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RANKIN-SELBERG METHOD FOR SIEGEL CUSP FORMS

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Introduction

Let G_n (resp. Γ_n) be the real symplectic (resp. Siegel modular) group of degree n. The Siegel cusp form is a holomorphic function on the Siegel upper half plane which satisfies functional equations relative to Γ_n and vanishes at the cusps. For an integer r, $1 \leq r \leq n$, there exists a maximal parabolic subgroup P_r of G_n defined by

$$P_r = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n \,|\, a_{21} = c_{21} = 0,\ c_{11} = 0,\ c_{12} = 0 \right\},$$

in which we decompose an $n \times n$ matrix x into $r \times r$, $r \times (n-r)$, $(n-r) \times r$ and $(n-r) \times (n-r)$ submatrices x_{11}, x_{12}, x_{21} and x_{22}, respectively. Let F and H be Siegel cusp forms of the same weight l. For any half-integral positive definite symmetric matrix S of size r, we denote by f_S and h_S the S-th Fourier-Jacobi coefficients relative to P_r of F and H, respectively. Then they are Jacobi cusp forms of weight l and index S and we denote their Petersson inner product by (f_S, h_S). Consider a Dirichlet series defined by

$$D_r(F, H : s) = \sum_{S/\sim} \frac{1}{\varepsilon(S)} \frac{(f_S, h_S)}{(\det S)^s},$$

in which the summation is taken over the set of equivalence classes of S and $\varepsilon(S)$ denotes the order of its automorphism group. This is an obvious generalization of the symmetric square for the elliptic cusp forms ([8]). Our main objective is to show that the Rankin-Selberg method is applicable to the study of the analytic properties of $D_r(F, H : s)$.

We remark that, in the special case where $r = n$, this type of Dirichlet series has been examined by Maass [5] for $n = 2$ and by Kurokawa for general n (unpublished). Also Kohnen-Skoruppa [4] recently investigated

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the case where $n = 2$ and $r = 1$. Among other things, they showed that if $F = H$ is in the Maass space and is a common eigen function of the Hecke operators, then $D_1(F, F : s)$ has Euler product and, up to some elementary facts, coincides with Andrianov's spinor zeta function [1].

Now we give a brief account of the paper. In Section 1, we collect standard facts about Fourier-Jacobi expansion of the Siegel modular forms. In Section 2, following Kalinin [3] we closely examine the Eisenstein series $E_r(s : g)$ for the symplectic group. It is a function on $\mathbf{C} \times G_n$ and is a non-holomorphic automorphic form of weight zero with respect to g. We show that, as a function on $\mathbf{C}$, it can be continued meromorphically to the entire complex plane and satisfies a functional equation (Theorem 2.2). In a special case where $r = 1$, it has a nice holomorphy property (Theorem 2.3). In Section 3, we calculate the Petersson inner product (FE_r, H). It turns out that, up to some elementary factors, it is equal to a translate of $D_r(F, H: s)$ (Theorem 3.2). Then, applying the Rankin-Selberg method, we get analytic continuation and a functional equation for $D_r(F, H: s)$ (Theorem 3.4).

Notation. As usual we denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For an associative ring A with identity element, $A^\times$ denotes the group of invertible elements of A.

We denote by $M_{m,n}$ the set of $m \times n$ matrices. We put $M_m = M_{m,m}$. If x is a matrix, ${}^t x$, $\det(x)$ and $\mathrm{tr}(x)$ stand for its transpose, determinant and trace, respectively. The identity and zero matrix in M_m are denoted by 1_m and 0_m, respectively. If $x_1, \cdots, x_r$ are square matrices, $\mathrm{diag}(x_1, \cdots, x_r)$ denotes the matrix with $x_1, \cdots, x_r$ in the diagonal blocks and zero matrices in all other blocks.

For an algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$ and a commutative ring A, we denote by $\mathbf{G}(A)$ the group of A-valued points of $\mathbf{G}$.

We put $\mathrm{Sym}_m = \{S \in M_m \mid {}^t S = S\}$. For $S \in \mathrm{Sym}_m$ and $x \in M_{m,n}$, we write $S[x] = {}^t x S x$. Two symmetric matrices $S, T \in \mathrm{Sym}_m(\mathbf{Q})$ are said equivalent and written as $S \sim T$, if there exists $g \in GL_m(\mathbf{Z})$ such that $S[g] = T$.

The symplectic group Sp_n of degree n is defined by

$$Sp_n = \{g \in M_{2n} \mid {}^t g J_n g = J_n\},$$

in which $J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$. The Siegel upper half plane H_n of degree n

is the set of symmetric matrices $\tau = \mathrm{Sym}_n(\mathbf{C})$ with positive definite imaginary parts $\mathrm{Im}(\tau) > 0$.

For a real number x, we denote by $[x]$ the largest integer such that $[x] \leq x$. For a complex number s, we write $\mathbf{e}(s) = e^{2\pi i s}$. We also write $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, in which Γ denotes the gamma function and ζ denotes the Riemann zeta function.

1. Preliminaries

The purpose of this section is to summarize those items that we shall need in the following. Let us start at the Siegel cusp forms. Let $\mathbf{G}_n$ be the symplectic group of degree n. We put $G_n = \mathbf{G}_n(\mathbf{R})$ and $\Gamma_n = \mathbf{G}_n(\mathbf{Z})$. Then G_n operates transitively on the Siegel upper half plane, namely for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G_n and τ in H_n, we define

$$g\langle\tau\rangle = (a\tau + b)(c\tau + d)^{-1},$$

and the canonical automorphic factor is given by

$$j(g, \tau) = c\tau + d.$$

The isotropy subgroup K at $\tau_0 = i1_n$ is a maximal compact subgroup of G_n. Let us fix a natural number l and consider a function F on G_n which satisfies the functional equation

$$\text{(S1)} \qquad F(\gamma gk) = \det j(k, \tau_0)^{-l} F(g),$$

for all γ in Γ_n and k in K. For any function F on G_n which satisfies (S1), we put

$$F^o(\tau) = \det j(g_\tau, \tau_0)^l F(g_\tau),$$

in which for any τ in H_n we take an element g_τ in G_n such that $g_\tau\langle\tau_0\rangle = \tau$. Then F^o does not depend on the choice of g_τ and defines a function on H_n. For a function F on G_n satisfying (S1), we consider the following conditions.

(S2) The associated function F^o on H_n is holomorphic.

(S3) The function F is bounded on G_n.

The functions on G_n which satisfy the conditions (S1), (S2) and (S3) are called the Siegel cusp forms of weight l, and we denote by $S(l)$ the totality of such functions. We also define the Petersson inner product on $S(l)$ by

$$(F_1, F_2) = \int_{\Gamma_n/G_n} F_1(g)\overline{F_2(g)}dg\,,$$

in which dg denotes the Haar measure on G_n.

Secondly we shall briefly recall the basic facts about the Jacobi forms. For more details, we refer to Murase [7]. Let m and r be natural numbers. For $h = (\lambda, \mu, \kappa) \in M_{r,m} \times M_{r,m} \times \mathrm{Sym}_r$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}_m$, we put

$$(h, g) = \begin{bmatrix} 1_r & 0 & \kappa & \mu \\ 0 & 1_m & {}^t\mu & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & 1_m \end{bmatrix} \times \begin{bmatrix} 1_r & \lambda & 0 & 0 \\ 0 & 1_m & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & -{}^t\lambda & 1_m \end{bmatrix} \times \begin{bmatrix} 1_r & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1_r & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

Then $\mathbf{G}_{m,r} = \{(h, g) | h \in M_{r,m} \times M_{r,m} \times \mathrm{Sym}_r, g \in \mathbf{G}_m\}$ forms a $\mathbf{Q}$-algebraic subgroup of $\mathbf{G}_{m+r}$, and it is a semi-direct product of the Heisenberg group $\mathbf{H}_{m,r} = \{(h, 1_{2m}) | h \in M_{r,m} \times M_{r,m} \times \mathrm{Sym}_r\}$ and $\mathbf{G}_m$. Note that the center of $\mathbf{G}_{m,r}$ is $\mathbf{Z}_{m,r} = \{((0, 0, \kappa), 1_{2m}) | \kappa \in \mathrm{Sym}_r\}$. For simplicity, we wrtie hg for each element (h, g) of $\mathbf{G}_{m,r}$. Let $D_{m,r}$ denote the complex domain $H_m \times M_{r,m}(\mathbf{C})$. Then $G_{m,r} = \mathbf{G}_{m,r}(\mathbf{R})$ acts on $D_{m,r}$ transitively by

$$\eta\langle Z\rangle = (g\langle\tau\rangle,\ zj(g, \tau)^{-1} + \lambda g\langle\tau\rangle + \mu)\,,$$

in which $\eta = (\lambda, \mu, \kappa)g \in G_{m,r}$ and $Z = (\tau, z) \in D_{m,r}$. The stabilizer of $Z_0 = (\tau_0, 0) \in D_{m,r}$ in $G_{m,r}$ coincides with $\mathbf{Z}_{m,r}(\mathbf{R})K$. We shall fix a natural number l and a half-integral positive definite symmetric matrix S of size r. The automorphic factor $J_{l,S}\colon G_{m,r} \times D_{m,r} \to \mathbf{C}^\times$ of weight l and index S is defined by

$$J_{l,S}(\eta, Z) = \det j(g, \tau)^l J_S(\eta, Z)\,,$$

where for $\eta = (\lambda, \mu, \kappa)g \in G_{m,r}, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Z = (\tau, z) \in D_{m,r}$ we put

$$\begin{aligned} &J_S(\eta, Z) \\ &\quad = \mathbf{e}(-\operatorname{tr}(S\kappa) + \operatorname{tr}(S[z]j(g, \tau)^{-1}c) - 2\operatorname{tr}({}^t\lambda Szj(g, \tau)^{-1}) - \operatorname{tr}(S[\lambda]g\langle\tau\rangle))\,. \end{aligned}$$

We also define a character ψ_S of $\mathrm{Sym}_r(\mathbf{R})/\mathrm{Sym}_r(\mathbf{Z})$ by

$$\psi_S(\kappa) = \mathbf{e}(\operatorname{tr}(S\kappa))\,.$$

Let f be a function on $G_{m,r}$ satisfying

(J1) $$f((0, 0, \kappa)\gamma\eta k) = \det j(k, \tau_0)^{-l}\psi_S(\kappa)f(\eta)$$

for $\kappa \in \mathrm{Sym}_r(\mathbf{R})$, $\gamma \in G_{m,r}(\mathbf{Z})$ and $k \in K$. For each $Z \in D_{m,r}$, take an element $\eta_z \in G_{m,r}$ so that $\eta_z\langle Z_0\rangle = Z$ and put

$$f^o(Z) = f(\eta_z)J_{l,s}(\eta_z, Z_z)\,.$$

Then $f^o(Z)$ does not depend on the choice of η_z and defines a function on $D_{m,r}$.

Let $S(l, S)$ be the space of functions f on $G_{m,r}$ satisfying the following conditions (J2) and (J3) as well as (J1).

(J2) The associated function f^o is holomorphic on $D_{m,r}$.

(J3) The function f is bounded on $G_{m,r}$.

Each element of $S(l, S)$ is called a Jacobi cusp form of weight l and index S. The Petersson inner product is defined by

$$(f_1, f_2) = \int_{G_{m,r}(\mathbf{Z})\backslash G_{m,r}} f_1(\eta)\overline{f_2(\eta)}d\eta\,.$$

Finally let us explain about Fourier-Jacobi expansions of automorphic forms relative to a parabolic subgroup. Take integers r, n such that $1 \leq r \leq n$ and put $m = n - r$. Then we have the maximal parabolic subgroup P_r of G_n defined by (see Section 2)

$$P_r = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n \,\middle|\, a_{21} = 0, c_{11} = 0, c_{12} = 0, c_{21} = 0,\ d_{12} = 0\right\},$$

in which a, b, c, and d are $n \times n$ matrices and decompose an $n \times n$ matrix x into $r \times r$, $r \times m$, $m \times r$ and $m \times m$ blocks $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. We shall always consider $G_{m,r}$ as a subgroup of P_r. For any element w in $GL(r, R)$, we define $\tilde{w} = \mathrm{diag}(w, 1_m, {}^tw^{-1}, 1_m)$. Then any element in P_r can be written uniquely as $\eta\tilde{w}$, where $\eta \in G_{m,r}$ and $w \in GL(r, \mathbf{R})$. Let F be a Siegel cusp form of weight l for Γ_n. For any positive definite half-integral matrix $S \in \mathrm{Sym}_r(\mathbf{Q})$, we define a function f_S on $G_{m,r}$ by

$$f_S(\eta) = \int_{\mathrm{Sym}_r(\mathbf{R})/\mathrm{Sym}_r(\mathbf{Z})} F((0, 0, x)\eta)\mathbf{e}(-\mathrm{tr}(S(i1_m + x)))dx\,.$$

Then f_S is a Jacobi cusp form of weight l and index S for Γ_m and we

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call it the S-th Fourier-Jacobi coefficient of F relative to P_r. The Fourier-Jacobi expansion of F relative to P_r is given by

$$F(\eta\tilde{w}) = \sum_{S>0} \mathbf{e}(i\,\mathrm{tr}(S[w]))(\det w)^l f_S(\eta),$$

in which the summation is taken over the set of positive definite half-integral symmetric matrices $S \in \mathrm{Sym}_r(\mathbf{Q})$. We note that, by the uniqueness of the Fourier-Jacobi expansion we have

$$f_{{}^tuSu}((\lambda, \mu, \kappa)g) = (\det u)^l f_S((u\lambda, u\mu, u\kappa\,{}^tu)g)$$

for all $S > 0$, $u \in GL(r, \mathbf{Z})$ and $(\lambda, \mu, \kappa)g \in G_{m,r}$.

In terms of the associated functions F^o and f_S^o with F and f_S, the Fourier-Jacobi expansion may be written as

$$F^o = \sum_{S>0} f_S^o(\tau_{22}, \tau_{12})\, \mathbf{e}(\mathrm{tr}(S\tau_{11})),$$

in which we decompose $\tau \in H_n$ into blocks $\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$ with $\tau_{11} \in H_r$, $\tau_{12} \in M_{r,m}(\mathbf{C})$ and $\tau_{22} \in H_m$.

§ 2. Eisenstein series

This section is devoted to a discussion of the Eisenstein series for the symplectic group. Since we essentially follow Kalinin [3], and since many of the statements can be proved in the similar way as [3], we omit most of the proofs.

As in the previous section, let G_n be the real symplectic group of degree n and let Γ_n be the Siegel modular group in G_n. Since we fix n all through this section, for simplicity we drop the index n and write just G and Γ for example. Let $\mathfrak{g}$ be the Lie algebra of G. We denote by e_{ij}, $(i, j = 1, \cdots, 2n)$ the matrix unit of size $2n$, and put $h_i = e_{ii} - e_{i+n,i+n}$ for $1 \leq i \leq n$. Then the Lie subalgebra $\mathfrak{a}$ spanned by h_i, $(1 \leq i \leq n)$ is a Cartan subalgebra of $\mathfrak{g}$. In the dual vector space $\mathfrak{a}^*$ we choose basis ε_i, $(1 \leq i \leq n)$ which is dual to h_i. As a system of positive roots relative to the Cartan subalgebra $\mathfrak{a}$, we may choose the set

$$\Sigma = \{2\varepsilon_i (1 \leq i \leq n),\ \varepsilon_i \pm \varepsilon_j (1 \leq i < j \leq n)\}.$$

With this choice of order, the set of simple roots is given by

$$\Sigma^o = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} (1 \leq i \leq n-1),\ \alpha_n = 2\varepsilon_n\}.$$

The Weyl group W is generated by the orthogonal reflections w_{α_i} for $1 \leq i \leq n$. We set

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Sigma} \mathfrak{n}_\alpha ,$$

in which $\mathfrak{n}_\alpha$ is the root subspace corresponding to α. Then $(\mathfrak{p}, \mathfrak{a})$ is a Borel pair in $\mathfrak{g}$ in the sense of [2]. Let (P, A) be the Borel pair in G corresponding to $(\mathfrak{p}, \mathfrak{a})$, and let $P = UAM$ be its Langlands decomposition. Let K be a maximal compact subgroup of G. Then we have $G = PK = UAMK$. Therefore any element g in G can be written as $g = uamk$, with $u \in U$, $a \in A$, $m \in M$ and $k \in K$, and the A-part a is uniquely determined. We denote it by $a(g)$. Let $\mathfrak{a}_{\mathbf{C}}^*$ be the dual of the complexified vector space $\mathfrak{a}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{a}$. For any λ in $\mathfrak{a}_{\mathbf{C}}^*$ and for any a in A, we put

$$\omega_\lambda(a) = e^{\lambda(\log a)} ,$$

in which log denotes the inverse of the exponential map of $\mathfrak{a}$ to A. We introduce coordinates on $\mathfrak{a}_{\mathbf{C}}^*$ as follows. We set for $1 \leq i \leq n$,

$$\bar{\omega}_i = \varepsilon_1 + \cdots + \varepsilon_i .$$

Note that $\bar{\omega}_i$, $i = 1, \cdots, n$ are the fundamental weights. For $(z_1, \cdots, z_n) \in \mathbf{C}^n$ we set

$$\lambda(z_1, \cdots, z_n) = \sum_{i=1}^{n} z_i \bar{\omega}_i .$$

In terms of these coordinates the vector $\lambda(1, \cdots, 1)$ is the half-sum ρ of the positive roots.

Now we define the Eisenstein series associated to the constant function on M. For any $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$ and for any $g \in G$, we set

$$\begin{aligned} E(z\colon g) &= E(\lambda(z)\colon g) \\ &= \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \omega_{\lambda(z)+\rho}(a(\gamma g)) . \end{aligned}$$

We remark that from the general theory of the Eisenstein series, $E(z\colon g)$ is holomorphic for $\mathrm{Re}(z_i) > 1$, $1 \leq i \leq n$. Let us fix an integer r such that $1 \leq r \leq n$. We set

$$\tilde{E}_r(z_r\colon g) = \mathrm{Res}_{z_n=1} \cdots \widehat{\mathrm{Res}_{z_r=1}} \cdots \mathrm{Res}_{z_1=1} E(z_1, \cdots, z_n\colon g) ,$$

in which we take residues at $z_i = 1$, $1 \leq i < n$ except at $z_r = 1$.

We shall need another type of Eisenstein series. We know that for any subset $F_r = \Sigma^o - \{\alpha_r\}$ of Σ^o, these exists a parabolic pair $(\mathfrak{p}_r, \mathfrak{a}_r)$ such that $\mathfrak{p}_r \supset \mathfrak{p}$ and $\mathfrak{a}_r \subset \mathfrak{a}$. In particular, by definition we have

$$\mathfrak{a}_r = \{H \in \mathfrak{a} \,|\, \alpha_i(H) = 0 \text{ for } i \neq r\}$$
$$= \mathbf{R}\cdot\left(\sum_{i=1}^{r} h_i\right).$$

We denote by Σ_r, the set of elements $\alpha \in \Sigma$ which are not identically equal to zero on $\mathfrak{a}_r$, and we set

$$\mathfrak{n}_r = \sum_{\alpha \in \Sigma_r} \mathfrak{n}_\alpha .$$

Then we have

$$\mathfrak{p}_r = \mathfrak{z}(\mathfrak{a}_r) + \mathfrak{n}_r ,$$

in which $\mathfrak{z}(\mathfrak{a}_r)$ is the centralizer of $\mathfrak{a}_r$ in $\mathfrak{g}$. Let (P_r, A_r) be the parabolic pair in G corresponding to $(\mathfrak{p}_r, \mathfrak{a}_r)$. Take a Langlands decomposition $P_r = U_r A_r M_r$ of P_r. Then we have $G = P_r K = U_r A_r M_r K$, and for any g in G we denote by $a_r(g)$ the A_r-part of g.

For any $s \in \mathbf{C}$ and $g \in G$, we define

$$E_r(s; g) = \sum_{\gamma \in \Gamma \cap P_r \backslash \Gamma} \varphi_s^{(r)}(\gamma g),$$

where we write $\varphi_s^{(r)}(g) = \omega_{2s\bar{\omega}_r}(a_r(g))$. It follows from the general theory of the Eisenstein series that the sum in the right hand side converges absolutely for $\mathrm{Re}(s) > n - (r-1)/2$. The relation between the two Eisenstein series $\tilde{E}_r$ and E_r is given by the following

LEMMA 2.1. *There exists a domain* $V \subset \{s \in \mathbf{C} \,|\, \mathrm{Re}(s) > n - (r-1)/2\}$ *such that for all* $s \in V$

$$E_r(s: g) = c\cdot\tilde{E}_r(2s - 2n + r: g),$$

in which c *is a non-zero constant given by*

$$c = \prod_{j=1}^{n-r} \xi(2j) \prod_{j=2}^{r} \xi(j) .$$

THEOREM 2.2. *Let*

$$\mathscr{E}_r(s: g) = \prod_{i=1}^{r} \xi(2s + 1 - i) \prod_{i=1}^{[r/2]} \xi(4s - 2n + 2r - 2i)\cdot E_r(s: g) .$$

For any $g \in G$ *the function* $\mathscr{E}_r(s: g)$ *is meromorphic in* s *on the entire complex plane and holomorphic for* $\mathrm{Re}(s) > (2n - r + 1)/2$. *It satisfies a*

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functional equation

$$\mathscr{E}_r(s:g) = \mathscr{E}_r\left(\frac{2n-r+1}{2} - s:g\right).$$

It has a simple pole at $s = n - (r-1)/2$ *with residue*

$$\frac{1}{2}\prod_{j=2}^{r}\xi(j)\prod_{j=1}^{[r/2]}\xi(2n-2r+2j+1).$$

Proof. In the Weyl group W consider an element w for which we have $w\varepsilon_j = \varepsilon_{r+1-j}$ for $1 \leq j \leq r$. Then our theorem follows from the functional equation of the Eisenstein series $E(\lambda(z):g)$ for w. For more details see the proof of [3] Theorem 2′. Q.E.D.

If $r \geq 2n - 2r + 1$, then cancellations of elementary factors occur and we can replace $\mathscr{E}_r(s:g)$ by

$$\prod_{i=1}^{2n-2r+1}\xi(2s+1-i)\prod_{i=1}^{[r/2]}\xi(4s-2n+2r-2i)\cdot E_r(s:g).$$

Of course the resiaue at $s = n - (r-1)/2$ would be

$$\frac{1}{2}\prod_{j=2}^{2n-2r+1}\xi(j)\prod_{j=1}^{[r/2]}\xi(2n-2r+2j+1).$$

By definition $E_r(s:g)$ is right K-invariant as a function on G. Hence it may be considered as a function on the Siegel upper half plane. We define a function $E_r^o(s:\tau)$ on H_n by

$$E_r^o(s:g\langle i1_n\rangle) = E_r(s:g),$$

for all $g \in G$. If we put $\tau = g\langle i1_n\rangle$, we have

$$\varphi_s^{(r)}(g) = \left(\frac{\det \operatorname{Im}(\tau)}{\det \operatorname{Im}(\tau_{22})}\right)^s,$$

in which we decompose τ into blocks $\begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{pmatrix}$ with $\tau_{11} \in H_r$, $\tau_{22} \in H_{n-r}$. Therefore we have another expression for $E_r^o(s:\tau)$:

$$E_r^o(s:\tau) = \sum_{\gamma \in \Gamma \cap P_r \backslash \Gamma}\left(\frac{\det \operatorname{Im}(\gamma\langle\tau\rangle)}{\det \operatorname{Im}(\gamma\langle\tau\rangle)_{22}}\right)^s.$$

All the statements about E_r in this section are easily reformulated in terms of E_r^o.

In the special where $r=1$, we can say much more. For another extreme case where $r=n$, see [3] Theorem 2′.

THEOREM 2.3. *Let*

$$\mathscr{E}_1(s:g)=\xi(2s)E_1(s:g).$$

For any $g \in G$, the function $\mathscr{E}_1(s:g)$ is holomorphic in s on the entire complex plane except for simple poles at $s=n$ and $s=0$ with residues $\frac{1}{2}$, $-\frac{1}{2}$, respectively. It satisfies the functional equation

$$\mathscr{E}_1(s:g)=\mathscr{E}_1(n-s:g).$$

Proof. All we have to do is to prove that $\mathscr{E}_1(s:g)$ is holomorphic in the half plane $0<\mathrm{Re}(s)$ except for simple pole at $s=n$. For that purpose it suffices to consider the constant term $\mathscr{E}_{1,P}(s:g)$ is the Fourier expansion of $\mathscr{E}_1(s:g)$ relative to the Borel subgroup P (see [3] Lemma 2.3). It is easy to see that for any $a \in A$ and $m \in M$ we have

$$\mathscr{E}_{1,P}(s:am)=\xi(2s)\sum_{w\alpha_i<0,\,i>1}\tilde{c}(w:s)\omega_{w\langle 2s-2n+1,1,\cdots,1\rangle+\rho}(a),$$

where the summation is taken over the set of $w \in W$ such that $w\alpha_i<0$ for $i>1$,

$$\tilde{c}(w:s)=\prod_{\alpha\in\Sigma',\,w\alpha<0}c(\alpha,s),$$

$$\Sigma'=\{2\varepsilon_1,\ \varepsilon_1\pm\varepsilon_j,\ 1<j\leq n\},$$

and

$$c(\alpha,s)=\begin{cases}\dfrac{\xi(2s-n)}{\xi(2s-n+1)} & \text{if } \alpha=2\varepsilon_1\\[2ex] \dfrac{\xi(2s-2n+j-1)}{\xi(2s-2n+j)} & \text{if } \alpha=\varepsilon_1-\varepsilon_j\\[2ex] \dfrac{\xi(2s-j+1)}{\xi(2s-j+2)} & \text{if } \alpha=\varepsilon_1+\varepsilon_j.\end{cases}$$

Now consider an element w in W such that $w\alpha_i<0$ for $2\leq i\leq n$. If $w\alpha_1<0$, then such $w=w_0$ is unique and $w_0\alpha<0$ for all $\alpha\in\Sigma$. Therefore in this case we have

$$\tilde{c}(w_0:s)=\frac{\xi(2s-2n+1)}{\xi(2s)}.$$

So let us assume that $w\alpha_1=w(\varepsilon_1-\varepsilon_2)>0$. If $w(2\varepsilon_1)<0$, then $w(\varepsilon_1+\varepsilon_j)$

< 0 for all $1 < j \leq n$. Suppose that $1 \leq j < k \leq n$. Since $\varepsilon_1 - \varepsilon_k = (\varepsilon_1 - \varepsilon_j) + (\varepsilon_j - \varepsilon_k)$, it is easy to see that if $w(\varepsilon_1 - \varepsilon_k) > 0$, then $w(\varepsilon_1 - \varepsilon_j) > 0$. Take the largest integer k such that $w(\varepsilon_1 - \varepsilon_k) > 0$, then we have

$$\tilde{c}(w\colon s) = \frac{\xi(2s - 2n + k)}{\xi(2s)}.$$

We note that the above condition determines the signatures of $w\alpha$ for all possitive roots α, so such an element u_k in W is unique. Actually it is given by $u_k = w_{k-1} \cdots w_1 w_0$, where for $1 \leq i \leq k$, w_i denotes the reflection defined by the simple root α_i.

On the other hand, if $w(2\varepsilon_1) > 0$ then $w(\varepsilon_1 - \varepsilon_j) > 0$ for all j. Similarly take the largest integer k such that $w(\varepsilon_1 + \varepsilon_k) < 0$, then we have

$$\tilde{c}(w\colon s) = \frac{\xi(2s - k + 1)}{\xi(2s)}.$$

Therefore we know that the singularities of $\mathscr{E}_{1,P}(s\colon g)$ for $\mathrm{Re}(s) \geq n/2$ are at most simple poles at $s = (n + j)/2$, $0 \leq j \leq n$. An easy calculation shows that

$$\begin{aligned} u_k \lambda(1 - k, 1, \cdots, 1) &= u_{k+1} \lambda(1 - k, 1, \cdots, 1) \\ &= \bar{\omega}_k - \rho. \end{aligned}$$

Since $\xi(s)$ has simple poles at $s = 1$ and $s = 0$ with residues 1 and -1 respectively, it follows that $\mathscr{E}_{1,P}$ is holomorphic at $s = n - k/2$, $1 \leq k < n$. Similarly, by considering the element $w_n \cdots w_1 w_0$, we can show that $\mathscr{E}_{1,P}$ is holomorphic at $s = n/2$. On the other hand, the functional equation shows that $\mathscr{E}_{1,P}$ is holomorphic for $0 < \mathrm{Re}(s) < n$ as well. Q.E.D.

§ 3. Rankin-Selberg convolution

Let F and H be Siegel cusp forms of weight l for Γ_n. We fix an integer r, $1 \leq r \leq n$, and consider the parabolic subgroup P_r. For any positive definite half-integral matrix $S \in \mathrm{Sym}_r(\mathbf{Q})$, we denote by f_S and h_S the S-th Fourier-Jacobi coefficient relative to P_r of F and H, respectively (see Section 1). We shall consider a Dirichlet series defined by

$$D_r(F, H\colon s) = \sum_{S/\sim} \frac{1}{\varepsilon(S)} \frac{(f_S, h_S)}{(\det S)^s},$$

in which the summation is taken over the set of representatives of the $GL(r, \mathbf{Z})$-equivalence class of positive definite half-integral symmetric matrices and, for any such S, $\varepsilon(S)$ denotes the order of its automorphism group.

LEMMA 3.1. *The series*

$$\sum_{S/\sim} \frac{1}{\varepsilon(S)} \frac{(f_S, h_S)}{(\det S)^s}$$

converges absolutely for $\mathrm{Re}(s) > l + (r+1)/2$ *and represents a holomorphic function there,*

Proof. Since F and H are cusp forms we have

$$|f_S| \leq c_F \cdot (\det S)^{l/2}, \qquad |h_S| \leq c_H \cdot (\det S)^{l/2},$$

in which c_F and c_H are constants depending only on F and H, respectively. Therefore we have $|(f_S, h_S)| \leq c \cdot \det S^l$, with a positive constant c. On the other hand it is well known that the series

$$\sum_{S/\sim} \frac{1}{\varepsilon(S)} \frac{1}{(\det S)^s}$$

is absolutely convergent for $\mathrm{Re}(s) > (r+1)/2$ (see [9]). Q.E.D.

It is a general philosophy due to Rankin and Selberg, that the analytic properties of $D_r(F, H\colon s)$ follow from those of the Eisenstein series E_r via the convolution $(FE_r(s\colon *), H)$.

THEOREM 3.2. *For* $\mathrm{Re}(s) > n - l - (r-1)/2$, *we have*

$$(FE_r(s\colon *), H) = c \cdot (4\pi)^{-r(s+l-n+(r-1)/2)} \prod_{k=1}^{r} \Gamma\left(s + l - n + \frac{k-1}{2}\right) \cdot D_r\left(F, H\colon s + l - n + \frac{r-1}{2}\right),$$

with a positive constant c.

Proof. Since $E_r(s\colon *)$ is an automorphic form in the sense of [2], and since F and H are cusp forms, the integral $(FE_r(s\colon *), H)$ converges absolutely if $\mathrm{Re}(s)$ is sufficiently large. It follows from the definition that

$$(FE_r(s\colon *), H) = \int_{\Gamma\backslash G} F(g) E_r(s\colon g) \overline{H(g)} dg = \int_{\Gamma \cap P_r \backslash G} F(g) \varphi_s^{(r)}(g) \overline{H(g)} dg\,.$$

Since $G = P_r K$, we can normalize the Haar measures on G, P_r, and K so that

$$dg = dpdk\,,$$

where dk is the Haar measure on K such that $\int_K dk = 1$ and dp is a left Haar measure on P_r. The integrand $F(g)\varphi_s^{(r)}(g)\,\overline{H(g)}$ is K-invariant on the right, therefore we have

$$(FE_r(s:*), H) = \int_{\Gamma\cap P_r\backslash P_r} F(p)\varphi_s^{(r)}(p)\overline{H(p)}dp\,.$$

Let $P_r = U_r A_r M_r$ be the Langlands decomposition of P_r. It is well known that a left Haar measure dp on P_r is given by

$$dp = e^{-2\rho_r(a)}dudadm\,,$$

in which $2\rho_r = (2n - r + 1)\bar{\omega}_r$ is the sum of roots in Σ_r (see Section 2) and du, da, dm represent Haar measures on U_r, A_r and M_r, respectively. We shall change our notation slightly. Write an element p in P_r in the form $p = \eta\tilde{w}$ in which $\eta \in G_{n-r,r}$, $w \in GL(r, \mathbf{R})$ and $\tilde{w} = \mathrm{diag}(w, 1_{n-r}, {}^t w^{-1}, 1_{n-r})$. Then, in terms of the new coordinates, we have

$$dp = |\det w|^{-2n+r-1}d\eta dw\,,$$

in which $d\eta$ and dw are the Haar measures on $G_{n-r,r}$ and $GL(r, \mathbf{R})$, respectively. Also by definition we get $\varphi_s^{(r)}(\eta\tilde{w}) = |\det w|^{2s}$. Substitute the Fourier-Jacobi expansions into the integrand. Concerning about the Petersson inner product of Jacobi forms, we remark that $(f_T, h_T) = (f_S, h_S)$ if T and S are equivalent and $(f_T, h_S) = 0$ if $T \neq S$. Therefore we have

$$\begin{aligned}&(FE_r(s:*), H)\\ &\quad = \sum_{S/\sim}\frac{1}{\varepsilon(S)}\int_{GL(r,\mathbf{R})}|\det w|^{2s+2l-2n+r-1}e^{-4\pi tr(S[w])}dw\int_{G_{m,r}(\mathbf{Z})\backslash G_{m,r}} f_S(\eta)\overline{h_S(\eta)}d\eta\,.\end{aligned}$$

Then our theorem follows from the following lemma.

LEMMA 3.3 ([6]). *Let S be a positive definite symmetric matrix of degree r. Then we have for* $\mathrm{Re}(s) > r - 1$

$$\int_{GL(r,\mathbf{R})}|\det w|^s e^{-4\pi tr(S[w])}dw = c_r\cdot(\det S)^{-s/2}(4\pi)^{-(r/2)s}\prod_{k=1}^{r}\Gamma\Big(\frac{s}{2} - \frac{k-1}{2}\Big)\,,$$

where c_r is a positive constant depending on the normalization of the Haar measure.

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Combining Theorem 2.2 and Theorem 3.2 we obtain the following

THEOREM 3.4. *Let*

$$\mathscr{D}_r(F, H: s) = \left(F\mathscr{E}_r\left(s + n - l - \frac{r-1}{2} : *\right), H\right).$$

Then $\mathscr{D}_r(F, H: s)$ *can be continued meromorphically to the entire complex plane and holomorphic for* $\mathrm{Re}(s) > l$. *It has a simple pole at* $s = l$. *It satisfies a functional equation*

$$\mathscr{D}_r(F, H: s) = \mathscr{D}_r\left(F, H: 2l - n + \frac{r-1}{2} - s\right).$$

Remark. Note that $\mathscr{D}_r(F, H: s)$ is a constant multiple of

$$(4\pi)^{-rs} \prod_{k=1}^{r} \Gamma\left(s - \frac{k-1}{2}\right) \xi(2s - 2l + 2n + 2 - r - k)$$
$$\times \prod_{i=1}^{[r/2]} \xi(4s - 4l + 2n + 2 - 2i) \times D_r(F, H: s).$$

In the special case where $r = 1$, we have a better result by Theorem 2.3.

THEOREM 3.5. *Assume that* $r = 1$. *Then* $\mathscr{D}_1(F, H: s)$ *is holomorphic on* $\mathbf{C}$ *except for simple poles at* $s = l$ *and* $s = l - n$. *The residue at* $s = l$ *is* $\frac{1}{2}(F, H)$. *It satisfies the functional equation*

$$\mathscr{D}_1(F, H: s) = \mathscr{D}_1(F, H: 2l - n - s).$$

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