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RADII OF CONVEXITY OF TWO CLASSES OF REGULAR FUNCTIONS

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This paper establishes the radii of convexity of the following two classes of regular functions,

$$R_{\gamma a} = \left\{f(z) = z - 2az^2 + \ldots ; \left|\frac{f(z)}{z} - \gamma\right| < \gamma, \right.$$
$$\left. \gamma \geq 1,\ 0 \leq a \leq 1-(2\gamma)^{-1},\ |z| < 1\right\} ,$$

$$T_\gamma(G) = \left\{f(z) = z + a_2z^2 + \ldots ; \left|\frac{f(z)}{g(z)} - \gamma\right| < \gamma , \right.$$
$$\left. g(z) \in G,\ \gamma \geq 1,\ |z| < 1\right\} ,$$

where

$$G = \left\{g(z) = z + a_2z^2 + \ldots ; |g'(z)-1| < 1,\ |z| < 1\right\} .$$

1. Introduction

Let N be the class of functions $f(z)$ regular in the unit disc $\Delta = \{z;\ |z| < 1\}$ with the normalisation $f(0) = 0$, $f'(0) = 1$. The classes of functions $f(z) \in N$ which are univalent, univalent convex, univalent starlike are denoted by S, S^c, S^* , respectively. Let F be a subclass of N . By $T(F)$ and $T_\gamma(F)$ we shall mean the classes

$$T(F) = \left\{f(z) \in N;\ \mathrm{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0,\ g(z) \in F,\ z \in \Delta\right\} ,$$

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$$T_\gamma(F) = \left\{f(z) \in N;\ \left|\frac{f(z)}{g(z)} - \gamma\right| < \gamma,\ g(z) \in F,\ \gamma \geq 1,\ z \in \Delta\right\} .$$

We note that $T_\infty(F) \equiv T(F)$ and $T(S^*)$ is the well-known class of close-to-starlike functions introduced by Reade [8].

The problem of determining the radius of starlikeness of $T(F)$ or $T_\gamma(F)$ when F varies in a subclass of N or S has been extensively studied. For example, MacGregor [4], [5] obtained the radii of starlikeness of $T(F)$ and $T_1(F)$ when $F \equiv S^c$ or $F \equiv S^*$; Krzyż and Reade [3] found those of $T(S)$ and $T_1(S)$. A more difficult question which arises naturally is that of determining the radii of convexity of these classes. Sakaguchi [10] established the radius of convexity of $T(S^*)$. Reade, Ogawa and Sakaguchi [9] obtained the radius of convexity for a subclass of $T(S^*)$, namely the class

$$R = \{f(z) \in N;\ \mathrm{Re}\{f(z)/z\} > 0,\ z \in \Delta\} .$$

The method of [9] and [10], which is based on certain coefficient inequalities, does not apply to the classes under consideration; therefore we shall take a different course.

The problem of finding the radii of convexity of the classes $T_\gamma(F)$ may be transformed into that of establishing bounds for certain functionals over the class

$$P = \left\{p(z) = 1 + p_1z + p_2z^2 + \ldots;\ \mathrm{Re}\{p(z)\} > 0,\ z \in \Delta\right\} .$$

In fact, let B be the class of functions $\omega(z)$ regular in Δ and satisfying $\omega(0) = 0$, $|\omega(z)| < 1$ there. Let $f(z) \in T_\gamma(F)$, then writing $\psi(z) = 1 - \gamma^{-1}f(z)/g(z)$, we have $|\psi(z)| < 1$ in Δ and $\psi(0) = 1 - \gamma^{-1} = \psi_0$. Put $\omega(z) = [\psi(z)-\psi_0]/[1-\psi_0\psi(z)]$; then $\omega(z) \in B$ and $\psi(z) = [\omega(z)+\psi_0]/[1+\psi_0\omega(z)]$. In view of this and the fact that every $\omega(z) \in B$ can be represented by $\omega(z) = [p(z)-1]/[p(z)+1]$ for some $p(z) \in P$ (see Nehari [7], p. 169), we get

$$f(z) = \frac{2\gamma g(z)}{1+(2\gamma-1)p(z)} , \quad z \in \Delta . \tag{1.1}$$

This representation yields

$$(1.2)\quad 1+\frac{zf''(z)}{f'(z)} = 1 - \left[\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}\right]\left[\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\right]^{-1} - \frac{2zp'(z)}{p(z)+\mu} ,$$

where $\mu = (2\gamma-1)^{-1}$. Hence the problem now is to find the sharp upper bounds on $|z| = r$ for

$$\left|\left[\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}\right]\left[\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\right]^{-1}\right|$$

and $|zp'(z)/(p(z)+\mu)|$ and to check that these bounds are attained at the same point.

In this paper we shall employ the method described above to establish the radii of convexity of the classes

$$R_{\gamma a} = \left\{f(z) = z - 2az^2 + \ldots;\ \left|\frac{f(z)}{z} - \gamma\right| < \gamma,\right.$$
$$\left.\gamma \ge 1,\ 0 \le a \le 1-(2\gamma)^{-1},\ z \in \Delta\right\}$$

and $T_\gamma(G)$, where $G = \{g(z) \in N;\ |g'(z)-1| < 1,\ z \in \Delta\}$. Letting $\gamma \to \infty$ in these results we obtain the radii of convexity of the classes

$$R_a = \{f(z) = z - 2az^2 + \ldots;\ \mathrm{Re}\{f(z)/z\} > 0,\ 0 \le a \le 1,\ z \in \Delta\}$$

and $T(G)$, respectively. The result for R_a , which involves the second coefficient in the series expansion of functions in the class, refines that obtained by Reade, Ogawa and Sakaguchi [9].

2. Radius of convexity of the class $R_{\gamma a}$

We first remark that there is no loss of generality in assuming the second coefficient of functions $f(z) \in R_{\gamma a}$ to be real and negative, for, if this is not the case, we may consider the functions

$$e^{i\theta}f\left(e^{-i\theta}z\right) = z - |a_2|z^2 + \ldots ,\ \text{where}\ \theta = \arg a_2 + \pi .$$

We next define the subclass

$$P_b = \{p(z) \in P;\ p'(0) = 2b,\ 0 \le b \le 1\} .$$

Then for $p(z) \in P_b$, we may write $p(z) = [1+w(z)]/[1-w(z)]$ for some $w(z) \in B$ so that

$$w(z) = \frac{p(z)-1}{p(z)+1} = bz + \ldots = z\psi(z) ,$$

where $\psi(z)$ is regular and $|\psi(z)| \le 1$ in Δ with $\psi(0) = b$. Now, since $0 \le b \le 1$, we have $[\psi(z)-b]/[1-b\psi(z)] \prec z$ in Δ ($\prec$ reads "is subordinate to"). Hence $\psi(z) \prec (z+b)/(1+bz)$ in Δ. This yields

$$(2.1) \quad \operatorname{Re}\{\psi(z)\} \ge \frac{b-|z|}{1-b|z|} , \quad |\psi(z)| \le \frac{|z|+b}{1+b|z|} , \quad |w(z)| \le |z| \frac{|z|+b}{1+b|z|} .$$

We now put $A = (r+b)/(1+br)$, $0 < r < 1$, and define $H_r(z) = (1+Az)/(1-Az)$; then it is clear that for $p(z) \in P_b$, $p(z) \prec H_r(z)$, $|z| \le r$. And so, $p(z)$ maps $|z| \le r$ into the disc $|p(z)-a| \le d$, where

$$a = \frac{1+B^2}{1-B^2} , \quad d = \frac{2B}{1-B^2} , \quad B = \frac{r(r+b)}{1+br} .$$

It follows immediately that for $p(z) \in P_b$, $|z| = r < 1$,

$$(2.2) \qquad \frac{1-B}{1+B} \le \operatorname{Re}\{p(z)\} \le |p(z)| \le \frac{1+B}{1-B} .$$

The first inequality is sharp for the function

$$p(z) = \frac{1-z^2}{1-2bz+z^2} \quad \text{at} \quad z = -r ,$$

while the third inequality is sharp for the function

$$p(z) = \frac{1+2bz+z^2}{1-z^2} \quad \text{at} \quad z = r .$$

Returning to the class $R_{\gamma a}$, then in view of (1.1), every $f(z) = z - 2az^2 + \ldots \in R_{\gamma a}$ can be written as

$$(2.3) \qquad f(z) = \frac{2\gamma z}{1+(2\gamma-1)p(z)} , \quad z \in \Delta .$$

From the power series expansion of $f(z)$ and (2.3) we find $p(z) = 1 + 2bz + \ldots$, where $b = 2\gamma a/(2\gamma-1)$. Also, $0 \le b \le 1$ as $0 \le a \le 1-(2\gamma)^{-1}$. Thus $p(z) \in P_b$.

For $f(z) \in R_{\gamma a}$, (1.2) becomes

$$(2.4) \qquad 1 + \frac{zf''(z)}{f'(z)} = 1 - \frac{z^2p''(z)}{p(z)+\mu-zp'(z)} - \frac{2zp'(z)}{p(z)+\mu} .$$

The upper bounds for $|zp'(z)/(p(z)+\mu)|$ and $|z^2p''(z)/(p(z)+\mu-zp'(z))|$ on $|z| = r < 1$ are established in the following lemmas.

LEMMA 2.1. *If* $p(z) \in P_b$, $\mu > 0$, *then on* $|z| = r < 1$,

$$(2.5) \qquad \left|\frac{p(z)}{p(z)+\mu}\right| \le \frac{1+2br+r^2}{1+\mu+2br+(1-\mu)r^2} .$$

Proof. Since $p(z) = [1+w(z)]/[1-w(z)]$ for $w(z) = bz + \ldots \in B$, we have

$$(2.6) \qquad \frac{p(z)}{p(z)+\mu} = \frac{1}{1+\mu} \cdot \frac{1+w(z)}{1+[(1-\mu)/(1+\mu)]w(z)} .$$

From (2.1), $|w(z)| \le r(r+b)/(1+br)$. Hence, in view of the Subordination Principle, the image of $|z| \le r$ under the transformation of the right-hand side is contained in the disc

$$(2.7) \qquad \left|\frac{1+w(z)}{1+Cw(z)} - \frac{1-CB^2}{1-C^2B^2}\right| \le \frac{(1-C)B}{1-C^2B} ,$$

where $C = (1-\mu)/(1+\mu)$, $B = r(r+b)/(1+br)$. The assertion now follows from (2.6) and (2.7).

Equality in (2.5) occurs for the function

$$p(z) = (1+2bz+z^2)/(1-z^2) \quad \text{at} \quad z = r .$$

LEMMA 2.2. *If* $p(z) \in P_b$, *then for* $z \in \Delta$,

$$(2.8) \qquad |p'(z)| \le 2\,\frac{\mathrm{Re}\{p(z)\}}{1-|z|^2} \cdot \frac{b+2|z|+b|z|^2}{1+2b|z|+|z|^2} .$$

Proof. Write $p(z) = [1+z\psi(z)]/[1-z\psi(z)]$, where $\psi(z) = b + \ldots$ is such that $|\psi(z)| \le 1$ in $|z| < 1$. Then, from (2.1),

$$(2.9) \qquad |\psi(z)| \le \frac{|z|+b}{1+b|z|} .$$

Also,

$$p'(z) = 2\,\frac{z\psi'(z)+\psi(z)}{(1-z\psi(z))^2} , \quad z \in \Delta .$$

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Thus

$$(2.10)\qquad \begin{aligned} |p'(z)| &= \frac{2|z\psi'(z)+\psi(z)|}{1-|z\psi(z)|^2} \cdot \frac{1-|z\psi(z)|^2}{|1-z\psi(z)|^2} \\ &= \frac{2|z\psi'(z)+\psi(z)|}{1-|z\psi(z)|^2} \cdot \mathrm{Re}\{p(z)\} \\ &\le 2\,\mathrm{Re}\{p(z)\} \cdot \frac{|z\psi'(z)|+|\psi(z)|}{1-|z\psi(z)|^2} \\ &\le 2\,\mathrm{Re}\{p(z)\} \cdot \frac{|z|\left(1-|\psi(z)|^2\right)/\left(1-|z|^2\right)+|\psi(z)|}{1-|z\psi(z)|^2} \\ &= \frac{2\mathrm{Re}\{p(z)\}}{1-|z|^2} \cdot \frac{|\psi(z)|+|z|}{1+|z|\,|\psi(z)|} . \end{aligned}$$

The second last inequality follows from Carathéodory's inequality (see Carathéodory [1], p. 18). The function $\left(|\psi(z)|+|z|\right)/\left(1+|z|\,|\psi(z)|\right)$ is monotonically increasing with respect to $|\psi(z)|$; hence from (2.9) and (2.10) the result follows.

LEMMA 2.3. *If* $p(z) \in P_b$, $\mu > 0$, *then on* $|z| = r < 1$,

$$(2.11)\qquad \left|\frac{zp'(z)}{p(z)+\mu}\right| \le \frac{2r}{1-r^2} \cdot \frac{b+2r+br^2}{1+\mu+2br+(1-\mu)r^2} .$$

Proof. For $\mu > 0$, we have

$$\begin{aligned} \left|\frac{zp'(z)}{p(z)+\mu}\right| &\le \frac{|zp'(z)|}{\mathrm{Re}\{p(z)\}+\mu} = \frac{|zp'(z)|}{\mathrm{Re}\{p(z)\}} \cdot \frac{1}{1+\mu/\mathrm{Re}\{p(z)\}} \\ &\le \frac{2r}{1-r^2} \cdot \frac{b+2r+br^2}{1+2br+r^2}\left\{1+\frac{\mu\left(1-r^2\right)}{1+2br+r^2}\right\}^{-1} , \text{ from (2.8) and (2.2)} \\ &= \frac{2r}{1-r^2} \cdot \frac{b+2r+br^2}{1+\mu+2br+(1-\mu)r^2} . \end{aligned}$$

Equality in (2.11) is attained for the function

$$p(z) = \left(1+2bz+z^2\right)/\left(1-z^2\right) \quad \text{at} \quad z = r .$$

The next lemma establishes an inequality which involves the second derivative of $p(z)$. This is based on the well-known result that if $p(z) = 1 + p_1z + p_2z^2 + \ldots \in P$, then $|p_2| \le 2$. The bound also holds true for functions in P with a fixed first coefficient. Indeed, let

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$p(z) = 1 + 2bz + p_2z^2 + \ldots \in P_b$, $0 \le b \le 1$, then from the representation $p(z) = [1+z\psi(z)]/[1-z\psi(z)]$, where $\psi(z) = b + b_1z + \ldots$ and satisfies $|\psi(z)| \le 1$ in Δ , we get after equating the coefficients of the same powers of z ,

$$2b_1 = p_2 - 2b^2 . \tag{2.12}$$

It follows from Carathéodory's inequality that

$$b_1 \le 1 - |b|^2 .$$

Thus, in view of (2.12), we have

$$|p_2-2b|^2 \le 2 - 2b^2 ,$$

that is, $|p_2| \le 2$, which is sharp for the function

$$p(z) = \frac{1+2bz+z^2}{1-z^2} = 1 + 2bz + 2z^2 + \ldots .$$

Now let ξ be a complex number such that $0 < |\xi| < 1$ and $p(z) \in P_b$. Then the function $q(z)$ defined by

$$q(z) = p\left(\frac{z+\xi}{1+\bar{\xi}z}\right) = p(\xi) + \left(1-|\xi|^2\right)p'(\xi)z$$
$$+ \tfrac{1}{2}\left(1-|\xi|^2\right)\left[\left(1-|\xi|^2\right)p''(\xi)-2\bar{\xi}p'(\xi)\right]z^2 + \ldots$$

is regular and satisfies $\mathrm{Re}\{q(z)\} > 0$ in Δ . Hence from the above remark, the following lemma follows.

LEMMA 2.4. *If* $p(z) \in P_b$, *then for* $|z| < 1$,

$$\left|zp''(z) - \frac{2|z|^2}{1-|z|^2}\,p'(z)\right| \le \frac{4|z|}{\left(1-|z|^2\right)^2}\,|p(z)| . \tag{2.13}$$

In view of inequality (2.13) we get for $|\ \ | < 1$,

$$\left|\frac{z^2p''(z)}{p(z)+\mu}\right| \le \frac{2|z|^2}{1-|z|^2}\left|\frac{zp'(z)}{p(z)+\mu}\right| + \frac{4|z|^2}{\left(1-|z|^2\right)^2}\left|\frac{p(z)}{p(z)+\mu}\right| .$$

Thus an application of (2.11) and (2.5) to the right-hand side yields

LEMMA 2.5. *If* $p(z) \in P_b$, $\mu > 0$, *then on* $|z| = r < 1$,

$$\left|\frac{z^2p''(z)}{p(z)+\mu}\right| \le \frac{4r^2\left(1+3br+3r^2+br^3\right)}{\left(1-r^2\right)^2\left[1+\mu+2br+(1-\mu)r^2\right]} . \tag{2.14}$$

Equality occurs for the function $p(z) = \left(1+2bz+z^2\right)/\left(1-z^2\right)$ at $z = r$.

We are now in a position to prove the main result of this section.

THEOREM 2.6. *The radius of convexity of* $R_{\gamma a}$ *is given by the smallest root in* $(0, 1]$ *of the equation*

$$(1+\mu)^2 - 2b(1+\mu)r - 3(1+\mu)(5+\mu)r^2 - 4b(6+5\mu)r^3 - \left(1+2\mu-3\mu^2+16b^2\right)r^4 - 6b(1-\mu)r^5 - (1-\mu)^2r^6 = 0 ,$$

where $b = 2\gamma a/(2\gamma-1)$, $\mu = 1/(2\gamma-1)$.

Proof. As derived earlier in (2.4),

$$1 + \frac{zf''(z)}{f'(z)} = 1 - \frac{2zp'(z)}{p(z)+\mu} - \frac{z^2p''(z)}{p(z)+\mu-zp'(z)} ,$$

where $p(z) \in P_b$. Hence

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \ge 1 - \left|\frac{2zp'(z)}{p(z)+\mu}\right| - \left|\frac{z^2p''(z)}{p(z)+\mu}\right|\left|1 - \frac{zp'(z)}{p(z)+\mu}\right|^{-1} . \tag{2.15}$$

Now, from (2.11), we have

$$\begin{aligned} \left|1 - \frac{zp'(z)}{p(z)+\mu}\right| &\ge 1 - \left|\frac{zp'(z)}{p(z)+\mu}\right| \\ &\ge 1 - \frac{2r}{1-r^2} \cdot \frac{b+2r+br^2}{1+\mu+2br+(1-\mu)r^2} \\ &= \frac{1+\mu-2(2+\mu)r^2-4br^3-(1-\mu)r^4}{\left(1-r^2\right)\left[1+\mu+2br+(1-\mu)r^2\right]} . \end{aligned} \tag{2.16}$$

It is easy to check that the numerator has a root in $(0, 1)$. Let σ be its smallest root in $(0, 1)$; then for $|z| < \sigma$, we obtain

$$\left|1 - \frac{zp'(z)}{p(z)+\mu}\right|^{-1} \le \frac{\left(1-r^2\right)\left[1+\mu+2br+(1-\mu)r^2\right]}{1+\mu-2(2+\mu)r^2-4br^3-(1-\mu)r^4} . \tag{2.17}$$

Applying the bounds in (2.11), (2.14) and (2.17) to (2.15) we get

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \frac{F(r)}{\left[1+\mu+2br+(1-\mu)r^2\right]\left[1+\mu-2(2+\mu)r^2-4br^3-(1-\mu)r^4\right]} ,$$

where

$$F(r) = (1+\mu)^2 - 2b(1+\mu)r - 3(1+\mu)(5+\mu)r^2 - 4b(6+5\mu)r^3$$
$$- \left(1+2\mu-3\mu^2+16b^2\right)r^4 - 6b(1-\mu)r^5 - (1-\mu)^2r^6 .$$

Since $F(0) = (1+\mu)^2 > 0$, $F(1) = -16 - 16\mu - 16b\mu - 32b - 16b^2 < 0$, $F(r)$ has a root in $(0, 1)$. Denote its smallest root in $(0, 1)$ by ρ ; then the condition $\operatorname{Re}\{1+zf''(z)/f'(z)\} > 0$ is satisfied in $|z| < \min(\rho, \sigma)$. We further note that, for $f(z)$ as defined,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{1-\frac{zp'(z)}{p(z)+\mu}\right\} \geq 1 - \left|\frac{zp'(z)}{p(z)+\mu}\right| .$$

Thus, in view of (2.16), we have $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ in $|z| < \sigma$. In other words, $f(z)$ is starlike in $|z| < \sigma$. Since the radius of starlikeness of $f(z)$ is greater than or equal to its radius of convexity, we get $\rho \leq \sigma$ and the assertion follows.

To see that the result is sharp, we consider the function

$$f(z) = \frac{\gamma z\left(1-z^2\right)}{\gamma+(2\gamma-1)bz+(\gamma-1)z^2} .$$

The case $b = 1$, $\gamma \to \infty$ corresponds to the theorem of Reade, Ogawa and Sakaguchi [9].

3. Radius of convexity of the class $T_\gamma(G)$

We require the following lemmas:

LEMMA 3.1. *If* $w(z) \in B$, *then* $|w'(z)| \leq 1$ *for* $|z| \leq \sqrt{2}-1$.

LEMMA 3.2. *If* $p(z) \in P$, $\mu > 0$, *then on* $|z| = r < 1$,

$$\left|\frac{zp'(z)}{p(z)+\mu}\right| \leq \frac{2r}{(1-r)[1+\mu+(1-\mu)r]} , \tag{3.1}$$

$$\left|\frac{z^2p''(z)}{p(z)+\mu}\right| \leq \frac{4r^2}{(1-r)^2[1+\mu+(1-\mu)r]} . \tag{3.2}$$

A proof for Lemma 3.1, which is due to Dieudonné, may be found in Carathéodory [1], p. 19. Inequalities (3.1) and (3.2) are derived from

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(2.11) and (2.14) respectively by putting $b = 1$ in the latter results. Equalities in (3.1) and (3.2) occur for the function $p(z) = (1+z)/(1-z)$ at $z = r$.

LEMMA 3.3. *Let* $g(z) \in N$ *be such that* $|g'(z)-1| < 1$ *in* Δ. *Then on* $|z| = r$,

$$\left|\frac{z^2g''(z)}{g(z)}\right| \le \frac{2r}{2-r}, \text{ for } r < \sqrt{2}-1 \tag{3.3}$$

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{2(1-r)}{2-r}, \text{ for } r < \tfrac{1}{2}. \tag{3.4}$$

The results are sharp.

Proof. For $g(z)$ as defined, we have $g'(z) - 1 = w_1(z)$ for some $w_1(z) \in B$; hence, in view of Lemma 3.1,

$$|g''(z)| \le 1, \quad |z| \le \sqrt{2}-1. \tag{3.5}$$

Also, from Section 2 of MacGregor [6],

$$\left|\frac{g(z)}{z} - 1\right| \le \tfrac{1}{2}|z|;$$

hence we may write

$$g(z) = z + \tfrac{1}{2}zw_2(z), \quad w_2(z) \in B, \quad z \in \Delta. \tag{3.6}$$

This implies

$$|g(z)| \ge |z| - \tfrac{1}{2}|z|^2 \tag{3.7}$$

and (3.3) now follows from (3.5) and (3.7).

From the representation (3.6) and Dieudonné's Lemma (see Duren [2]) we get

$$\begin{aligned} \mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} &= 1 + \mathrm{Re}\left\{\frac{zw_2'(z)}{2+w_2(z)}\right\} \\ &\ge 1 + \mathrm{Re}\left\{\frac{w_2(z)}{2+w_2(z)}\right\} - \frac{r^2-|w_2(z)|^2}{(1-r^2)\,|2+w_2(z)|}. \end{aligned} \tag{3.8}$$

Put $2 + w_2(z) = Re^{i\theta}$ and denote the right-hand side of (3.8) by $S(R, \theta)$; then $2-r \le R \le 2+r$ and

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$$S(R,\ \theta) = 2 - \left(\frac{2}{R} + \frac{4}{1-r^2}\right)\cos\theta + \frac{4-r^2}{1-r^2} \cdot \frac{1}{R} + \frac{R}{1-r^2} .$$

Since $\partial S/\partial\theta = \sin\theta T(R)$ and

$$T(R) = \frac{2}{R} + \frac{4}{1-r^2} > 0 ,$$

the minimum of $S(R,\ \theta)$ occurs when $\theta = 0$ and $R \in [2-r,\ 2+r]$. Now

$$S(R,\ 0) = \frac{1}{1-r^2}\left[-2\left(1+r^2\right)+\left(2+r^2\right)\frac{1}{R} + R\right] ,$$

which yields $dS(R,\ 0)/dR = 0$ at $R = \left(2+r^2\right)^{\frac{1}{2}}$. This point is outside the range of values of R if $\left(2+r^2\right)^{\frac{1}{2}} < 2 - r$, that is, if $r < \frac{1}{2}$. Thus, for $r < \frac{1}{2}$, the minimum of $S(R,\ 0)$ is attained at the end-point $R = 2 - r$, its value being

$$S(2-r,\ 0) = \frac{2(1-r)}{2-r} .$$

The sharpness of both results is easily verified for the function $g_0(z) = z + z^2/2$.

We now prove the main result of this section.

THEOREM 3.4. *The radius of convexity of* $T_\gamma(G)$ *is given by the only root in* $(0,\ \frac{1}{4})$ *of the equation*

$$4(1-\gamma)^2r^4 - (1-\gamma)(3+8\gamma)r^3 + 9\gamma r^2 - 2\gamma(7\gamma-1)r + 2\gamma^2 = 0 .$$

Proof. In view of (1.2), we get for $f(z) \in T_\gamma(G)$,

$$\text{(3.9)}\quad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}$$

$$= 1 - \operatorname{Re}\left\{\left[\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}\right]\left[\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\right]^{-1} - \frac{2zp'(z)}{p(z)+\mu}\right\}$$

$$\geq 1 - \left|\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}\right|\left|\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\right|^{-1} - \left|\frac{2zp'(z)}{p(z)+\mu}\right|$$

$$\geq 1 - \left|\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}\right|\left[\operatorname{Re}\left\{\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\right\}\right]^{-1} - \left|\frac{2zp'(z)}{p(z)+\mu}\right|$$

provided that $\operatorname{Re}\left\{\left(zg'(z)/g(z)\right)-\left(zp'(z)/[p(z)+\mu]\right)\right\} > 0$. From (3.4) and

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(3.1) we have

$$\text{Re}\left\{\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\right\} \geq \frac{2[\mu+1-3(\mu+1)r+3\mu r^2+(1-\mu)r^3]}{(1-r)(2-r)[1+\mu+(1-\mu)r]} . \tag{3.10}$$

It is easy to check that the numerator has a single root in $(0, 1)$; furthermore, this root is located in $(\frac{1}{4}, \frac{1}{2})$. Thus the right-hand side of (3.10) is positive for $r < \frac{1}{4}$. This fact together with (3.1), (3.2), (3.3) and (3.10) applied to (3.9) will give, for $r < \frac{1}{4}$,

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \frac{G(r)}{\{(1-r)^2[1+\mu+(1-\mu)r]-r(2-r)\}[1+\mu+(1-\mu)r]} ,$$

where

$$G(r) = 2(\mu-1)^2r^4 + (1-\mu)(7\mu+4)r^3 + 9\mu(1+\mu)r^2 - (1+\mu)(5\mu+7)r + (1+\mu)^2 .$$

Now $G(0) = (1+\mu)^2$, $G(\frac{1}{4}) = \left(27\mu^2-44\mu-87\right)/128 < 0$ for $0 < \mu \leq 1$. Thus $G(r)$ has a zero, which is unique, in $(0, \frac{1}{4})$. The proof of the theorem is therefore completed.

The result is sharp for the function

$$f(z) = \frac{(1+z)\left(z+\left(z^2/2\right)\right)}{1+\left((1/\gamma)-1\right)z} .$$

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