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NON-LINEAR PREDICTORS OF TRANSFORMED STATIONARY PROCESSES

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§1. Introduction

One of the authors discussed the best non-linear predictor of the process $X(t) = f(U(t))$, which is obtained by transforming an Ornstein-Uhlenbeck process $U(t)$ with a measurable function $f(u)$ (see [5]). We denote by $\mathscr{B}_t(X)$ the completion of the σ-field generated by $\{X(s, \omega); s \leq t\}$, and by $\tilde{\mathscr{B}}_t(X)$ the completion of the σ-field generated by $\{X(s, \omega); s \geq t\}$. Assuming that $X(t)$ is square integrable, he defined the best non-linear predictor $\hat{X}(t; \tau, \omega)$ of $X(t + \tau, \omega)$ for the given data $\{X(s, \omega); s \leq t\}$ by

$$\hat{X}(t; \tau) \equiv E[X(t + \tau) \mid \mathscr{B}_t(X)], \tag{1.1}$$

which gives the least mean square error. It can be rewritten as

$$\hat{X}(t; \tau) = E[(T_\tau^U f)(U(t)) \mid \mathscr{B}_t(X)] \text{ a.s.}, \tag{1.2}$$

where $T_\tau^U, \tau > 0$, is the semi-group of the Ornstein-Uhlenbeck process $U(t)$. He has shown that $\mathscr{B}_t(X) = \mathscr{B}_t(U)$ holds in many cases and that algorithms determining the value of $U(t, \omega)$ by the observed data can be given.

In this paper, we shall see that his methods can be applied to more general base processes $\{V(t, \omega), t \in R\}$. Throughout this paper, we assume that each base process $\{V(t)\}$ is a strictly stationary process with continuous paths defined on a probability space $(\Omega, \mathscr{B}, P)$ and that $V(t, \omega)$ takes its values in an interval (ℓ, r) with $-\infty \leq \ell < r \leq \infty$, and satisfies

$$\varliminf_{t \to -\infty} V(t, \omega) = \ell \quad \text{and} \quad \varlimsup_{t \to -\infty} V(t, \omega) = r \text{ a.s.} \tag{1.3}$$

Let $m(dv)$ be the distribution of $V(t)$ for each t and let $f(v)$ be a function in $L^2(dm)$. Then the transformed process $X(t, \omega) \equiv f(V(t, \omega))$, $t \in R$, is a strictly stationary process with the variance $\int_\ell^r f(v)^2 dm(v) - \left(\int_\ell^r f(v) dm(v)\right)^2$.

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The best non-linear predictor $\hat{X}(t;\tau,\omega)$ of $X(t+\tau,\omega)$ for the observed data $\{X(s,\omega); s \leq t\}$ is defined by

$$\hat{X}(t;\tau) \equiv E[f(V(t+\tau))\,|\,\mathscr{B}_t(X)] = E[E[f(V(t+\tau))\,|\,\mathscr{B}_t(V)]\,|\,\mathscr{B}_t(X)]\,, \tag{1.4}$$

which gives the least mean square error. Our problem is to give an algorithm to determine $\hat{X}(t;\tau,\omega)$ in terms of the observed data $\{X(s,\omega); s \leq t\}$.

In Section 2, we shall discuss the case that the base process $V(t)$ satisfies the local limit theorem

$$\overline{\lim_{h\to 0}}\,\frac{|V(t+h,\omega)-V(t,\omega)|}{\gamma(h)} = 1 \text{ a.s.} \qquad \text{for any } t \in R\,, \tag{1.5}$$

with a suitable $\gamma(h)$ of $\gamma(h) = o(1)$. If $f(v)$ is a piecewise monotone and differentiable function without any local symmetry (see Definition 2.3), then we can show $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ and $\hat{X}(t,\tau) = E[f(V(t+\tau))\,|\,\mathscr{B}_t(V)]$ (see Theorems 2.4 and 2.5). If $f(v)$ has a single bottom without peaks (see Definition 2.1) additionally, then we can give an algorithm to determine $V(t)$ by the observed data $\{X(s); s \leq t\}$. The condition (1.5) is weakened as (2.5).

In Section 3, we shall discuss the case that the base process $V(t)$ is a diffusion. The results are very similar to those in [5].

In Section 4, we shall discuss the case that $V(t)$ has smooth sample paths. If $V(t)$ has the property that for any fixed v

$$P(\{\omega\colon \exists t \in R,\ V(t,\omega) = v,\ V'(t,\omega) = 0\}) = 0 \tag{1.6}$$

holds, then we have a finer result (see Theorem 4.3).

In Section 5, stationary Gaussian processes will be discussed. In particular, we give precise results for double Markov, stationary Gaussian processes (see Theorem 5.2).

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§2. Prediction appealing to local limit theorems

Now we are going to discuss the best non-linear prediction problem of the transformed process $X(t) = f(V(t))$, assuming that the base process $V(t)$ satisfies the conditions already stated in Section 1. We restrict the

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class of functions $f(v)$ defined on the range (ℓ, r) of $V(t)$.

DEFINITION 2.1. A function $f(v)$ on (ℓ, r) is simply said to be *piecewise monotone*, if $f(v)$ is a continuous function*) with extremum points $v_1 < v_2 < \cdots < v_L$ finite in number and if $f(v)$ is strictly monotone on each interval $(v_j, v_{j+1}), 0 \leq j \leq L$, with $v_0 \equiv \ell$ and $v_{L+1} \equiv r$. We call v_j *a peak* (resp. *a bottom*), if v_j is a maximal (resp. minimal) point of $f(v)$.

Throughout this paper, we assume that $f(v)$ is piecewise monotone and piecewise continuously differentiable. Additionally, we assume that $f(v)$ belongs to $L^2(dm)$.

Let $\{v_1, \cdots, v_L\}$ be the set of all extremum points of $f(v)$. Set

$$I_j \equiv (v_j, v_{j+1}]\,, \qquad 0 \leq j \leq L-1\,,$$
$$I_L \equiv (v_L, v_{L+1}) \quad \text{and} \quad \bar{I}_j \equiv [v_j, v_{j+1}]\,, \quad 0 \leq j \leq L\,.$$

Put $f(v_0) \equiv f(\ell + 0)$ and $f(v_{L+1}) \equiv f(r-0)$. Define $f_j^{-1}(h)$ by

$$f_j^{-1}(h) \equiv \begin{cases} v & \text{if } f(v) = h \quad \text{and} \quad v \in \bar{I}_j\,, \\ \Delta & \text{if } f(v) \neq h \quad \text{for any } v \in \bar{I}_j\,, \end{cases}$$

with an extra point Δ. Define a process $j(s, \omega)$ by

$$j(s, \omega) \equiv j \qquad \text{if } V(s, \omega) \in I_j,\ 0 \leq j \leq L\,. \tag{2.1}$$

Obviously we have

$$V(s, \omega) = f_{j(s,\omega)}^{-1}(X(s, \omega))\,. \tag{2.2}$$

Firstly, we discuss $V(t)$ satisfying the following local limit theorem. Let $\gamma(h)$ be a continuous even function defined in a neighbourhood of $h = 0$ with $\gamma(h) = o(1)$ as $h \to 0$. Assume that

$$\overline{\lim_{h \to 0}} \frac{V(t+h, \omega) - V(t, \omega)}{\gamma(h)} = 1 \text{ a.s.} \tag{2.3}$$

Then the following lemma is obviously proved, while useful.

LEMMA 2.2. (i) *Let $D(f)$ be the set of all v at which $f(v)$ is differentiable. Then*

$$O^{\gamma}(X; t, \omega) \equiv \overline{\lim_{h \to 0}} \frac{|X(t+h, \omega) - X(t, \omega)|}{\gamma(h)} = |f'(V(t, \omega))$$

*) The continuity of $f(v)$ is not essential for our discussions. But we assume it for simplicity.

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for a.e. ω *in* $\{\omega;\ V(t, \omega) \in D(f)\}$.

(ii) *If* $T(\omega)$ *is a random variable taking its values only in rational numbers, then*

$$O^r(X; T(\omega), \omega) = \overline{\lim_{h \to 0}} \frac{|X(T(\omega) + h, \omega) - X(T(\omega), \omega)|}{\gamma(h)} \text{ a.s.}$$

Let us observe the two dimensional process $X(t, \omega) \equiv (X(t, \omega), O^r(x; t, \omega)) \equiv (f(V(t, \omega)), |f'(V(t, \omega))|)$, which is $\mathscr{B}_t(X)$-measurable. Obviously, $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ holds, if $f(v)$ and $|f'(v)|$ separate each point of the interval (ℓ, r). We shall give weaker conditions for $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ and algorithms to determine $V(t)$ by the observed data $\{X(s); s \leq t\}$. For the purpose, we introduce a terminology.

DEFINITION 2.3. We say that $f(v)$ has a *local symmetry with respect to* v^*, if $f(v^* + v) = f(v^* - v)$ holds in a neighbourhood of $v = 0$. We say that $f(v)$ has a *global symmetry* with respect to v^*, if $f(v^* + v) = f(v^* - v)$ holds for any v with $\ell < v^* + v < r$.

Now we suppose that $f(v)$ has a unique bottom v_1 without peaks and that it does not have any local symmetry with respect to v_1. Let $G(f)$ be the interior of the set of all h's such that $f_0^{-1}(h) \neq f_1^{-1}(h)$ holds, and $-f'(f_0^{-1}(h)) \neq f'(f_1^{-1}(h))$ holds whenever both $f_0^{-1}(h)$ and $f_1^{-1}(h)$ are in $D(f)$. Then $G(f)$ is non empty. Define random times $T_j(t, \omega), j = 0, 1$, by

$$T_j(t, \omega) \equiv \begin{cases} \sup\limits_{\substack{k \in Z \\ n \geq 1}} \left\{ \begin{array}{l} k \cdot 2^{-n} \leq t;\ X(k \cdot 2^{-n}, \omega) \in G(f), \\ \qquad\qquad O^r(X; k \cdot 2^{-n}, \omega) = |f'(f_j^{-1}(X(k \cdot 2^{-n}, \omega)))| \end{array} \right\} \\ -\infty \qquad \text{if the above set is empty.} \end{cases}$$

By Lemma 2.2 and the assumption (1.3), we can see that $T_j(t, \omega) > -\infty$ a.s. for $j = 0, 1$. We have obviously that

$$j(t, \omega) = \begin{cases} 0 & \text{if } T_0(t, \omega) \geq T_1(t, \omega)\,, \\ 1 & \text{if } T_0(t, \omega) < T_1(t, \omega) \end{cases} \tag{2.4}$$

almost surely. Moreover $T_{1-j}(t, \omega)$ is the time when $V(s, \omega)$ $(s \leq t)$ passes through v_1 lastly, if $j(t, \omega) = j$. Thus we have

THEOREM 2.4. *If* $V(t)$ *satisfies the local limit theorem* (2.3) *and if* $f(v)$ *has a unique bottom without peaks and has no local symmetry with respect to* v_1, *then* $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ *holds, and hence* $\hat{X}(t; \tau) = E[f(V(t + \tau)) \,|\, \mathscr{B}_t(V)]$ a.s. *holds. Actually the value of* $V(t, \omega)$ *is determined almost surely by the*

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observed data $\{X(s, \omega); s \leq t\}$ *as follows*

$$V(t, \omega) = f^{-1}_{j(t,\omega)}(X(t, \omega))$$

with $j(t, \omega)$ *given by* (2.4).

Proof. The assertion follows from (2.2) and the above discussions.

THEOREM 2.5. *Let* $f(v)$ *be a piecewise monotone and piecewise differentiable function which does not have any local symmetry. Then* $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ *holds and hence* $\hat{X}(t; \tau) = E[f(V(t + \tau)) \mid \mathscr{B}_t(V)]$.

Proof. Since $f(v)$ has no local symmetry, we can recognize the times when $V(s, \omega)$ passes through one of peaks (or bottoms), by the same reason as in the proof of Theorem 2.4. Hence $j(s, \omega)$ defined by (2.1), can be traced for $s \leq t$, if $j(s', \omega)$ is determined once at a certain time s' $(\leq t)$. Therefore we can determine $|j(s_1, \omega) - j(s_2, \omega)|$ for any $s_1, s_2 < t$, by the observed data $\{X(s, \omega); s \leq t\}$. By the condition (1.3), there exist s_1 and s_2 such that $|j(s_1, \omega) - j(s_2, \omega)| = L$, $s_1 < s_2$. Then $V(s_1, \omega)$ is in either I_0 or I_L. If L is even, we can determine $j(s_1, \omega)$ by observing which does $X(s, \omega)$ pass a peak or a bottom firstly after s_1. If L is odd, we can find a time s_3 $(s_1 < s_3 < s_2)$ when $V(s, \omega)$ passes $v_{(L+1)/2}$ and can see the interval containing $V(s, \omega)$ for each s in a neighbourhood of s_3. Thus we can trace $j(s, \omega)$, $s \leq t$. Obviously $V(s, \omega) = f^{-1}_{j(s,\omega)}(X(s, \omega))$ is $\mathscr{B}_t(X)$-measurable.

The local limit theorem (2.3) may seem a condition too strong. A weaker condition does work well. Suppose that there exists a positive continuous function $a(v)$ on (ℓ, r) admitting the following local limit theorem

$$O^r(V, t, \omega) \equiv \overline{\lim_{h \to 0}} \frac{|V(t + h, \omega) - V(t, \omega)|}{\gamma(h)} = a(V(t, \omega)) \text{ a.s.} \tag{2.5}$$

Define an increasing function $\varphi(v)$ on (ℓ, r) by

$$\varphi(v) \equiv \int_{v^*}^{v} \frac{du}{a(u)}, \qquad v \in (\ell, r) \tag{2.6}$$

with a fixed v^* in (ℓ, r). Put $\bar{\ell} \equiv \varphi(\ell + 0)$, $\bar{r} \equiv \varphi(r - 0)$ and $g(w) \equiv f \circ \varphi^{-1}(w)$ for $w \in (\bar{\ell}, \bar{r})$. Since $\varphi(v)$ is one-to-one map from (ℓ, r) onto $(\bar{\ell}, \bar{r})$,

$$W(t, \omega) \equiv \varphi(V(t, \omega)), \qquad t \in R,$$

satisfies $\mathscr{B}_t(W) = \mathscr{B}_t(V)$ and

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$$X(t, \omega) \equiv g(W(t, \omega)) . \tag{2.7}$$

Since

$$O^{\gamma}(W; t, \omega) \equiv \overline{\lim_{h \to 0}} \frac{|W(t+h, \omega) - W(t, \omega)|}{\gamma(h)} = 1 \text{ a.s.}$$

for any t, we can apply Theorem 2.4 and Theorem 2.5 to the new base process $W(t)$. Define $F_j(h)$ and $\eta(f, a; v_{j-1})$ by

$$F_j(h) \equiv \begin{cases} |f'(f_j^{-1}(h))| a(f_j^{-1}(h)) & \text{if } f_j^{-1}(h) \in D(f) , \\ \infty & \text{otherwise,} \end{cases} \tag{2.8}$$

$$\eta(f, a; v_j) \equiv \inf \{|h - f(v_j)|; F_j(h) \neq F_{j-1}(h)\} \tag{2.9}$$

for $i = 1, 2, \cdots, L$.

THEOREM 2.6. *Suppose that the base process $V(t)$ satisfies* (2.5) *and that $f(v)$ is piecewise differentiable and piecewise monotone with peaks and bottoms $v_1, \cdots, v_L$. If $\eta(f, a; v_j) = 0$ for every j, $1 \leq j \leq L$, then $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ holds, and hence $\hat{X}(t; \tau) = E[f(V(t+\tau)) | \mathscr{B}_t(V)]$. Moreover, the value of $V(t, \omega)$ can be determined in a similar way to Theorem* 2.4 *and Theorem* 2.5.

§ 3. Diffusion processes

In this section, we consider the case that the strictly stationary process $V(t)$ is a conservative one-dimensional diffusion process with natural boundaries ℓ and r ($\ell < r$). We assume that $V(t)$ satisfies a stochastic differential equation

$$dV(t) = a(V(t))\, dB(t) + b(V(t))dt \tag{3.1}$$

with a suitable Brownian motion $B(t)$ and with C^1-coefficients $a(v)$ (> 0) and $b(v)$ on (ℓ, r). Then we know that the distribution $dm(v)$ of $V(t)$ is the speed measure and that the speed measure m and the scale measure s are given by

$$\begin{cases} m(v) \equiv c \displaystyle\int_{v^*}^{v} \frac{2}{a^2(u)} \exp\left[\int_{v^*}^{u} \frac{2b(w)}{a^2(w)} dw\right] du, \\ s(v) \equiv \dfrac{1}{c} \displaystyle\int_{v^*}^{v} \exp\left[\int_{v^*}^{u} \frac{-2b(w)}{a^2(w)} dw\right] du , \end{cases} \tag{3.2}$$

with a v^* in (ℓ, r) and a normalizing constant c as

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$$m(r-0)-m(\ell+0)=1\,. \tag{3.3}$$

Since ℓ and r are natural boundaries, we have

$$s(r-0)=-s(\ell+0)=\int^{r-0} s(v)dm(v)=-\int_{\ell+0} s(v)dm(v)=\infty\,. \tag{3.4}$$

As well known, the semigroup $\{T_t^V\}$ of the process $V(t)$ is self adjoint in $L^2(dm)$ and the transition probability of $V(t)$ has a symmetric density with respect to the speed measure m (see McKean [7]). The time reversed process $\tilde{V}(t)=V(-t)$ is also a diffusion process with the same semigroup, and hence $V(t)$ satisfies the time reversed stochastic differential equation

$$\tilde{d}V(t)=a(V(t))\tilde{d}\tilde{B}(t)-b(V(t))dt\,, \tag{3.5}$$

with a suitable Brownian motion $\tilde{B}(t)$. Here we denote by $\int_s^t \psi(r,\omega)\tilde{d}V(r)$ the time reversed stochastic integral

$$\int_s^t \psi(r,\omega)\tilde{d}V(r)\equiv\lim_{n\to\infty}\sum_{s\leq r_j\leq t}\psi(r_{j+1},\omega)(V(r_{j+1},\omega)-V(r_j,\omega)) \tag{3.6}$$

with $r_j\equiv j\cdot 2^{-n}$. Put $\gamma^*(h)\equiv[2|h|\log_2|h|^{-1}]^{1/2}$. Then we can prove that

$$O^{\gamma^*}(V;t,\omega)\equiv\overline{\lim_{h\to 0}}\frac{V(t+h,\omega)-V(t,\omega)}{\gamma^*(h)}=a(V(t,\omega))\ \text{a.s.}$$

for any t (see McKean [6] p. 57 Problem 3) and that

$$\overline{\lim_{t\to-\infty}}\ V(t,\omega)=r\,,\ \underline{\lim_{t\to-\infty}}\ V(t,\omega)=\ell\ \text{a.s.}$$

(see Ikeda-Watanabe [3] Chapter VI § 3). Thus we can apply Theorem 2.6. By using the strong Markov properties of $V(t)$ and its time reversed process $\tilde{V}(t)$, we can state more precise results (c.f. Lee [5]).

PROPOSITION 3.1. *Let $f(v)$ be a function in $L^2(dm)$. Then the best non-linear predictor of $X(t+\tau)=f(V(t+\tau))$ for the given data $\{X(s,\omega)\equiv f(V(s,\omega));s\leq t\}$ is given by*

$$\hat{X}(t;\tau)=E[(T_\tau^V f)(V(t))\,|\,\mathscr{B}_t(X)]\ \text{a.s.}$$

Proof. By the Markov property, we have

$$\hat{X}(t;\tau)=E[E[f(V(t+\tau))\,|\,\mathscr{B}_t(V)]\,|\,\mathscr{B}_t(X)]=E[(T_\tau^V f)(V(t))\,|\,\mathscr{B}_t(X)]\,.$$

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Now, we suppose that $f(v)$ has a unique bottom without peaks. Let $\eta(f, a, v_1)$ be the number defined by (2.9). Firstly, we discuss the case that $\eta(f, a, v_1) < \max\{f(\ell + 0), f(r - 0)\} - f(v_1)$. Then we can find a decreasing sequence $\{h_n\}$ in the $G(f)$ defined in Section 2 such that $\lim_{n\to\infty} h_n = \eta(f, a, v_1) + f(v_1)$. Define stopping times of the time reversed process $\{\tilde{V}(t) = V(-t)\}$ by

$$\begin{cases} T(t, h, \omega) \equiv \sup\{s \leq t; X(s, \omega) = h\}, \\ T(\omega) = T(t, \{h_n\}, \omega) \equiv \sup_n T(t, h_n, \omega). \end{cases} \tag{3.7}$$

For almost every ω in $A \equiv \{\omega; T(t, f(v_1), \omega) \leq T(\omega)\}$, we have the dichotomy

$$V(t, \omega) = \begin{cases} f_0^{-1}(X(t, \omega)) & \text{if } O^{r*}(X, T(\omega), \omega) = F_0(X(T(\omega), \omega)) \\ f_0^{-1}(X(t, \omega)) & \text{if } O^{r*}(X, T(\omega), \omega) = F_1(X(T(\omega), \omega)). \end{cases} \tag{3.8}$$

Thus we have

$$\hat{X}(t; \tau, \omega) = (T_\tau^V f)(f_j^{-1}(X(t, \omega))) \tag{3.9}$$

if $O^{r*}(V, T(\omega), \omega) = F_j(X(T(\omega), \omega))$ for a.e. $\omega \in A$. In the contrary case $T(t, f(v_1), \omega) > T(\omega)$, we can not determine the value of $V(t, \omega)$ by the observed data $\{X(s, \omega); s \leq t\}$. Under the condition A^c, we have an expression

$$\hat{X}(t; \tau, \omega) = \sum_{j=0,1} p_j(\omega)(T_\tau^V f)(f_j^{-1}(X(t, \omega))) \text{ a.e. } \omega \in A^c, \tag{3.10}$$

where $p_j(\omega)$ is the conditional probability of the event $V(t) \in I_j$ under the conditions A^c and $\mathscr{B}_t(X)$. Let us now calculate $p_j(\omega)$, $j = 0, 1$. Let functions $\varphi(v)$ and $g(w)$ be as in Section 2 with $v^* = v_1$. Then $g(w)$ is symmetric on an interval $[-q, q]$ with $g(q) = \eta(f, a, v_1) + f(v_1)$. The process $W(t) = \varphi(V(t))$ is a conservative diffusion process with the natural boundaries $\bar{\ell}$ and $\bar{r}$. Moreover, it satisfies the stochastic differential equations.

$$\begin{cases} dW(t) = dB(t) + \beta(W(t))dt, \\ \tilde{d}W(t) = \tilde{d}\tilde{B}(t) - \beta(W(t))dt, \end{cases} \tag{3.11}$$

with $\beta(w) \equiv b(\varphi^{-1}(w))/a(\varphi^{-1}(w)) - \frac{1}{2}a'(\varphi^{-1}(w))$. Put $T^*(\omega) \equiv T(t, f(v_1), \omega) = \sup\{s < t; W(s, \omega) = 0\}$. Then either $|W(s, \omega)| = W(s, \omega)$, $T^*(\omega) < s \leq t$ or $|W(s, \omega)| = -W(s, \omega)$, $T^*(\omega) < s \leq t$ holds. Therefore the following stochastic integral makes a sense and we have the relation

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$$\int_{T^*(\omega)}^{t} \psi(s, \omega)\tilde{d}|W(s, \omega)| = \begin{cases} \int_{T^*(\omega)}^{t} \psi(s, \omega)\tilde{d}W(s, \omega) & \text{if } W(t, \omega) > 0\,, \\ -\int_{T^*(\omega)}^{t} \psi(s, \omega)\tilde{d}W(s, \omega) & \text{if } W(t, \omega) < 0\,, \end{cases}$$

or a $\tilde{\mathscr{B}}_t(V)$-adapted $\psi(s, \omega)$ in a suitable class.

LEMMA 3.2. *Let $p_0(\omega)$ and $p_1(\omega)$ be the conditional probabilities in* 3.10). *Then it holds that*

$$\begin{aligned} p_1(\omega) = \Big\{1 + \exp\Big[\int_{T^*(\omega)}^{t} (\beta(|W(u)|) + \beta(-|W(u)|))\tilde{d}|W(u)| \\ + \frac{1}{2}\int_{T^*(\omega)}^{t} (\beta^2(-|W(u)|) - \beta^2(|W(u)|))du \\ - 2\int_{-|W(t)|}^{|W(t)|} \beta(w)dw\Big]\Big\}^{-1}. \end{aligned}$$

nd $p_0(\omega) = 1 - p_1(\omega)$ *on* A^c.

Proof. Since $W(t)$ is a stationary Markov process satisfying (3.11), he density of the distribution of $W(t)$ is given by

$$c' \exp\Big[\int_0^w 2\beta(u)du\Big]$$

ith a normalizing constant c', we have

$$\text{3.12)} \qquad \frac{P(W(t) < 0 \,|\, |W(t)|)}{P(W(t) > 0 \,|\, |W(t)|)} = \exp\Big[-\int_{-|W(t)|}^{|W(t)|} 2\beta(u)du\Big].$$

n the set A^c, the data $X(s, \omega)$, $T^*(\omega) < s \leq t$, determine the values $W(s, \omega)|$, $T^*(\omega) < s \leq t$. By the strong Markov property, $\{W(s, \omega), s \leq$ $^*(\omega)\}$ is independent of $\{W(s, \omega); T^*(\omega) < s \leq t\}$. Hence it is sufficient or a computation of $p_1(\omega)$ that we consider the σ-field $\mathscr{B}_t(W) \cap \tilde{\mathscr{B}}_{T^*}(W)$. et $P_w(\cdot)$ be the conditional measure defined on $\mathscr{B}_t(W) \cap \tilde{\mathscr{B}}_{T^*}(W)$ condi- oned by $W(t) = w$. Define

$$\begin{aligned} M(\omega) &\equiv \exp\Big[\int_{T^*}^{t} \beta(W(u))\tilde{d}W(u) + \frac{1}{2}\int_{T^*}^{t} \beta^2(W(u))du\Big], \\ Y(s, \omega) &\equiv W((t - s)^\vee T(\omega)^\vee T^*(\omega)) - W(t) \\ &= B((t - s)^\vee T^\vee T^*) - B(t) - \int_t^{(t-s)^\vee T^\vee T^*} \beta^2(W(u))du\,. \end{aligned}$$

hen $(Y(s, \omega), P_W)$ is a stopped Brownian motion at $\{q - w, -q - w,$ $- w\}$, where $P_W = MP_w$ (see Ikeda-Watanabe [3], Chapter IV, § 4). Since

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the measure of a Brownian motion is symmetric, we can conclude the assertion by (3.12).

THEOREM 3.3. (i) *If* $\eta(f, a, v_1) = 0$, *then* $\mathscr{B}_t(X) = \mathscr{B}_t(V)$, *and hence* $\hat{X}(t; \tau) = (T_\tau^V f)(V(t))$. *The value of* $V(t, \omega)$ *is determined by* (3.11).

(ii) *If* $f(v_1) < \eta(f, a, v_1) + f(v_1) < \max\{f(\ell + 0), f(r - 0)\}$, *then we have*

$$\hat{X}(t; \tau, \omega) = \begin{cases} (T_\tau^V f)(V(t, \omega)) & \textit{for a.e. } \omega \textit{ in } A\,, \\ \sum_{j=0,1} p_j(\omega)(T_\tau^V f)(f_j^{-1}(X(t, \omega))) & \textit{for a.e. } \omega \textit{ in } A\,, \end{cases}$$

where $A = \{\omega; T^*(\omega) \leq T(\omega)\}$.

(iii) *If* $\eta(f, a, v_1) + f(v_1) = \max\{f(\ell + 0), f(r - 0)\}$, *then*

$$\hat{X}(t; \tau, \omega) = \sum p_j(\omega)(T_\tau^V f)(f_j^{-1}(X(t, \omega))) \text{ a.e. } \omega\,.$$

We now remark another interesting fact. Suppose that the generator $\mathscr{L} = \frac{1}{2}a^2(v)\partial/\partial v^2 + b(v)\partial/\partial v$ of $V(t)$ has an eigenvalue $\lambda(< 0)$ and a corresponding eigenfunction $f_\lambda(v)$ in $L^2(dm)$; that is,

$$(T_t^V f_\lambda)(v) = e^{\lambda t} f_\lambda(v)\,. \tag{3.13}$$

Then the best non-linear predictor of the transformed process $X(t) = f_\lambda(V(t))$ is given by

$$\hat{X}(t; \tau) = e^{\lambda \tau} X(t)\,, \tag{3.14}$$

by virtue of Proposition 3.1 and (3.13). The expression (3.14) means that the best non-linear predictor coincides with the best linear predictor. If $V(t)$ is Ornstein-Uhlenbeck process, then Hermite polynomials serve the eigenfunctions (c.f. Lee [5]).

§ 4. Processes with differetiable paths

In Section 2, we have discussed the cases that the base processes satisfy a local limit theorem either (2.3) or (2.5). If a base process $V(t)$ has differentiable paths, then such a local limit theorem does not hold. However we may expect that the N-th derivative $V^{(N)}(t, \omega)$ satisfies the local limit theorem (2.3). Actually, we have many examples in Gaussian stationary processes.

We assume that $V(t, \omega)$ is N-times continuously differentiable ($N \geq 1$) and its N-th derivative $V^{(N)}(t, \omega)$ satisfies

$$O^j(V^{(N)}; t, \omega) \equiv \overline{\lim_{h \to 0}} \frac{|V^{(N)}(t + h, \omega) - V^{(N)}(t, \omega)|}{\gamma(h)} = 1 \text{ a.s.} \tag{4.1}$$

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for any t. If $f(v)$ is $(N+1)$-times differentiable and piecewise monotone, the process $X(t) = f(V(t))$ has continuously N-times differentiable paths and its N-th derivative $X^{(N)}(t)$ satisfies the local limit theorem

$$(4.2) \qquad O^{\gamma}(X^{(N)}, t, \omega) \equiv \overline{\lim_{h \to 0}} \frac{|X^{(N)}(t+h, \omega) - X^{(N)}(t, \omega)|}{\gamma(h)} = |f'(V(t))| \text{ a.s.}$$

for any t. In a similar manner to Theorems 2.4 and 2.5, we can show:

THEOREM 4.1. *Suppose that $V(t)$ has continuously N-times differentiable paths satisfying* (4.1) *and that $f(v)$ is $(N+1)$-times differentiable and piecewise monotone $(N \geq 1)$. If $f(v)$ has no local symmetry, then $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ and $\hat{X}(t; \tau) = E[f(V(t+\tau)) | \mathscr{B}_t(V)]$ hold. In particular, if $f(v)$ has a single bottom without any peaks, then the value of $V(t, \omega)$ is determined by the observed data $\{X(s, \omega); s \leq t\}$ as follows*

$$V(t, \omega) = \begin{cases} f_0^{-1}(X(t, \omega)) & \text{if } T_0(T, \omega) \geq T_1(t, \omega), \\ f_1^{-1}(X(t, \omega)) & \text{if } T_0(t, \omega) < T_1(t, \omega). \end{cases}$$

where $T_j(t, \omega)$, $j = 0, 1$, are defined by

$$T_j(t, \omega) = \sup_{\substack{k \in Z \\ n \geq 1}} \left\{ \frac{k}{2^n} \leq t; \begin{array}{l} X\left(\frac{k}{2^n}, \omega\right) \in G(f), \\ O^{\gamma}\left(X^{(N)}; \frac{k}{2^n}, \omega\right) = \left| f'\left(f_j^{-1}\left(X\left(\frac{k}{2^n}, \omega\right)\right)\right)\right| \end{array} \right\}.$$

The theorem is merely a correspondence to the former result. But the following owes to the differentiability of paths essentially. We assume that

$$(4.3) \qquad P(\{\exists t \in R,\ V(t, \omega) = v^*,\ V'(t, \omega) = 0\}) = 0 \qquad \text{for any } v^*$$

and that $f(v)$ is piecewise monotone with peaks and bottoms at $\{v_1, \cdots, v_L\}$. Put $C(f) = \{v; \exists j, f(v_j) = f(v)\}$ and $\bar{C}(f) = \{f(v_j); 1 \leq j \leq L\}$. Then the set $\hat{\Omega} = \{\omega;\ V(t, \omega)$ is continuously differentiable, $\exists t \in R,\ V(t, \omega) \in C(f),\ V'(t, \omega) = 0\}$ has probability one. For any ω in $\hat{\Omega}$, $X(t, \omega)$ does not take its maximal (or minimal) values in $\bar{C}(f)$ except the times when $V(t, \omega)$ passes through the points $\{v_1, \cdots, v_L\}$. Moreover, $V(t, \omega)$ does not turn its moving direction at the times. Thus we can have the time sequence when $V(s, \omega)$, $s \leq t$, moves into an interval I_j from the other, by the observed data $\{X(s, \omega); s \leq t\}$. In the same way as proof of Theorem 2.5, we get the value of $j(s, \omega)$ defined by (2.1), if L is even. If L is odd, then we get an

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estimation $\hat{j}(s, \omega)$ such that

$$j(s, \omega) = \begin{cases} \hat{j}(s, \omega) & \text{for } s \leq t, \quad \text{if } j(t, \omega) \geq (L+1)/2\,, \\ L - \hat{j}(s, \omega) & \text{for } s \leq t, \quad \text{if } j(t, \omega) < (L+1)/2\,. \end{cases} \tag{4.4}$$

Define

$$\hat{V}(s, \omega) \equiv f^{-1}_{\hat{j}(s,\omega)}(X(s, \omega))\,, \qquad s \leq t\,. \tag{4.5}$$

If $f(v)$ has a symmetry with respect to v^*, then the following dichotomy holds;

$$V(s, \omega) = \begin{cases} \hat{V}(s, \omega) & \text{for } s \leq t\,, \quad \text{if } j(t, \omega) \geq (L+1)/2\,, \\ 2v^* - \hat{V}(s, \omega) & \text{for } s \leq t\,, \quad \text{if } j(t, \omega) < (L+1)/2\,. \end{cases} \tag{4.6}$$

THEOREM 4.2. *Suppose that $V(t)$ has continuously N-times differentiable paths ($N \geq 1$) and that $V(t)$ satisfies the condition* (4.3). *Suppose that $f(v)$ is piecewise monotone.*

(i) *If L is even, then $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ and $\hat{X}(t; \tau) = E[f(V(t+\tau)) \mid \mathscr{B}_t(V)]$ hold.*

(ii) *If $f(v)$ has a global symmetry with respect to a v^*, then the dichotomy* (4.6) *holds. Let $S(\omega)$ be a $\mathscr{B}_t(X)$-measurable random time such that $V(S(\omega), \omega) \neq 0$, $S(\omega) < t$. We have that*

$$\mathscr{B}_t(X) = \sigma(\{|V(s) - v^*|;\, s \leq t\}) \quad \textit{and}$$
$$\mathscr{B}_t(V) = \mathscr{B}_t(X) \vee \sigma(V(S(\omega), \omega))\,.$$

(iii) *If $f(v)$ has no global symmetry and is $(N+1)$-times differentiable and if $V(t)$ satisfies* (4.1), *then we have $\mathscr{B}_t(X) = \mathscr{B}_t(V)$ and $\hat{X}(t; \tau) = E[f(V(t+\tau)) \mid \mathscr{B}_t(V)]$.*

Proof. (i) By the above discussion, we know $j(s, \omega)$, since L is even. Therefore $V(s, \omega)$ is determined by $V(s, \omega) = f^{-1}_{j(s,\omega)}(X(s, \omega))$.

(ii) We have seen the dichotomy (4.6) whenever $f(v)$ has a global symmetry with respect to v^*. Therefore $|V(s, \omega) - v^*| = |\hat{V}(s, \omega) - v^*|$ is $\mathscr{B}_s(X)$-measurable. Thus we have the first equality. If we know once the actual value of $V(s', \omega)$ at a time $s' < t$, then the dichotomy (4.6) determines the values of $V(s, \omega)$, $s \leq t$.

(iii) If $f(v)$ has no global symmetry, we can find i and h such that $|f'(f_i^{-1}(h))| \neq |f'(f_{L-i}^{-1}(h))|$. Appealing to (4.2), we can determine $j(s, \omega)$ at the time s when $\hat{j}(s, \omega) = i$ and $X(s, \omega) = h$ hold.

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§ 5. Stationary Gaussian processes

In this section, we assume that process $V(t)$ is a stationary Gaussian process with continuous paths and that $V(t)$ has a canonical representation

$$V(t) = \int_{-\infty}^{t} G(t-u)dB(u) \tag{5.1}$$

with a suitable Brownian motion $\{B(t), t \in R\}$ and with a canonical kernel $G(t-u)$ (see Hida [2]). Let $\{T_\tau\}$ be the semi-group of the Brownian motion; that is,

$$(T_\tau f)(v) \equiv \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\tau}} f(u) \exp\left[-\frac{1}{2\tau}(v-u)^2\right] du\,. \tag{5.2}$$

Then we can easily see that

$$\begin{cases} \sigma^2 \equiv E[V(t)^2] = \int_0^{\infty} G(u)^2 du \\ \hat{V}(t;\tau) \equiv E[V(t+\tau) \mid \mathscr{B}_t(v)] = \int_{-\infty}^{t} G(t+\tau-u)dB(u) \qquad \text{a.s.}\,, \\ \delta^2 \equiv E[(V(t+\tau) - \hat{V}(t;\tau))^2 \mid \mathscr{B}_t(V)] = \int_0^{\tau} G(u)^2 du \qquad \text{a.s.} \end{cases} \tag{5.3}$$

For any $f(v)$ in $L^2((2\pi\sigma^2)^{-1/2} \exp[-v^2/2\sigma^2]dv)$, $X(t) = f(V(t))$ is a strictly stationary process with mean $(T_{\sigma^2}f)(0)$ and with variance $(T_{\sigma^2}f^2)(0) - (T_{\sigma^2}f)(0)^2$. The best non-linear predictor of $X(t+\tau)$ for the given data $\{X(s,\omega); s \leqq t\}$ is expressed in the form

$$\hat{X}(t;\tau) = E[(T_{\delta^2}f)(\hat{V}(t;\tau)) \mid \mathscr{B}_t(X)]\,. \tag{5.4}$$

In particular, if $\mathscr{B}_t(X) = \mathscr{B}_t(V)$, then we have

$$\hat{X}(t;\tau) = (T_{\delta^2}f)(\hat{V}(t;\tau))\,. \tag{5.4$'$}$$

Since $V(t)$ has the canonical representation (5.1), $V(t)$ is ergodic. Hence we have that

$$\lim_{s\to\infty} \frac{1}{s} \int_{t-s}^{t} V(u,\omega)du = 0 \qquad \text{a.s.}\,, \tag{5.5}$$

$$\overline{\lim_{s\to-\infty}}\, V(s,\omega) = -\underline{\lim_{s\to-\infty}}\, V(s,\omega) = \infty \qquad \text{a.s.} \tag{5.6}$$

Moreover, if $0 < \overline{\lim}_{h\to 0} |V(t+h,\omega) - V(t,\omega)|/\gamma(h) < \infty$ a.s., then the value must be constant a.s. Therefore the assumption (2.5) is abandon. We can

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apply Theorem 2.4 and Theorem 2.5. Sufficient conditions for (2.3) have been discussed by many authors (for example, see Kōno [4]).

We now come to the case that the base process $V(t)$ has differentiable paths and that the N-th derivative of $V(t)$ satisfies the local limit theorem (2.3). In this case, we can apply Theorem 4.1. More interesting is the case that (4.3) is applied. Suppose that $f(v)$ is piecewise monotone and globally symmetric with respect to a v^*. As discussed in Section 4, we have the estimation $\{\hat{V}(s, \omega); s \leqq t\}$ with the dichotomy (4.6). Together with (5.5), we have the dichotomy

$$\bar{v} \equiv \lim_{s\to\infty} \frac{1}{s} \int_{t-s}^{t} \hat{V}(u, \omega)du = 0 \text{ or } 2v^* \qquad \text{a.s.} \tag{5.6}$$

Hence we can determine $V(s, \omega)$ as follows

$$V(s, \omega) = \begin{cases} \hat{V}(s, \omega) & \text{if } \bar{v} = 0 \\ 2v^* - \hat{V}(s, \omega) & \text{if } \bar{v} = 2v^*, \end{cases}$$

if $v^* \neq 0$. Thus we have $\mathscr{B}_t(X) = \mathscr{B}_t(V)$. If $v^* = 0$, then either $\hat{V}(s, \omega) = V(s, \omega)$ for $s \leqq t$, or $\hat{V}(s, \omega) = -V(s, \omega)$ for $s \leqq t$, holds. Since $\hat{V}(t; \tau)$ is a linear functional of $\{V(s, \omega); s \leqq t\}$, $|\hat{V}(t; \tau)|$ is $\{\hat{V}(s, \omega); s \leqq t\}$-measurable; that is, $\mathscr{B}_t(X)$-measurable. Since $(T_{\delta^2}f)(v)$ is an even function, We have

$$\hat{X}(t; \tau) = (T_{\delta^2}f)(\hat{V}(t; \tau)) = (T_{\delta^2}f)(|\hat{V}(t; \tau)|) \tag{5.7}$$

with δ^2 defined by (5.3).

THEOREM 5.1. *Suppose that a Gaussian stationary process $V(t)$ with the canonical representation* (5.1) *has N-times continuously differentiable paths and that the N-th derivative $V^{(N)}(t, \omega)$ satisfies the local limit theorem* (2.3).

(i) *If $f(v)$ is $(N+1)$-times continuously differentiable and piecewise monotone, then*

$$\hat{X}(t; \tau) = (T_{\delta^2}f)(\hat{V}(t; \tau)) \quad \textit{with} \quad \delta^2 \equiv \int_0^\tau G(u)^2 du\,. \tag{5.8}$$

(ii) *If $f(v)$ is piecewise monotone and globally symmetric, then* (5.8) *holds again.*

For example, we now discuss double Markov, stationary Gaussian processes in the sense of Hida [2]. Such processes are classified in three cases by canonical representations:

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Case 1. $V(t) = \int_{-\infty}^{t} (b_1 e^{-a_1(t-u)} + b_2 e^{-a_2(t-u)}) dB(u)$,
with $a_1 \neq a_2 > 0$, $b_1 + b_2 \neq 0$, $(a_1 b_2 + a_2 b_1)/(b_1 + b_2) \leqq 0$.

Case 2. $V(t) = b \int_{-\infty}^{t} (e^{-a_1(t-u)} - e^{-a_2(t-u)}) dB(u)$
with $a_1 \neq a_2 > 0$, $b \neq 0$.

Case 3. $V(t) = b \int_{-\infty}^{t} (t-u) e^{-a(t-u)} dB(u)$
with $a > 0$, $b \neq 0$.

In Case 1, we can easily see that

$$\overline{\lim_{h \to 0}} \frac{V(t+h, \omega) - V(t, \omega)}{|b_1 + b_2| \gamma^*(h)} = 1 \qquad \text{a.s. for any } t,$$

where $\gamma^*(h) = [2|h| \log_2 (1/|h|)]^{1/2}$. Thus we can apply Theorems 2.4 and 2.5. The quantities in (5.3) are calculated;

$$\sigma^2 \equiv E[V(t)^2] = \frac{b_1^2}{2a_1} + \frac{2b_1 b_2}{a_1 + a_2} + \frac{b_2^2}{2a_2},$$

$$\hat{V}(t; \tau) = \int_{-\infty}^{t} (b_1 e^{-a_1(t+\tau-u)} + b_2 e^{-a_2(t+\tau-u)}) dB(u),$$

$$\delta^2 \equiv (1 - e^{-2a_1\tau}) \frac{b_0^2}{2a_1} + (1 - e^{-(a_1+a_2)\tau}) \frac{2b_1 b_2}{a_1 + a_2} + (1 - e^{-2a_2\tau}) \frac{b_2^2}{2a_2}.$$

In both Case 2 and Case 3, the process $V(t)$ has continuously differentiable paths. Moreover $(V(t), V'(t))$ is a two-dimensional diffusion process.

Case 2. The generator $\mathscr{L}$ is given by

$$\mathscr{L} = \frac{1}{2} b^2 (a_2 - a_1)^2 \frac{\partial^2}{\partial v_2^2} - (a_1 a_2 v_1 + (a_1 + a_2) v_2) \frac{\partial}{\partial v_2} + v_2 \frac{\partial}{\partial v_1}$$

and the $V(t)$ satisfies the stochastic differential equation;

$$\text{(5.9)} \qquad \begin{cases} dV(t) = V'(t) dt \\ dV'(t) = b(a_2 - a_1) dB(t) - a_1 a_2 V(t) dt - (a_1 + a_2) V'(t) dt. \end{cases}$$

The transition probability density $p(t, v_1, v_2; u_1, u_2)$ is given by

$$\text{(5.10)} \qquad (4\pi^2 \det \Gamma(t))^{-1/2} \exp \left[\frac{-1}{2 \det \Gamma(t)} \{\Gamma_{22}(t) \xi_1^2 - 2\Gamma_{12}(t) \xi_1 \xi_2 + \Gamma_{11}(t) \xi_2^2\} \right],$$

where $\Gamma(t) = (\Gamma_{ij}(t))_{1 \leqq i, j \leqq 2}$,

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$$\Gamma_{11}(t) = b^2 \int_0^t (e^{-a_1 u} - e^{-a_2 u})^2 du\,,$$

$$\Gamma_{12}(t) = \Gamma_{21}(t) = b^2 \int_0^t (e^{-a_1 u} - e^{-a_2 u})(-a_1 e^{-a_1 u} + a_2 e^{-a_2 u}) du\,,$$

$$\Gamma_{22}(t) = b^2 \int_0^t (-a_1 e^{-a_1 u} + a_2 e^{-a_2 u})^2 du\,,$$

$$\xi_1 = u_1 - \frac{a_2 e^{-a_1 t} - a_1 e^{-a_2 t}}{a_2 - a_1} v_1 - \frac{e^{-a_1 t} - e^{-a_2 t}}{a_2 - a_1} v_2\,,$$

$$\xi_2 = u_2 + \frac{a_1 a_2 (e^{-a_1 t} - e^{-a_2 t})}{a_2 - a_1} v_1 + \frac{a_1 e^{-a_1 t} - a_2 e^{-a_2 t}}{a_2 - a_1} v_2\,.$$

As $t \to 0$, we have

$$\Gamma_{11}(t) = \frac{b^2}{3}(a_2 - a_1)^2 t^3 + O(t^4)\,,$$

$$\Gamma_{12}(t) = \Gamma_{21}(t) = \frac{b^2}{2}(a_2 - a_1)^2 t^2 + O(t^3)\,,$$

$$\Gamma_{22}(t) = b^2 (a_2 - a_1)^2 t + O(t^2)\,,$$

$$\det \Gamma(t) = \frac{b^4}{12}(a_2 - a_1)^4 t^4 + O(t^5)\,.$$

Case 3. The generator $\mathscr{L}$ is given by

$$\mathscr{L} = \frac{1}{2} b^2 \frac{\partial^2}{\partial v_2^2} - (a^2 v_1 + 2a v_2)\frac{\partial}{\partial v_2} + v_2 \frac{\partial}{\partial v_1}$$

and the $V(t)$ satisfies the stochastic differential equation;

$$(5.9)' \qquad \begin{cases} dV(t) = V'(t)dt \\ dV'(t) = b\,dB(t) - a^2 V(t)dt - 2a V'(t)dt. \end{cases}$$

The transition probability density $p(t, v_1, v_2; u_1, u_2)$ is given by the same formula as (5.10) with the replacement;

$$\Gamma_{11}(t) = b^2 \int_0^t u^2 e^{-2au} du\,,$$

$$\Gamma_{12}(t) = \Gamma_{21}(t) = b^2 \int_0^t u(1 - au) e^{-2au} du\,,$$

$$\Gamma_{22}(t) = b^2 \int_0^t (1 - au)^2 e^{-2au} du\,,$$

$$\xi_1 = u_1 - (at + 1)e^{-at} v_1 - t e^{-at} v_2\,,$$

$$\xi_2 = u_2 + a^2 t e^{-at} v_1 - (1 - at) e^{-at} v_2\,.$$

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Then their orders as $t \to 0$ are given by

$$\Gamma_{11}(t) = \frac{b^2}{3}t^3 + O(t^4)\,, \qquad \Gamma_{12}(t) = \Gamma_{21}(t) = \frac{b^2}{2}t + O(t^3)\,,$$

$$\Gamma_{22}(t) = b^2 t + O(t^2) \quad \text{and} \quad \det \Gamma(t) = \frac{b^4}{12}t^4 + O(t^5)\,.$$

In both cases, we can show that the Green function

$$g_\alpha(v_1, v_2, u_1, u_2) = \int_0^\infty p(t, v_1, v_2; u_1, u_2)e^{-\alpha t}dt\,, \qquad \alpha > 0\,,$$

is positive, continuous except on the diagonal $\{(v_1, v_2) = (u_1, u_2)\}$, and satisfies

$$\mathscr{L} g_\alpha = \alpha g_\alpha \qquad \text{for } (v_1, v_2) \neq (u_1, u_2)$$

and

$$\lim_{(v_1, v_2) \to (u_1, 0)} g_\alpha(v_1, v_2, u_1, 0) = \infty\,.$$

Therefore we can apply Lemma 2.1 of Chapter 11 in Friedman [1]. Thus we can check in both Case 2 and Case 3, that $V(t)$ satisfies the condition (4.3).

By the proposition, we know that Theorem 5.1 is applicable to the double Markov, stationary Gaussian processes in the strict sense (Case 2 and Case 3).

THEOREM 5.2. *Let $V(t)$ be a double Markov, stationary Gaussian process in the strict sense and let $f(v)$ be piecewise monotone.*

(i) *If $f(v)$ is 3-times continuously diffentiable, then*

$$\hat{X}(t; \tau) = \begin{cases} (T_{\Gamma_{11}(\tau)}f)\left(\dfrac{a_2 e^{-a_1\tau} - a_1 e^{-a_2\tau}}{a_2 - a_1} V(t) + \dfrac{e^{-a_1\tau} - e^{-a_2\tau}}{a_2 - a_1} V'(t)\right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for Case 2}, \\ (T_{\Gamma_{11}(\tau)}f)((1 + a\tau)V(t) + \tau e^{-a\tau} V'(t)) \qquad \textit{for Case 3}. \end{cases} \tag{5.11}$$

(ii) *If $f(v)$ has a global symmetry with respect to a v^* with $v^* \neq 0$, then* (5.11) *holds. If $f(v)$ has a global symmetry with respect to* 0, *then* (5.11) *holds replacing $V(t, \omega)$ with $\hat{V}(t, \omega)$ defined by* (4.5).

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