}''t_{14}''}.$$

We now have generators for $13^{1+2}{:}(3 \times 4A_4)$. These are best taken as the generators a_1', a_2', g_5', g_7' given above, together with the conjugate of w_1 by $(l_3Tk_3T)^{-1}$, that is

$$w_1' = l_3Tk_3Tw_1T^{-1}k_3^{-1}T^{-1}l_3^{-1}.$$

7. *Finding* $3^3{:}13$

The element g_8 of order 13 lies inside a subgroup $6{\cdot}\mathrm{Suz}$ of $2^{1+24}{\cdot}\mathrm{Co}_1$. Now $6{\cdot}\mathrm{Suz}$ also lies in a (unique) subgroup

$$3^{1+12}{\cdot}3{\cdot}\mathrm{Suz},$$

of index 2 in a maximal subgroup $3^{1+12}{\cdot}3{\cdot}\mathrm{Suz}{:}2$ of the Monster. If we can find generators for this subgroup, then we can write down generators for a group $3^3{:}13$ containing g_8.

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The strategy is to find such a subgroup $6{}^{\cdot}\mathrm{Suz}$, which may be taken to be the centralizer in the Monster of the element za_2', and then move to the centralizer of a suitable non-central involution, where we can find an element of order 3 extending $3 \times 2^{1+6}{}^{\cdot}2{}^{\cdot}\mathrm{U}_4(2)$ to

$$(3^{1+4}{:}2 \times 2^{1+6}){}^{\cdot}\mathrm{U}_4(2),$$

and thereby extending $6{}^{\cdot}\mathrm{Suz}$ to $3^{1+12}{}^{\cdot}2{}^{\cdot}\mathrm{Suz}$. It is then easy to write down a word for the element we want.

7.1. *The subgroup* $6{}^{\cdot}\mathrm{Suz}$

The element

$$s_1 = abababab^2abab^2ab^2$$

has order 66, so s_1^{22} is conjugate to a_2'. In the quotient Co_1, the elements s_1^{22} and a_2' generate a subgroup A_4, and $a_2's_1^{22}$ conjugates s_1^{22} to a_2' modulo the 2-group. Let

$$s_1' = (a_2's_1^{22})^5 s_1 a_2' s_1^{22}.$$

Then, modulo the 2-group, both s_1' and $g_1 = c_1^2$ commute with a_2', and generate $3{}^{\cdot}\mathrm{Suz}$.

Hence, applying the formula, we obtain the following elements centralizing a_2', and generating $6{}^{\cdot}\mathrm{Suz}$:

$$\begin{aligned} s_1'' &= a_2's_1a_2's_1^{-1}a_2's_1, \\ s_2 &= a_2'c_1^2a_2'c_1^{-2}a_2'c_1^2. \end{aligned}$$

7.2. *Changing post again*

The element

$$j_2 = (s_2^2 s_1'')^6$$

turns out to be an involution mapping to Co_1-class $2A$, and forming a 2^2-group of Monster-type $2BAB$ with the central involution z of $2^{1+24}{}^{\cdot}\mathrm{Co}_1$. The 'standard' involution of this type is

$$j_0 = ((hi)^4 i)^{15}.$$

Moreover, j_0j_2 has order 5, so

$$j_3 = (j_0j_2)^2$$

conjugates j_2 to j_0. Hence

$$j_3Tj_4T^{-1}$$

conjugates j_2 to z.

7.3. *Identifying the element of order* 3

Writing

$$j_5 = ((s_1'')^{22})^{j_3Tj_4T^{-1}}$$

we want to find words for the centralizer of the element zj_5 of order 6. This centralizer is a group of shape

$$(2^{1+6} \times 3^{1+4}{:}2){}^{\cdot}\mathrm{U}_4(2).$$

In particular, we want to find a non-central element of the normal 3^{1+4}.

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We begin as usual in the Co_1 quotient, and look for $3A$-elements which commute with j_5. Writing

$$\begin{aligned} j_6 &= (ab)(ab^2)^6(ababab^2ab)^9(ababab^2)^9, \\ j_7 &= (ab)(ab^2)^8(ababab^2ab)^{11}(ababab^2)^4, \\ j_8 &= (ab)^3(ab^2)^5(ababab^2ab)^{13}(ababab^2)^6, \\ j_9 &= (ab)^3(ab^2)^{30}(ababab^2ab)^{10}(ababab^2)^4, \\ j_{10} &= (ab)^6(ab^2)^{17}(ababab^2ab)^{12}(ababab^2)^9, \end{aligned}$$

we have that $j_n' = (a_2')^{j_n}$ is such a $3A$-element for $n \in \{6, 7, 8, 9, 10\}$. Moreover

$$j_5' = (j_6'j_7'j_8'j_9'j_{10}')^{12}$$

is congruent to j_5 modulo the 2-group.

To find out which of the p_n and d_n to multiply by, we apply the element $j_5(j_5')^{-1}$ to 13 carefully selected coordinate vectors, as described in [12], and read off the answer from the result. We find that the correct answer is

$$j_5'' = p_1p_6p_8p_9p_{10}p_{11}p_{12}d_1d_2d_5d_6d_{10}d_{11}d_{12}j_5'.$$

That is, j_5'' is actually equal to j_5 in the Monster.

7.4. *Finding* 3^{1+12}

Now apply the formula so that we get generators for the centralizer of j_5'' as follows. For $n \in \{6, 7, 8, 9, 10\}$, define

$$j_n'' = j_5''j_n'j_5''j_n'^{-1}j_5''j_n'.$$

Then $j_6''j_7''(j_8''j_9''j_{10}'')^2$ has order 10 and we find that

$$j = (j_6''j_7''(j_8''j_9''j_{10}'')^2)^5(j_8''j_9''j_{10}''j_6''j_7''j_8''j_9''j_{10}'')^5$$

is an element in the normal 3^{1+4} as required.

We now know that, under the action of $6{}^{\cdot}\mathrm{Suz}$ generated by s_1'' and s_2, the element

$$j' = j^{Tj_4^{-1}T^{-1}j_3^{-1}}$$

and its conjugates generate a group 3^{1+12}.

7.5. *Finding the right* 3^3

We know from the character table of $6{}^{\cdot}\mathrm{Suz}$ that the element g_8 of order 13 acts fixed-point-freely on the natural 3^{12} quotient of 3^{1+12}. Elementary linear algebra then tells us that if we want a 3^3 on which the minimum polynomial of the action of g_8 is $x^3 - x - 1$, then we compute

$$\frac{x^{13}-1}{(x-1)(x^3-x-1)} = x^9 + x^8 - x^7 + x^5 - x^3 - x^2 - 1,$$

and compute the image of the linear map obtained by substituting g_8 for x. In other words, (modulo the central 3) the element

$$j'' = j'^{-1}g_8^4j'g_8j'g_8j'^{-1}g_8^2j'g_8^2j'^{-1}g_8j'^{-1}g_8^2$$

is an element in the 3^3 that we want. In fact, it turned out that no correction for the centre was required.

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8. *Proof of the main theorem*

8.1. *Analysing the cases*

The 78 ways of extending 13 to D_{26} are obtained by taking the six non-central involutions in $4A_4$, and conjugating by suitable elements of 13^{1+2}. First take the involution

$$i_4 = a_1' g_5'^3$$

and check that $w_1 i_4$ has order 2, so that the 13 conjugates of i_4 by powers of w_1' can be written as

$$(w_1')^n i_4$$

for $0 \leqslant n \leqslant 12$. Then conjugate these 13 involutions by suitable elements of $\langle a_1', a_2' \rangle \cong 2A_4$ to get the full set of 78. For example, we may conjugate in turn by each of the six elements

$$1, a_2', a_2'^2, a_2'a_1', a_2'a_1'a_2', a_2'a_1'a_2'^2.$$

However, a subgroup $3 \times 2 \times 3$ (generated by a_2', $a_1'^2$ and g_7') of $13^{1+2}{:}(3 \times 4A_4)$ normalizes $3^3{:}13$, and we have already observed that it fuses the 78 cases into six orbits, of lengths $3, 3, 18, 18, 18, 18$. These six cases are represented by the elements

$$\begin{aligned} &i_4, \\ &i_5 = w_1' i_4, \\ &i_6 = w_1'^2 i_4, \\ &i_7 = i_4^{a_2'a_1'}, \\ &i_8 = (w_1' i_4)^{a_2'a_1'}, \\ &i_9 = (w_1'^2 i_4)^{a_2'a_1'}. \end{aligned}$$

In each of these six cases we perform the following test. Given the fixed generators j'', g_8 for $3^3{:}13$, and the six involutions i, test each of the 13 words

$$j'' i g_8^m$$

(for $0 \leqslant m \leqslant 12$) on a random vector to see if it has order 3. Since j'' is a word involving exactly 28 occurrences of T or T^{-1}, and i averages just four such occurrences, each test involves on average 96 applications of T or T^{-1}, and a slightly larger number of applications of elements of $2^{1+24}{\cdot}\mathrm{Co}_1$. On my rather old laptop, such a test takes around 15 minutes, and therefore the total calculation takes around 1.5 hours.

8.2. *Proofs*

Most of the computer calculations were performed without proof, and therefore it is necessary to provide proofs for the few statements which we actually need in order to prove our main theorem.

THEOREM 4. *There is no subgroup of the Monster isomorphic to* $\mathrm{PSL}_2(27)$.

Proof. By Theorems 1 and 2, there is a unique class of $3^3{:}13$ which could lie in a subgroup $\mathrm{PSL}_2(27)$. We prove computationally that the elements j'' and g_8 generate a group $3^3{:}13$, by checking relations on two vectors whose joint stabilizer in the Monster is known to be trivial. The relations can be taken as

$$\alpha^3 = \beta^{13} = [\alpha, \alpha^\beta] = [\alpha, \alpha^{\beta^2}] = 1, \alpha^{\beta^3} = \alpha^\beta \alpha,$$

with $\alpha = j''$ and $\beta = g_8$. Moreover, g_8 lies in $2^{1+24}{\cdot}\mathrm{Co}_1$, and so is in Monster-class $13B$.

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Next we check that w_1', a_1', a_2' centralize g_8, while g_5' inverts it and g_7' cubes it. In particular, they all lie in the normalizer of the subgroup $\langle g_8 \rangle$ of order 13, and it follows easily that the given elements generate $13^{1+2}{:}(3 \times 4A_4)$, as required.

Therefore, the test runs through all the involutions inverting g_8, and since the test fails in every one of the six cases, the proof is complete. $\square$

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