BULL. AUSTRAL. MATH. SOC. 47D20, 52A41, 46B20
VOL. 50 (1950) [481–499]

REGULARISATIONS OF CONVEX FUNCTIONS AND SLICEWISE SUPREMA

S. SIMONS

For a number of years, there has been interest in the *regularisation* of a given proper convex lower semicontinuous function on a Banach space, defined to be the episum (=inf-convolution) of the function with a scalar multiple of the norm. There is an obvious geometric way of characterising this regularisation as the *lower envelope* of cones lying *above* the graph of the original function. In this paper, we consider the more interesting problem of characterising the regularisation in terms of approximations from *below*, expressing the regularisation as the *upper envelope* of certain *subtangents* to the graph of the original function. We shall show that such an approximation is sometimes (but not always) valid. Further, we shall give an extension of the whole procedure in which the scalar multiple of the norm is replaced by a more general sublinear functional. As a by-product of our analysis, we are led to the consideration of two senses stronger than the pointwise sense in which a function on a Banach space can be expressed as the upper envelope of a family of functions. These new senses of suprema lead to some questions in Banach space theorey.

1. INTRODUCTION

Let E be a real Banach space with adjoint E^*. If $h, k : E \to \mathbb{R} \cup \{\infty\}$ are proper and convex, we define the *episum* (or *inf-convolution*) of h and k by

$$\Big(h \underset{e}{+} k\Big)(x) := \inf_{y \in E}\Big(h(y) + k(x-y)\Big).$$

(By saying that h is *proper* we mean that $\operatorname{dom}(h) := \big\{x : x \in E,\ h(x) \in \mathbb{R}\big\} \neq \emptyset$.) For a number of years, there has been some interest in the *Baire-Wijsman- Hausdorff-Pasch regularisation* of a given proper convex lower semicontinuous function $f : E \to \mathbb{R} \cup \{\infty\}$, defined for $n \geqslant 1$ to be $f \underset{e}{+} n\|\ \|$. In more recent years, Hiriart-Urruty conducted a systematic investigation of the regularisation in [7]. In [6], Fitzpatrick and Phelps used the regularisation to motivate their approximation scheme for locally maximal monotone operators which (by contrast with the Moreau-Yosida scheme) is valid in non-reflexive spaces. They also gave a history of the regularisation. Finally,

Received 16th February, 1994

Beer in [2] and Borwein and Vanderwerff in [3] discussed the connection between the regularisation and epigraphical convergence. There is an obvious way of characterising the regularisation in terms of cones lying above the graph of f, which we discuss later on in this introduction. More interesting is the problem of characterising the regularisation in terms of approximations from below, expressing $f \underset{e}{+} n\|\ \|$ as the upper envelope of certain subtangents to the graph of f. We shall see that such an approximation is sometimes (but not always) valid. Further, we shall give an extension of the whole procedure in which $n\|\ \|$ is replaced by a more general sublinear functional. As a by-product of our analysis, we are led to the consideration of two senses stronger than the pointwise sense in which a function on E can be expressed as the upper envelope of a family of functions. We call these senses "graphwise supremum" and "slicewise supremum".

We shall suppose in the discussion that follows that there exists $x \in E$ such that $\left(f \underset{e}{+} n\|\ \|\right)(x) > -\infty$ or, equivalently (see Lemma 6) that $f \underset{e}{+} n\|\ \| : E \to \mathbb{R}$. It is immediate from the definitions that $f \underset{e}{+} n\|\ \|$ can be described in terms of its strict epigraph as follows: let K be the open cone $\left\{(y,\lambda) : y \in E,\ \lambda \in \mathbb{R},\ n\|y\| < \lambda\right\}$. Then

$$\left\{(y,\lambda) : y \in E,\ \lambda \in \mathbb{R}, \left(f \underset{e}{+} n\|\ \|\right)(y) < \lambda\right\} = \bigcup_{x \in \operatorname{dom}(f)} \left\{(x, f(x)) + K\right\},$$

that is to say, the strict epigraph of $f \underset{e}{+} n\|\ \|$ is obtained by sliding the vertex of the cone K along the graph of f and then taking the lower envelope. Another potential way of visualising $f \underset{e}{+} n\|\ \|$ is motivated by the fact that f itself is the upper envelope of its subtangents. In order to explain this, we shall need to introduce some more notation. If $x \in E$, the *subdifferential* of f at x is defined by

$$\partial f(x) := \left\{x^* : x^* \in E^*,\ \text{for all } y \in E, f(x) + \langle y - x, x^*\rangle \leqslant f(y)\right\}.$$

We write $\partial f := \left\{(x, x^*) : x \in E,\ x^* \in \partial f(x)\right\} \subset E \times E^*$. If $(x, x^*) \in \partial f$, we write $\sigma f(x, x^*)$ for the *subtangent to* f at $(x, f(x))$ with slope x^*,that is to say

$$\text{for all } y \in E, \quad \sigma f(x, x^*)(y) := f(x) + \langle y - x, x^*\rangle.$$

The "fact" referred to above is then that

$$f = \sup\left\{\sigma f(z, z^*) : (z, z^*) \in \partial f\right\} \tag{0.1}$$

and the corresponding way of visualising $f \underset{e}{+} n\|\ \|$ is then that

$$f \underset{e}{+} n\|\ \| = \sup\left\{\sigma f(z, z^*) : (z, z^*) \in \partial f,\ \|z^*\| \leqslant n\right\}. \tag{0.2}$$

We first give an example where the formula (0.2) fails.

EXAMPLE 1. Let $E := \mathbb{R}$ and $f(x) := e^x + x$. Then $\left(f \underset{e}{+} \| \|\right)(x) = x$ but, since there exists no $(z, z^*) \in \partial f$ such that $|z^*| \leqslant 1$, the supremum in (0.2) is $-\infty$.

It will follow from Corollaries 9 and 14 that the formula (0.2) is true if f is bounded below. The next example shows that (0.2) may be true even if f is not bounded below.

EXAMPLE 2. Let $E := \mathbb{R}$ and $f(x) := x$. Then we have equality in (0.2).

2. GRAPHWISE SUPREMA

The suprema in (0.1) and (0.2) are, of course, to be interpreted in the pointwise sense. There are, however, stronger senses in which a given function $g : E \to \mathbb{R} \cup \{\infty\}$ can be the "supremum" of a family $\{s_\omega\}_{\omega \in \Omega}$ where, for each $\omega \in \Omega$, $s_\omega : E \to \mathbb{R} \cup \{\infty\}$ and $s_\omega \leqslant g$ on E. We now discuss two of these senses. We say that an extended real-valued function on E is *boxed above* if its domain is bounded and it is bounded above by an element of $\mathbb{R}$. We shall say that

$$g = \sup_{\omega \in \Omega} s_\omega \textit{ graphwise}$$

if, for each proper, concave, boxed above function $b : E \to \mathbb{R} \cup \{-\infty\}$ such that $\inf_E [g - b] > 0$, there exists $\omega \in \Omega$ such that $\inf_E [s_\omega - b] > 0$. It follows easily that if $g = \sup_{\omega \in \Omega} s_\omega$ graphwise then $g = \sup_{\omega \in \Omega} s_\omega$ pointwise. There are trivial examples that show that the converse of this fails. Let $E = \mathbb{R}$, $s(x) := x$ and $t(x) := -x$. Then $|\ |$ is the pointwise but not the graphwise supremum of s and t. In Theorem 17, we characterise (for E a general Banach space) those nonempty bounded *convex* subsets Ω of E^* for which the pointwise supremum is actually a graphwise supremum and, in Remark 18, we consider which Banach spaces have the property that the pointwise supremum of every bounded convex subset of E^* is a graphwise supremum.

3. SLICEWISE SUPREMA

If A and B are nonempty subsets of $E \times \mathbb{R}$, we say that A is *separated from* B if the distance between A and B (with respect to any norm on $E \times \mathbb{R}$ that gives the product topology) is strictly positive. We use "epi(f)" to stand for the epigraph of f, the set of points in $E \times \mathbb{R}$ that lie on or above the graph of f and "hypo(f)" to stand for the hypograph of f, the set of points in $E \times \mathbb{R}$ that lie on or below the graph of f. Let $g : E \to \mathbb{R} \cup \{\infty\}$ and, for each $\omega \in \Omega$, $s_\omega : E \to \mathbb{R} \cup \{\infty\}$ and $s_\omega \leqslant g$ on E. We shall say that

$$g = \sup_{\omega \in \Omega} s_\omega \textit{ slicewise}$$

if, for each proper, concave, boxed above function $b : E \to \mathbb{R} \cup \{-\infty\}$ such that epi(g) is separated from hypo(b), there exists $\omega \in \Omega$ such that epi(s_ω) is separated from hypo(b). In Theorem 16, we give other characterisations of slicewise suprema.

The concepts of graphwise and slicewise supremum are related by the following result.

Lemma 3. *Let $g : E \to \mathbb{R}$ and, for each $\omega \in \Omega$, $s_\omega : E \to \mathbb{R}$ and $s_\omega \leqslant g$ on E. Suppose that g and all the s_ω are Lipschitz. Then*

$$g = \sup_{\omega\in\Omega} s_\omega \text{ graphwise} \quad \iff \quad g = \sup_{\omega\in\Omega} s_\omega \text{ slicewise.}$$

Proof: $(\Longrightarrow)$ Suppose that b is as above, and epi(g) is separated from hypo(b). Then $\inf_E[g-b] > 0$. By hypothesis, there exists $\omega \in \Omega$ such that $\delta := \inf_E[s_\omega - b] > 0$. Let M be strictly greater than the Lipschitz constant of s_ω. If $(x, \lambda) \in$ epi(s_ω), $(y, \mu) \in$ hypo(b) and $\|x - y\| \leqslant \delta/2M$ then

$$|\lambda - \mu| \geqslant \lambda - \mu \geqslant s_\omega(x) - b(y) \geqslant s_\omega(y) - b(y) - M\,\|x - y\| \geqslant \delta - \frac{\delta}{2} = \frac{\delta}{2},$$

hence epi(s_ω) is separated from hypo(b). Thus $g = \sup_{\omega\in\Omega} s_\omega$ graphwise.

$(\Longleftarrow)$ The proof of this is similar to the proof of $(\Longrightarrow)$, only using the Lipschitz property of g. $\square$

4. Regularisation of convex functions and slicewise suprema

The definition of "slicewise supremum" is motivated by a result of Beer (see [**1**, Lemma 4.10]) which implies that (0.1) holds slicewise. These observations leads naturally to the question whether there is a corresponding strengthening of (0.2). We shall prove in the separation form of Theorem 8 that there is such a strengthening which is, in fact, true for sublinear functionals T more general than $n\|\ \|$. (We refer the reader to the statement of the episum form of Theorem 8 for the definition of $\ll$.) Theorems 7 and 8 form the central part of the analysis of this paper. We have stated both of them in an "episum form" and a "separation form". The separation form is the most convenient for applications, while the somewhat more obscure episum form seems to be the most convenient for computation, and enables us to exploit directly the associativity property of $\underset{e}{+}$. (The computational device contained in the proof of Theorem 7 is motivated by [**10**, Theorem 4.4]. In Theorem 11, we give a geometric form of the ideas of Theorem 7, which we phrase in terms of cones. In Theorem 12, we bootstrap Theorem 8 into the following generalisation: let C^* be a nonempty weak*-compact convex subset of E^* such that dom $f^* \cap$ int$(C^*) \neq \emptyset$. For all $x \in E$, let $U(x) := \max\langle x, C^*\rangle$. Then

$$f \underset{e}{+} U = \sup\Big\{\sigma f(z, z^*) : (z, z^*) \in \partial f,\ z^* \in \text{int}(C^*)\Big\} \text{ slicewise and graphwise.}$$

5. Results

Let $g : E \to \mathbb{R} \cup \{\infty\}$ be a proper, convex, lower semicontinuous function. If $z \in \operatorname{dom}(g)$ and $v \in E$ then we write

$$\mathrm{d}^+g(z)(v) := \lim_{\theta \to 0+} \frac{g(z+\theta v) - g(z)}{\theta}.$$

$\mathrm{d}^+g(z)(v)$ is the *directional derivative* of g at z in the direction v. Since the above limit can be replaced by an infimum, it follows that,

$$\text{(3.1)} \qquad \text{for all } z \in \operatorname{dom}(g) \text{ and } v \in E, \quad g(z+v) \geqslant \mathrm{d}^+g(z)(v) + g(z).$$

Lemma 4. *Let g be as above, $\beta > 0$ and g be bounded below. Then there exists $z \in \operatorname{dom}(g)$ such that, for all $v \in E$, $\mathrm{d}^+g(z)(v) \geqslant -\beta\,\|v\|$.*

Proof: From Ekeland's variational principle, (see [**5**, Theorem 1, p.444]), there exists $z \in \operatorname{dom}(g)$ such that, for all $w \in E$, $g(w) \geqslant g(z) - \beta\,\|z - w\|$. The required result follows from the definition of $\mathrm{d}^+g(z)(v)$ by putting $w := z + \theta v$ and letting $\theta \to 0+$. ∎

We now give two computational properties of $\underset{e}{+}$ which do not depend on the norm of E.

Lemma 5. *Let $g : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and bounded below, and $T : E \to [0,\infty)$ be convex. Then*

$$\Big(g \underset{e}{+} \eta T\Big)(0) \to \Big(g \underset{e}{+} T\Big)(0) \text{ as } \eta \to 1-.$$

Proof: Clearly $g \underset{e}{+} T$ is real-valued and bounded below (by $\inf_E g$). Let $\pi := \Big(g \underset{e}{+} T\Big)(0)$. For all $\eta \in (0,1)$, $\Big(g \underset{e}{+} \eta T\Big)(0) \leqslant \pi$. Let $\mu \in (-\infty, \pi)$. We shall complete the proof of the Lemma by finding $\eta \in (0,1)$ such that

$$\text{(5.1)} \qquad \Big(g \underset{e}{+} \eta T\Big)(0) \geqslant \mu.$$

If $\mu \leqslant \inf_E g$ then (5.1) follows with any $\eta \in (0,1)$. Suppose, on the other hand, that $\mu > \inf_E g$. Let

$$\eta := \frac{\mu - \inf_E g}{\pi - \inf_E g} \in (0,1).$$

For all $y \in \operatorname{dom}(g)$, $g(y) + T(-y) \geqslant \Big(g \underset{e}{+} T\Big)(0) = \pi$, hence

$$g(y) + \eta T(-y) \geqslant g(y) + \eta\big(\pi - g(y)\big) = (1-\eta)g(y) + \eta\pi \geqslant (1-\eta)\inf_E g + \eta\pi = \mu.$$

(5.1) now follows by taking the infimum over y. ☐

Lemma 6. *Let $g : E \to \mathbb{R} \cup \{\infty\}$ be proper and convex, $V : E \to \mathbb{R}$ be sublinear, and suppose that there exists $x \in E$ such that $\left(g \underset{e}{+} V\right)(x) > -\infty$. Then $g \underset{e}{+} V : E \to \mathbb{R}$ and,*

$$\text{for all } u, v \in E, \quad \left(g \underset{e}{+} V\right)(u) - \left(g \underset{e}{+} V\right)(v) \leqslant V(u - v).$$

Proof: Let $y, z \in E$. Then

$$g(y) + V(z - y) \geqslant g(y) + V(x - y) - V(x - z) \geqslant \left(g \underset{e}{+} V\right)(x) - V(x - z).$$

Taking the infimum over y,

$$\left(g \underset{e}{+} V\right)(z) \geqslant \left(g \underset{e}{+} V\right)(x) - V(x - z) > -\infty.$$

The results follows easily from this. ☐

We say that an extended real-valued function on E is *boxed below* if its domain is bounded and it is bounded below by an element of $\mathbb{R}$.

Theorem 7. **(Episum form)** *Let $h : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous, and $k : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and boxed below. Let $T : E \to \mathbb{R}$ be a sublinear functional such that,*

$$\text{(7.1)} \qquad \text{for some } M > m > 0, \; m\|\ \| \leqslant T \leqslant M\|\ \| \text{ on } E.$$

(That is to say, T is continuous and coercive.) Suppose, further, that

$$\left(h \underset{e}{+} k \underset{e}{+} T\right)(0) > \nu > -\infty$$

and $\theta > 1$. Then

$$\text{(7.2)} \qquad \text{there exists } z \in \operatorname{dom}(h) \text{ such that } h(z) + \left(\, d^+h(z) \underset{e}{+} k \underset{e}{+} \theta T\right)(-z) \geqslant \nu$$

from which it follows that

$$\text{(7.3)} \qquad \text{there exists } z^* \in \partial h(z) \text{ such that } z^* \leqslant \theta T \text{ on } E \text{ and } \left(\sigma h(z, z^*) \underset{e}{+} k\right)(0) \geqslant \nu.$$

Proof: Choose $\alpha > 0$ such that

$$\text{(7.4)} \qquad \left(h \underset{e}{+} k \underset{e}{+} T\right)(0) \geqslant \nu + \alpha.$$

and $\delta > 0$ such that $1 + 2\delta \leqslant \theta$ and δm diam$\big(\mathrm{dom}\,(k)\big) \leqslant \alpha$. Define

$$W(y) := \Big(k \underset{e}{+} (1+\delta)T\Big)(-y) \quad (y \in E).$$

From (7.1) and Lemma 6 with $g := k$ and $V := (1+\delta)T$, $W : E \to \mathbb{R}$, W is $M(1+\delta)$-Lipschitz, and thus W is continuous. Further, from (7.4),

$$\inf_E (h + W) = \Big(h \underset{e}{+} k \underset{e}{+} (1+\delta)T\Big)(0) \geqslant \nu + \alpha > -\infty.$$

Thus, from Lemma 4 with $g := h + W$ and $\beta := \delta m > 0$, there exists $z \in \mathrm{dom}\,(h + W) = \mathrm{dom}\,(h)$ such that,

$$\text{for all } v \in E,\ \mathrm{d}^+(h + W)(z)(v) \geqslant -\delta m \,\|v\|. \tag{7.5}$$

Let $y \in E$. Since $\theta \geqslant 1 + 2\delta$, for all $x \in \mathrm{dom}\,(k)$,

$$\begin{aligned} k(x) + \theta T(-y - x) &\geqslant k(x) + (1+\delta)T(-y-x) + \delta T(-y-x) \\ &\geqslant \Big(k \underset{e}{+} (1+\delta)T\Big)(-y) + \delta m D(y), \end{aligned}$$

where $D(y)$ stands for the distance from $-y$ to $\mathrm{dom}\,(k)$. Taking the infimum over $x \in \mathrm{dom}\,(k)$ and using the definition of W, we obtain

$$\Big(k \underset{e}{+} \theta T\Big)(-y) \geqslant W(y) + \delta m D(y). \tag{7.6}$$

Using an exactly analogous argument, we can prove that

$$W(z) := \Big(k \underset{e}{+} (1+\delta)T\Big)(-z) \geqslant \Big(k \underset{e}{+} T\Big)(-z) + \delta m D(z). \tag{7.7}$$

We now prove that (7.2) holds with the value of z chosen above. Let $v \in E$. From (7.6) with $y := z + v$,

$$h(z) + \mathrm{d}^+ h(z)(v) + \Big(k \underset{e}{+} \theta T\Big)(-z - v) \geqslant h(z) + \mathrm{d}^+ h(z)(v) + W(z + v) + \delta m D(z + v);$$

from (3.1) with $g := W$,

$$\begin{aligned} &\geqslant h(z) + \mathrm{d}^+ h(z)(v) + \mathrm{d}^+ W(z)(v) + W(z) + \delta m D(z + v) \\ &\qquad\qquad = h(z) + \mathrm{d}^+(h + W)(z)(v) + W(z) + \delta m D(z + v); \end{aligned}$$

from (7.5),

$$\geqslant h(z) - \delta m \,\|v\| + W(z) + \delta m D(z + v) = h(z) + W(z) + \delta m D(z + v) - \delta m \,\|v\|;$$

from (7.7),

$$\geqslant h(z) + \left(k \underset{e}{+} T\right)(-z) + \delta m D(z) + \delta m D(z+v) - \delta m \|v\|.$$

Thus we have proved that

$$\text{(7.8)} \quad h(z) + \mathrm{d}^+ h(z)(v) + \left(k \underset{e}{+} \theta T\right)(-z-v) \\ \geqslant \left[h(z) + \left(k \underset{e}{+} T\right)(-z)\right] + \delta m\left[D(z) + D(z+v) - \|v\|\right].$$

From (7.4),

$$\text{(7.9)} \qquad h(z) + \left(k \underset{e}{+} T\right)(-z) \geqslant \left(h \underset{e}{+} k \underset{e}{+} T\right)(0) \geqslant \nu + \alpha.$$

Furthermore, by direct computation,

$$\text{(7.10)} \qquad D(z) + D(z+v) - \|v\| \geqslant -\operatorname{diam}\left(\operatorname{dom}(k)\right).$$

Thus, since $\delta m \operatorname{diam}\left(\operatorname{dom}(k)\right) \leqslant \alpha$, we can substitute (7.9) and (7.10) into (7.8) and obtain:

$$h(z) + \mathrm{d}^+ h(z)(v) + \left(k \underset{e}{+} \theta T\right)(-z-v) \geqslant (\nu + \alpha) - \alpha = \nu.$$

Since this holds for all $v \in E$, this establishes that

$$h(z) + \left(\mathrm{d}^+ h(z) \underset{e}{+} k \underset{e}{+} \theta T\right)(-z) \geqslant \nu,$$

which is (7.2). Consequently, for all $x \in E$,

$$\left(\mathrm{d}^+ h(z) \underset{e}{+} \theta T\right)(x) \geqslant \nu - h(z) - k(-x-z).$$

Since k is proper, from Lemma 6 with $g := \mathrm{d}^+ h(z)$ and $V := \theta T$,

$$\mathrm{d}^+ h(z) \underset{e}{+} \theta T : E \to \mathbb{R}.$$

We write $S := \mathrm{d}^+ h(z) \underset{e}{+} \theta T$, and define $f : E \to \mathbb{R} \cup \{-\infty\}$ by $f(x) := \nu - h(z) - k(-x-z)$. It now follows from a routine computation using infima that S is a sublinear functional. Since f is concave, from the sandwich theorem (see König [**8**, Theorem 1.7, p.112]), there exists a linear functional z^* on E such that

$$f \leqslant z^* \leqslant S \text{ on } E.$$

https://doi.org/10.1017/S0004972700013599 Published online by Cambridge University Press

(The existence of z^* satisfying the above inequality can also be deduced from the Eidelheit separation theorem in $E \times \mathbb{R}$.) Since $z^* \leqslant S$ on E we have both $z^* \leqslant \theta T$ on E and $z^* \leqslant \mathrm{d}^+h(z)$ on E. The first of these inequalities implies that z^* is continuous and, combining this with the second, we obtain that $(z, z^*) \in \partial h$. Let $y \in E$. Then, since $f \leqslant z^*$ on E, $f(y - z) \leqslant \langle y - z, z^* \rangle$, which can be rewritten $\sigma h(z, z^*)(y) + k(-y) \geqslant \nu$. Taking the infimum over $y \in E$, we obtain that

$$\left(\sigma h(z, z^*) \underset{e}{+} k\right)(0) \geqslant \nu.$$

This establishes (7.3), and hence completes the proof of the theorem. ∎

Theorem 7. (Separation form) *Let $h : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex, and lower semicontinuous, $T : E \to \mathbb{R}$ be a sublinear functional satisfying (7.1), and $\theta > 1$. Let $b : E \to \mathbb{R} \cup \{-\infty\}$ be proper, concave and boxed above, and $\inf_E\left[\left(h \underset{e}{+} T\right) - b\right] > 0$. Then there exists $(z, z^*) \in \partial h$ such that $z^* \leqslant \theta T$ on E and $\inf_E\left[\sigma h(z, z^*) - b\right] > 0$.*

Proof: We define $k := E \to \mathbb{R} \cup \{\infty\}$ by $k(x) := -b(-x)$. It follows that $\left(h \underset{e}{+} k \underset{e}{+} T\right)(0) > 0$. From the episum form of the theorem, there exists $(z, z^*) \in \partial h$ such that $z^* \leqslant \theta T$ on E and $\left(\sigma h(z, z^*) \underset{e}{+} k\right)(0) > 0$. The result follows since $\inf_E[\sigma h(z, z^*) - b] = \left(\sigma h(z, z^*) \underset{e}{+} k\right)(0)$. ∎

Theorem 8. (Episum form) *Let $h : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex, lower semicontinuous and bounded below, $k : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and boxed below, and $T : E \to \mathbb{R}$ be a sublinear functional satisfying (7.1). Then*

$$\begin{aligned}\left(h \underset{e}{+} k \underset{e}{+} T\right)(0) &= \sup\left\{\left(\sigma h(z, z^*) \underset{e}{+} k\right)(0) : (z, z^*) \in \partial h,\ z^* \leqslant T \text{ on } E\right\} \\ &= \sup\left\{\left(\sigma h(z, z^*) \underset{e}{+} k\right)(0) : (z, z^*) \in \partial h,\ z^* \ll T \text{ on } E\right\},\end{aligned}$$

where "$z^* \ll T$ on E" means that there exists $\zeta \in (0, 1)$ such that $z^* \leqslant \zeta T$ on E.

Proof: Let $(z, z^*) \in \partial h$ and $z^* \leqslant T$ on E. Let $x, y \in E$. Then

$$\sigma h(z, z^*)(x) + k(-x) = \sigma h(z, z^*)(y) + \langle x - y, z^* \rangle + k(-x) \leqslant h(y) + T(x - y) + k(-x).$$

Taking the infimum over x and y, $\left(\sigma h(z, z^*) \underset{e}{+} k\right)(0) \leqslant \left(h \underset{e}{+} k \underset{e}{+} T\right)(0)$. Consequently,

$$\left(h \underset{e}{+} k \underset{e}{+} T\right)(0) \geqslant \sup\left\{\left(\sigma h(z, z^*) \underset{e}{+} k\right)(0) : (z, z^*) \in \partial h,\ z^* \leqslant T \text{ on } E\right\}.$$

https://doi.org/10.1017/S0004972700013599 Published online by Cambridge University Press

We complete the proof of the theorem by showing that

$$\sup\Big\{\Big(\sigma h(z,z^*) \underset{e}{+} k\Big)(0) : (z,z^*) \in \partial h,\ z^* \ll T \text{ on } E\Big\} \geqslant \Big(h \underset{e}{+} k \underset{e}{+} T\Big)(0).$$

Let $-\infty < \nu < \Big(h \underset{e}{+} k \underset{e}{+} T\Big)(0)$. From Lemma 5 with $g := h \underset{e}{+} k$ (which is bounded below since h and k are), there exists $\eta \in (0,1)$ such that $\Big(h \underset{e}{+} k \underset{e}{+} \eta T\Big)(0) > \nu$. The result follows from the episum form of Theorem 7 with T replaced by ηT and any $\zeta \in (\eta\ ,1)$. $(\theta := \zeta/\eta.)$ ∎

Theorem 8. (Separation form) *Let $h : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex, lower semicontinuous and bounded below, and $T : E \to \mathbb{R}$ be a sublinear functional satisfying (7.1). Then*

$$h \underset{e}{+} T = \sup\Big\{\sigma h(z,z^*) : (z,z^*) \in \partial h,\ z^* \ll T \text{ on } E\Big\} \text{ slicewise and graphwise.}$$

Proof: Let $b : E \to \mathbb{R} \cup \{-\infty\}$ be proper, concave and boxed above, and $\inf_E\Big[\Big(h \underset{e}{+} T\Big) - b\Big] > 0$. We define $k := E \to \mathbb{R} \cup \{\infty\}$ by $k(x) := -b(-x)$. It follows that $\Big(h \underset{e}{+} k \underset{e}{+} T\Big)(0) > 0$. From the episum form of the theorem, there exists $(z,z^*) \in \partial h$ such that $z^* \ll T$ on E and $\Big(\sigma h(z,z^*) \underset{e}{+} k\Big)(0) > 0$. The result follows since $\inf_E\big[\sigma h(z,z^*) - b\big] = \Big(\sigma h(z,z^*) \underset{e}{+} k\Big)(0)$. ∎

Corollary 9. *Let $h : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex, lower semicontinuous and bounded below, and $T : E \to \mathbb{R}$ be a sublinear functional satisfying (7.1). Then*

$$\begin{aligned} h \underset{e}{+} T &= \sup\Big\{\sigma h(z,z^*) : (z,z^*) \in \partial h,\ z^* \leqslant T \text{ on } E\Big\} \\ &= \sup\Big\{\sigma h(z,z^*) : (z,z^*) \in \partial h,\ z^* \ll T \text{ on } E\Big\}. \end{aligned}$$

Remark 10. Let $E := \mathbb{R}, h(x) := e^x$ and $T := 0$. Then $h \underset{e}{+} T = 0$, but (since $z^* \leqslant T$ implies that $z^* = 0$) there exists no $(z,z^*) \in \partial h$ such that $z^* \leqslant T$ on E. So Corollary 9 fails if T is only required to be continuous and we do not assume the coercivity condition.

We now give a geometric version of the ideas of Theorem 7, which we shall phrase in terms of cones. If $\theta > 0$, we write K_θ for the open cone $\{(y,\lambda) : y \in E,\ \lambda \in \mathbb{R},\ \theta T(y) < \lambda\}$ (so that, if $T = n\|\ \|$, then K_1 is the set K defined in the introduction). There is also a version in which (q,ρ) is replaced by a nonempty bounded closed convex subset of $E \times \mathbb{R}$ – the statement is somewhat more complicated since $\mathrm{diam}\big(\mathrm{dom}(k)\big) > 0$ in this case.

THEOREM 11. *Let $h : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous, $T : E \to \mathbb{R}$ be a sublinear functional satisfying (7.1), $(q, \rho) \in E \times \mathbb{R}$, $(q, \rho) - K_1$ be disjoint from the graph of h, and $\theta > 1$. Then there exists $(z, z^*) \in \partial h$ such that $z^* \leqslant \theta T$ on E and $(q, \rho) - K_\theta$ is disjoint from $\operatorname{epi}(\sigma h(z, z^*))$.*

PROOF: Our assumptions imply that, for all $y \in E$, $(q-y, h(q-y)) \notin (q, \rho) - K_1$, that is,

$$\text{for all } y \in E, \quad h(q-y) + T(y) - \rho \geqslant 0. \tag{11.1}$$

We define k by $k(-q) := -\rho$ and $k := \infty$ otherwise. Then (11.1) can be rewritten

$$\left(h \underset{e}{+} k \underset{e}{+} T\right)(0) \geqslant 0.$$

We now proceed as in the proof of the episum form of Theorem 7, but with $\alpha = 0$. This is permissible since $\operatorname{diam}(\operatorname{dom}(k)) = 0$. From (7.3),

$$\text{there exists } z^* \in \partial h(z) \text{ such that } z^* \leqslant \theta T \text{ on } E \text{ and } \left(\sigma h(z, z^*) \underset{e}{+} k\right)(0) \geqslant 0,$$

from which $\sigma h(z, z^*)(q) \geqslant \rho$. If now $(x, \lambda) \in (q, \rho) - K_\theta$ then $\theta T(q - x) < \rho - \lambda$. Thus

$$\lambda < \rho - \theta T(q-x) \leqslant \sigma h(z, z^*)(q) - \langle q - x,\, x^* \rangle = \sigma h(z, z^*)(x),$$

as required. ∎

THEOREM 12. *Let $f : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous and C^* be a nonempty weak*-compact convex subset of E^*. For all $x \in E$, let $U(x) := \max\langle x, C^* \rangle$. U is a continuous (but not necessarily positive) sublinear functional on E.*

(a) *If there exists $x \in E$ such that $\left(f \underset{e}{+} U\right)(x) > -\infty$ and*

$$f \underset{e}{+} U = \sup\left\{\sigma f(z, z^*) : (z, z^*) \in \partial f,\ z^* \in C^*\right\}$$

then $\operatorname{dom}(f^*) \cap C^* \neq \emptyset$.

(b) *If there exists $x \in E$ such that $\left(f \underset{e}{+} U\right)(x) > -\infty$ and*

$$f \underset{e}{+} U = \sup\left\{\sigma f(z, z^*) : (z, z^*) \in \partial f,\ z^* \in \operatorname{int}(C^*)\right\}$$

then $\operatorname{dom}(f^*) \cap \operatorname{int}(C^*) \neq \emptyset$.

(c) *If $\operatorname{dom}(f^*) \cap \operatorname{int}(C^*) \neq \emptyset$ then*

$$f \underset{e}{+} U = \sup\left\{\sigma f(z, z^*) : (z, z^*) \in \partial f,\ z^* \in \operatorname{int}(C^*)\right\} \text{ slicewise and graphwise.}$$

(d) *If* $\operatorname{dom}(f^*) \cap \operatorname{int}(C^*) \neq \emptyset$ *then*

$$f \underset{e}{+} U = \sup\Big\{\sigma f(z,z^*) : (z,z^*) \in \partial f,\ z^* \in C^*\Big\}$$
$$= \sup\Big\{\sigma f(z,z^*) : (z,z^*) \in \partial f,\ z^* \in \operatorname{int}(C^*)\Big\}.$$

PROOF: (a) Here there exists $(z,z^*) \in \partial f$ such that $z^* \in C^*$. This gives the required result since $f^*(z^*) = \langle z, z^*\rangle - f(z) \in \mathbb{R}$.

(b) The proof of this is similar to the proof of (a).

(c) Let $x^* \in \operatorname{dom}(f^*) \cap \operatorname{int}(C^*)$. Let $a : E \to \mathbb{R} \cup \{-\infty\}$ be proper, concave and boxed above, and $\inf_E\Big[\Big(f \underset{e}{+} U\Big) - a\Big] > 0$. We write $h := f - x^*$, $T := U - x^*$ and $b := a - x^*$. Then $\inf_E\Big[\Big(h \underset{e}{+} T\Big) - b\Big] = \inf_E\Big[\Big(f \underset{e}{+} U\Big) - a\Big] > 0$, and the fact that $x^* \in \operatorname{dom}(f^*)$ implies that h is bounded below. From the separation form of Theorem 8, there exists $(z,w^*) \in \partial h$ such that $w^* \ll T$ on E and $\inf_E[\sigma h(z,w^*) - b] > 0$. We write $z^* := w^* + x^*$. Then $\inf_E[\sigma f(z,z^*) - a] = \inf_E[\sigma h(z,w^*) - b] > 0$. Clearly, $(z,z^*) \in \partial f$. Finally, there exists $\zeta \in (0,1)$ such that $z^* - x^* = w^* \leqslant \zeta T = \zeta(U - x^*)$ on E. From the bipolar theorem, $z^* - x^* \in \zeta(C^* - x^*)$, that is, $z^* \in x^* + \zeta(C^* - x^*)$. It follows that $z^* \in \operatorname{int}(C^*)$, which (modulo Lemma 3) completes the proof of (c).

(d) follows immediately from (c). ∎

The following result extends [**2**, Lemma 2.2] in that C^* can be a subset of E^* more general than a multiple of a ball, (z,z^*) is restricted to be in ∂f, and the supremum is valid slicewise and graphwise rather than pointwise.

COROLLARY 13. *Let* f, C^* *and* U *be as in Theorem 12 and* $\operatorname{dom}(f^*) \cap \operatorname{int}(C^*) \neq \emptyset$. *Then*

$$f \underset{e}{+} U = \sup\Big\{z^* - f^*(z^*) : (z,z^*) \in \partial f,\ z^* \in \operatorname{int}(C^*)\Big\} \text{ slicewise and graphwise.}$$

PROOF: This follows from Theorem 12(c) since, for all $(z,z^*) \in \partial f$, $\sigma f(z,z^*) = z^* - f^*(z^*)$. ∎

The author is grateful to Professor J.-B. Hiriart-Urruty for providing him with a proof of Corollary 14 below using totally different ideas. This result is also used by Borwein and Vanderwerff in [**3**].

COROLLARY 14. *Let* $f : E \to \mathbb{R} \cup \{\infty\}$ *be proper, convex and lower semicontinuous and suppose that there exists* $x^* \in \operatorname{dom}(f^*)$ *such that* $\|x^*\| < n$. *Then*

$$f \underset{e}{+} n\|\ \| = \sup\Big\{\sigma f(z,z^*) : (z,z^*) \in \partial f,\ \|z^*\| \leqslant n\Big\}$$
$$= \sup\Big\{\sigma f(z,z^*) : (z,z^*) \in \partial f,\ \|z^*\| < n\Big\}.$$

https://doi.org/10.1017/S0004972700013599 Published online by Cambridge University Press

In particular, (0.2) is true.

PROOF: This follows from Theorem 12(d) with C^* the ball in E^* with centre at the origin and radius n. ∎

REMARK 15. Theorem 12(c)-(d) provide converses to Theorem 12(b). On the other hand, the example in Remark 10 shows that the converse to Theorem 12(a) fails: we may have $\operatorname{dom}(f^*) \cap C^* \neq \emptyset$, but there might still exist no $(z, z^*) \in \partial f$ such that $z^* \in C^*$.

6. MORE ON GRAPHWISE AND SLICEWISE SUPREMA

If F is a Banach space, we write $C(F)$ (respectively $BC(F)$) for the set of all nonempty closed convex (respectively nonempty bounded closed convex) subsets of F. We now fix a norm on $E \times \mathbb{R}$ that gives the product topology. For all $A, B \in C(E \times \mathbb{R})$, let $d(A, B)$ be the distance between A and B, measured by this norm.

Our next result is a characterisation of slicewise suprema. The possibility of the implication (d)$\Longrightarrow$(a) was suggested by a comment of Jon Vanderwerff.

THEOREM 16. *Let $g : E \to \mathbb{R} \cup \{\infty\}$ and, for all $\omega \in \Omega$, $s_\omega : E \to \mathbb{R} \cup \{\infty\}$ be proper and $s_\omega \leqslant g$ on E. Then (a)$\Longrightarrow$(b)$\Longleftrightarrow$(c)$\Longrightarrow$(d). If g is convex then (d)$\Longrightarrow$(a), so all four conditions are equivalent.*

(a) *$g = \sup_{\omega \in \Omega} s_\omega$ slicewise.*

(b) *If $B \in BC(E \times \mathbb{R})$ and B is separated from $\operatorname{epi}(g)$ then there exists $\omega \in \Omega$ such that B is separated from $\operatorname{epi}(s_\omega)$.*

(c) *If $B \in BC(E \times \mathbb{R})$ then*

$$d\big(B, \operatorname{epi}(g)\big) = \sup_{\omega \in \Omega} d\big(B, \operatorname{epi}(s_\omega)\big) \tag{16.1}$$

(d) *If $a : E \to \mathbb{R}$ is continuous and affine, $D \in BC(E)$ and $\big\{\big(y, a(y)\big) : y \in D\big\}$ is separated from $\operatorname{epi}(g)$ then there exists $\omega \in \Omega$ such that $\big\{\big(y, a(y)\big) : y \in D\big\}$ is separated from $\operatorname{epi}(s_\omega)$.*

PROOF: ((a)$\Longrightarrow$(b)) If $B \in BC(E \times \mathbb{R})$ and B is separated from $\operatorname{epi}(g)$ then the function $b : E \to \mathbb{R} \cup \{-\infty\}$ defined by

$$b(x) := \sup_{(x,\lambda) \in B} \lambda$$

($\sup \emptyset := -\infty$) is proper, concave and boxed above, and $\operatorname{hypo}(b)$ is separated from $\operatorname{epi}(g)$. From (a), there exists $\omega \in \Omega$ such that $\operatorname{hypo}(b)$ is separated from $\operatorname{epi}(s_\omega)$. Since $B \subset \operatorname{hypo}(b)$, it follows that B is separated from $\operatorname{epi}(s_\omega)$. Thus (b) is satisfied.

https://doi.org/10.1017/S0004972700013599 Published online by Cambridge University Press

((b)$\Longrightarrow$(c)) Let $B \in BC(E \times \mathbb{R})$. If $d(B, \operatorname{epi}(g)) = 0$ then

$$\text{for all } \omega \in \Omega, d(B, \operatorname{epi}(s_\omega)) = 0,$$

and (16.1) is immediate. Suppose now that $d(B, \operatorname{epi}(g)) > \mu \geqslant 0$. Write

$$B' := \Big\{(x,\lambda) : (x,\lambda) \in E \times \mathbb{R},\ \operatorname{dist}\big((x,\lambda), B\big) \leqslant \mu\Big\}.$$

Then $B' \in BC(E \times \mathbb{R})$ and $d(B', \operatorname{epi}(g)) \geqslant d(B, \operatorname{epi}(g)) - \mu > 0$. From (b), there exists $\omega \in \Omega$ such that $d(B', \operatorname{epi}(s_\omega)) > 0$. In particular, $B' \cap \operatorname{epi}(s_\omega) = \emptyset$, hence $d(B, \operatorname{epi}(s_\omega)) \geqslant \mu$. Thus (c) is satisfied. It is trivial that (c)$\Longrightarrow$(b).

((b)$\Longrightarrow$(d)) This is immediate since $\{(y, a(y)) : y \in D\} \in BC(E \times \mathbb{R})$.

((d)$\Longrightarrow$(a)) Let $b : E \to \mathbb{R} \cup \{-\infty\}$ be proper, concave, and boxed above, and $\operatorname{epi}(g)$ be separated from $\operatorname{hypo}(b)$. Let $D := \overline{\operatorname{dom}(b)} \in BC(E)$. From the Eidelheit separation theorem in $E \times \mathbb{R}$, there exist $x^* \in E^*$, and $\alpha, \beta, \gamma \in \mathbb{R}$ such that $(x^*, \alpha) \neq (0,0)$, $\beta < \gamma$

$$(16.2) \qquad (x,\lambda) \in \operatorname{hypo}(b) \text{ implies that } \langle x, x^*\rangle + \alpha\lambda \leqslant \beta$$

and

$$(16.3) \qquad (x,\lambda) \in \operatorname{epi}(g) \text{ implies that } \langle x, x^*\rangle + \alpha\lambda \geqslant \gamma.$$

Since $\operatorname{epi}(g)$ recedes vertically, $\alpha \geqslant 0$.

Case 1. $\alpha > 0$. Here we can divide by α in (16.2) and (16.3). Writing $y^* := -x^*/\alpha$, $\pi := \beta/\alpha$, and $\rho := \gamma/\alpha$, we obtain:

$$(16.4) \qquad b \leqslant y^* + \pi \text{ on } E$$

and

$$(16.5) \qquad y^* + \rho \leqslant g \text{ on } E.$$

We write $a := y^* + \pi$. Then a is continuous and affine. From (16.5), $\delta := \inf_E[g - a] \geqslant \rho - \pi > 0$. Let $B := \{(y, a(y)) : y \in D\}$ and M be strictly greater than the Lipschitz constant of a. If $(x,\lambda) \in \operatorname{epi}(g)$, $(y,\mu) \in B$ and $\|x - y\| \leqslant \delta/2M$ then

$$|\lambda - \mu| \geqslant \lambda - \mu \geqslant g(x) - a(y) \geqslant g(x) - a(x) - M\,\|x - y\| \geqslant \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

(This argument is *dual* to that of Lemma 3.) Consequently, B is separated from $\operatorname{epi}(g)$.

https://doi.org/10.1017/S0004972700013599 Published online by Cambridge University Press

CASE 2. $\alpha = 0$. Let $B := \left\{\left(x, \sup_E b\right) : x \in D\right\}$. In this case, $x^* \neq 0$, and so it follows from (16.2) and (16.3) that D is separated from $\operatorname{dom}(g)$ in E, from which B is separated from $\operatorname{epi}(g)$ in $E \times \mathbb{R}$.

In both cases, B is separated from $\operatorname{epi}(g)$, and so it follows from (d) that there exists $\omega \in \Omega$ such that B is separated from $\operatorname{epi}(s_\omega)$. Since B lies above $\operatorname{hypo}(b)$ (in case 1, this follows from (16.4) and, in case 2, it is obvious from the definition of B), $\operatorname{hypo}(b)$ is also separated from $\operatorname{epi}(s_\omega)$. Thus we have proved that (a) is satisfied. ∎

For the remainder of this paper, we consider the suprema of *continuous linear functionals* on E.

THEOREM 17. *Let Ω be a nonempty bounded convex subset of E^* and*

$$g = \sup_{\omega \in \Omega} \omega \text{ on } E.$$

Then $g = \sup_{\omega \in \Omega} \omega$ graphwise or slicewise $\iff$ the norm and the weak closures of Ω are identical.*

PROOF: Write $\overline{\Omega}^{\|\,\|}$ and $\overline{\Omega}^{w^*}$ for the norm- and weak*-closures of Ω.

$(\Longrightarrow)$ Suppose that $g = \sup_{\omega \in \Omega} \omega$ graphwise. Let $x^* \in \overline{\Omega}^{w^*}$ and $\varepsilon > 0$. Then $x^* \leqslant g$ on E. Let $b(x) := \langle x, x^* \rangle - \varepsilon$ if $\|x\| \leqslant 1$ and $b(x) := -\infty$ otherwise. Then $\inf_E [g - b] \geqslant \varepsilon > 0$. Consequently, there exists $\omega \in \Omega$ such that $\inf_E \left[\omega - b\right] \geqslant 0$, from which

$$\|x\| \leqslant 1 \Longrightarrow \langle x, x^* \rangle - \varepsilon \leqslant \langle x, \omega \rangle,$$

that is:

$$\|x\| \leqslant 1 \Longrightarrow \langle x, x^* - \omega \rangle \leqslant \varepsilon,$$

which is equivalent to the statement that $\|x^* - \omega\| \leqslant \varepsilon$. Thus $x^* \in \overline{\Omega}^{\|\,\|}$.

$(\Longleftarrow)$ Let $b : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and boxed above and $\operatorname{epi}(g)$ be separated from $\operatorname{hypo}(b)$. We argue as in the proof of Theorem 16((d)$\Longrightarrow$(a)). Since $\operatorname{dom}(g) = E$, Case 2 cannot arise, hence there exist $y^* \in E^*$ and $\pi, \rho \in \mathbb{R}$ such that $\pi < \rho$,

$$b \leqslant y^* + \pi \text{ on } E \tag{16.4}$$

and

$$y^* + \rho \leqslant g \text{ on } E. \tag{16.5}$$

Evaluating (16.5) at 0 gives that $\rho \leqslant 0$, hence $\pi < 0$. Using the positive homogeneity of y^* and g, it follows from (16.5) that $y^* \leqslant g$ on E. From the bipolar theorem,

https://doi.org/10.1017/S0004972700013599 Published online by Cambridge University Press

$y^* \in \overline{\Omega}^{w^*}$. By hypothesis, $y^* \in \overline{\Omega}^{\| \,\|}$. Let $M > \sup \|\mathrm{dom}\,(b)\|$. Then there exists $\omega \in \Omega$ such that $\|\omega - y^*\| \leqslant -\pi/4M$. Choose $\delta > 0$ such that $\delta \|y^*\| \leqslant -\pi/4$ and $\delta \leqslant M$. If $(x, \lambda) \in \mathrm{epi}\,(\omega)$, $(y, \mu) \in \mathrm{hypo}\,(b)$ and $\|x - y\| \leqslant \delta$ then

$$\begin{aligned} |\lambda - \mu| &\geqslant \lambda - \mu \geqslant \langle x, \omega\rangle - b(y) \\ &= \langle y, y^*\rangle + \langle x - y, y^*\rangle + \langle y, \omega - y^*\rangle + \langle x - y, \omega - y^*\rangle - b(y) \\ &\geqslant \langle y, y^*\rangle - \|x - y\|\,\|y^*\| - \|y\|\,\|\omega - y^*\| - \|x - y\|\,\|\omega - y^*\| - b(y), \end{aligned}$$

Since $y \in \mathrm{dom}\,(b)$, $\|x - y\| \leqslant \delta$ and $\delta \leqslant M$,

$$|\lambda - \mu| \geqslant \langle y, y^*\rangle - b(y) - \delta\,\|y^*\| - 2M\,\|\omega - y^*\| \geqslant \langle y, y^*\rangle - b(y) + \frac{\pi}{4} + 2\left(\frac{\pi}{4}\right).$$

From (16.4),

$$|\lambda - \mu| \geqslant -\frac{\pi}{4}.$$

Thus $\mathrm{epi}\,(\omega)$ is separated from $\mathrm{hypo}\,(b)$. Consequently, $g = \sup_{\omega \in \Omega} \omega$ slicewise. ∎

Remark 18. In the following discussion, Ω is a nonempty bounded convex subset of E^*. We consider two questions:

Question 1. When is it the case that

(18.1) $$\sup_{\omega \in \Omega} \omega = g \text{ pointwise on } E \Longrightarrow \sup_{\omega \in \Omega} \omega = g \text{ slicewise?}$$

Question 2. When is it the case that

(18.2) $$\sup_{\omega \in \Omega} \omega = \|\ \| \text{ pointwise on } E \Longrightarrow \sup_{\omega \in \Omega} \omega = \|\ \| \text{ slicewise?}$$

If E is reflexive then $\overline{\Omega}^{w^*}$ is identical with the weak closure of Ω. Since Ω is convex this is, in turn, identical with $\overline{\Omega}^{\| \,\|}$. Thus, from Theorem 17, both (18.1) and (18.2) are true. We write E_1^* for the unit ball of E^*.

The answer to Question 1 is:

(18.3) $$\text{(18.1) is true} \iff E \text{ is reflexive.}$$

Proof: Suppose that (18.1) is true and $x^{**} \in E^{**}$. Write

$$\Omega := \left\{x^* : x^* \in E_1^*,\ \langle x^*, x^{**}\rangle = 0\right\}.$$

Since Ω is norm-closed, $\Omega = \overline{\Omega}^{\|\ \|}$. From Theorem 17, $\Omega = \overline{\Omega}^{w^*}$, hence Ω is weak*-closed. Thus the kernel of x^{**} intersects E_1^* in a weak*-closed set. From the Krein-Smulian theorem, the kernel of x^{**} is itself weak*-closed, hence $x^{**} \in E$. This establishes (18.3). The author is grateful to Gilles Godefroy for showing him this argument. There are many other characterisations of reflexive spaces in terms of convex analysis in a recent paper by Borwein, Fitzpatrick and Vanderwerff [4]. $\square$

In order to discuss Question 2, we introduce the notion of a *slice* of E_1^*. If $x^{**} \in E^{**}$ and $\alpha > 0$, we write

$$S(x^{**},\alpha) := \left\{x^* : x^* \in E_1^*,\ \langle x^*, x^{**}\rangle > \|x^{**}\| - \alpha\right\}.$$

(See [**9**, p.24] – note that we are only considering slices of E_1^*.) If $x \in E$ and $\beta > 0$ then, of course,

$$S(x,\beta) := \left\{x^* : x^* \in E_1^*, \langle x, x^*\rangle > \|x\| - \beta\right\}.$$

Our answer to Question 2 is:

$$\text{(18.4)}\qquad \begin{gathered}\text{(18.2) is true} \iff \text{for all } x^{**} \in E^{**} \text{ and } \alpha > 0,\\ \text{there exists } x \in E \text{ and } \beta > 0 \text{ such that } S(x,\beta) \subset S(x^{**},\alpha).\end{gathered}$$

PROOF: $(\Longrightarrow)$ We suppose first that there exist $x^{**} \in E^{**}$ and $\alpha > 0$ such that,

$$\text{(18.5)}\qquad \text{for all } x \in E \text{ and } \beta > 0,\ S(x,\beta) \not\subset S(x^{**},\alpha).$$

Let $\Omega := E_1^* \backslash S(x^{**},\alpha)$. Ω is a proper, norm-closed subset of E_1^*, hence $\overline{\Omega}^{\|\ \|} \neq E_1^*$. From (18.5),

$$\text{(18.6)}\qquad \text{for all } x \in E \text{ and } \beta > 0,\ S(x,\beta) \cap \Omega \neq \emptyset.$$

Hence,

$$\text{(18.7)}\qquad \text{for all } x \in E,\ \sup\langle x, \Omega\rangle = \|x\|.$$

From the separation theorem in (E, w^*), $\overline{\Omega}^{w^*} = E_1^*$, and so $\overline{\Omega}^{w^*} \neq \overline{\Omega}^{\|\ \|}$. From (18.7) and Theorem 17, it is false that $\sup_{\omega\in\Omega} \omega = \|\ \|$ slicewise, hence (18.2) fails. This completes the proof of $(\Longrightarrow)$.

$(\Longleftarrow)$ Suppose now that (18.2) fails and Ω is such that $\sup_{\omega\in\Omega} \omega = \|\ \|$ pointwise on E but not slicewise. Then, again, (18.6) and (18.7) are satisfied and $\overline{\Omega}^{w^*} = E_1^*$. On the other hand, from Theorem 17, $\overline{\Omega}^{\|\ \|} \neq \overline{\Omega}^{w^*}$, hence $\overline{\Omega}^{\|\ \|}$ is a proper subset

https://doi.org/10.1017/S0004972700013599 Published online by Cambridge University Press

of E_1^*. From the separation theorem in $(E^*, \| \|)$, there exists $x^{**} \in E^{**}$ such that $\sup\langle \Omega, x^{**}\rangle < \|x^{**}\|$, hence there exists $\alpha > 0$ such that $S(x^{**}, \alpha) \cap \Omega = \emptyset$. From (18.6),

$$\text{for all } x \in E \text{ and } \beta > 0,\ S(x, \beta) \not\subset S(x^{**}, \alpha).$$

This completes the proof of $(\Longleftarrow)$ and also of (18.4). ∎

We do not know whether (18.2) can be put in terms of more standard Banach space concepts. Here we shall show that if $E = c_0$ then E satisfies (18.2), and if $E = \ell^1$ or $E = c$ then E does not satisfy (18.2). This last example is due to Isaac Namioka — the author would also like to thank Robert Phelps for providing him with it. Namioka's proof of the $E = c$ case also led to a simplification of our original proof of the $E = \ell^1$ case. Since c_0 and c are isomorphic, this shows that the property (18.2) is not preserved under isomorphism. The example $E = c$ shows that Asplund spaces do not necessarily satisfy (18.2). If (18.2) is true then E_1^* is the norm-closed convex hull of its set of extreme points. The examples $E = \ell^1$ and $E = c$ show that the converse of this assertion is false. In fact, from [9, Theorem 5.12, p.86], when $E = c$, E_1^* is even the norm-closed convex hull of its set of weak* strongly exposed points. We do not know whether (18.2) implies that E_1^* is the norm-closed convex hull of its set of exposed points.

If $E = c_0$ let $x^{**} \in E^{**} = \ell^\infty$ with $x^{**} \neq 0$, and let $\alpha > 0$. Choose p such that $x_p^{**} \neq 0$ and $|x_p^{**}| > \|x^{**}\| - \alpha/3$. Let $x := (\operatorname{sgn} x_p^{**})e_p \in E$. Then $\|x\| = 1$. Let $\beta \in (0, 1]$ and $\|x^{**}\|\beta < \alpha/3$. We shall show that $S(x, \beta) \subset S(x^{**}, \alpha)$. Indeed, let $x^* \in S(x, \beta)$. Then

$$(\operatorname{sgn} x_p^{**})x_p^* = \langle x, x^*\rangle > \|x\| - \beta = 1 - \beta,$$

from which $|x_p^*| > 1 - \beta$, hence

$$\sum_{n \neq p} |x_n^*| \leqslant 1 - |x_p^*| < \beta.$$

Then

$$\begin{aligned}\langle x^*, x^{**}\rangle = x_p^* x_p^{**} + \sum_{n\neq p} x_n^* x_n^{**} &= x_p^*(\operatorname{sgn} x_p^{**})\,|x_p^{**}| + \sum_{n\neq p} x_n^* x_n^{**}\\ &> (1-\beta)\,|x_p^{**}| - \sum_{n\neq p} |x_n^*|\,\|x^{**}\| \geqslant (1-\beta)\Big(\|x^{**}\| - \frac{\alpha}{3}\Big) - \beta\,\|x^{**}\|\\ &> \|x^{**}\| - 2\beta\,\|x^{**}\| - \frac{\alpha}{3} > \|x^{**}\| - \alpha,\end{aligned}$$

that is to say, $x^* \in S(x^{**}, \alpha)$.

If $E = \ell^1$, let x^{**} be a Banach limit on ℓ^∞. Then $\|x^{**}\| = 1$. If $x \in E$ and $\beta > 0$ then, since $\|x\| = \sup\{\langle x, x^*\rangle : x^* \in E_1^* \cap c_0\}$, there always exists $x^* \in S(x,\beta) \cap c_0 \subset S(x,\beta)\backslash S(x^{**},1)$. Thus (18.2) is not satisfied.

If $E = c$ then we can represent the norm-dual of E by ℓ^1, where

$$\text{for all } x = \{x_n\}_{n\geqslant 1} \in c \text{ and } x^* = \{x_n^*\}_{n\geqslant 1} \in \ell^1,\ \langle x, x^*\rangle = \Bigl(\lim_{n\to\infty} x_n\Bigr)x_1^* + \sum_{n\geqslant 1} x_n x_{n+1}^*.$$

Define $x^{**} \in E^{**}$ by

$$\langle x^*, x^{**}\rangle := x_1^* \qquad (x^* \in E^*).$$

Then $\|x^{**}\| = 1$. If $x \in E$ and $\beta > 0$ then, since $\|x\| = \sup\bigl\{\langle x, x^*\rangle : x^* \in E_1^*,\ x_1^* = 0\bigr\}$, there exists $x^* \in S(x,\beta)$ such that $x_1^* = 0$. Then $x^* \in S(x,\beta)\backslash S(x^{**},1)$. Thus, again, (18.2) is not satisfied.

We do not know whether, as is the case with the two examples discussed above, the failure of (18.2) implies that there exists $x^{**} \in E^{**}$ such that $\|x^{**}\| = 1$ and, for all $x \in E$, $\|x\| = \sup\bigl\{\langle x, x^*\rangle : x^* \in E_1^*,\ \langle x^*, x^{**}\rangle = 0\bigr\}$.

References

[1] G. Beer, 'The slice topology: a viable alternative to Mosco convergence in nonreflexive spaces', *Nonlinear Anal.* **19** (1992), 271-290.

[2] G. Beer, 'Baire-Wijsman regularization and the convergence of convex functions', (preprint).

[3] J.M. Borwein and J.D. Vanderwerff, 'Convergence of Lipschitz regularizations of convex functions', *J. Funct. Anal.* (to appear).

[4] J.M. Borwein, S. Fitzpatrick and J.D. Vanderwerff, 'Examples of convex functions and classifications of normed spaces', *Convex Analysis* (to appear).

[5] I. Ekeland, 'Nonconvex minimization problems', *Bull. Amer. Math. Soc.* **1** (1979), 443-474.

[6] S. Fitzpatrick and R.R. Phelps, 'Bounded approximants to monotone operators on Banach Spaces', *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9** (1992), 573-595.

[7] J.-B. Hiriart-Urruty, 'Lipschitz r-continuity of the approximate subdifferential of a convex function', *Math. Scand.* **47** (1980), 123-134.

[8] H. König, 'Some basic theorems in convex analysis', in *Optimization and operations research*, (B. Korte, Editor) (North-Holland, 1982).

[9] R.R. Phelps, *Convex functions, monotone operators and differentiablilty*, Lecture notes in Mathematics **1364** (Springer-Verlag, Berlin, Heidelberg, New York, 1993).

[10] S. Simons, 'Subtangents with controlled slope', *Nonlinear Anal.* **22** (1994), 1373-1389.

Department of Mathematics
University of California
Santa Barbara CA 93106-3080
United States of America